On Asymmetric TSP: Transformation to Symmetric TSP and Performance Bound *

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Abstract

We show that an instance of traveling salesman problem (TSP) of size n with an *asymmetric* distance matrix can be transformed into an instance of TSP of size $2n$ with a *symmetric* distance matrix. This is an improvement over earlier transformations of this kind which *triple* the size of the problem. Next we use this transformation to obtain a Hamiltonian tour of a general TSP (which may be asymmetric and/or non-Eucledian) with the worst case performance ratio of $\frac{20}{9}$ when the ratio $\gamma := \frac{d_{max}}{d_{min}}$ is smaller than $\frac{4}{3}$, and $(\frac{11\gamma-8}{3})$ otherwise.

Keywords: Algorithm analysis, algorithm approximation, Asymmetric Traveling salesman problem

1 Introduction

Traveling salesman problem (TSP) [4] is a well-known combinatorial optimization problem. An instance of TSP of size n consists of an n-node complete directed graph and a distance matrix $D = [d_{ij}]_{n\times n}$, where for each i, j, d_{ij} is the distance between nodes i and j. The problem then is to find an optimal *Hamiltonian tour*, i.e., a closed-path of shortest possible length that visits each node exactly once. An instance of TSP is called symmetric, denoted STSP, if D is a symmetric matrix, i.e., if $D^\top = D$, where $D^\top$ denotes the transpose of D; otherwise, it is called asymmetric, denoted ATSP. An instance of TSP is called Euclidean if the triangle inequality holds, i.e.,

$$d_{ij} \le d_{ik} + d_{kj}; \quad \forall i \ne j \ne k.$$

In this paper, we show that an instance of ATSP of size n can be transformed into an instance of STSP of size $2n$ in the sense that an optimal Hamiltonian tour for the ATSP instance can be obtained from the corresponding STSP instance. This represents an improvement over earlier such transformations which *triple* the size of the problem [5, 3].

Since TSP is a NP-complete problem [2], unless $P = NP$, there does not exist a polynomially computable optimal solution for it. So heuristic algorithms are used which provide polynomially computable solutions. It is desirable to have a heuristic algorithm whose *worst case performance ratio*, i.e., the supremal value of the ratio of the cost of the Hamiltonian tour found by of the heuristic algorithm to the cost of an optimal Hamiltonian tour, is as small as possible. Christofides algorithm [4], which is applicable to the symmetric and Eucledian case, provides a worst case performance ratio of $\frac{3}{2}$. However, no heuristic algorithm with a constant worst case performance ratio is known to exist for the general case. [1] studies the worst case performance ratio of some well known heuristic algorithms such as nearest neighbor, cheapest insertion, etc., for the asymmetric but Eucledian case and shows that they are of the order $\Omega(n)$. [1] The paper also presents a heuristic algorithm with worst case performance ratio of order $O(\log_2 n)$. The worst case performance ratio in this case depends on the problem size. It also discusses a few other heuristic algorithms and obtains their worst case performance ratios which depend on the data, i.e., entries in the distance matrix. As mentioned above all these algorithms apply to the asymmetric but Eucledian case. Vishwanathan studies

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[1]Given functions f and g, f is said to be of order $\Omega(g)$ (respectively, $O(g)$), if there exists c such that for all n, $f(n) \ge cg(n)$ (respectively, $f(n) \le cg(n)$).

instances of ATSP where the entries of the distance matrix are either one or two [6] and obtains an algorithm with worst case performance ratio of $\frac{17}{12}$.

In this paper we present a heuristic algorithm for the general non-Eucledian and asymmetric case. The algorithm uses the asymmetric to symmetric transformation. We show that the worst case performance ratio of the algorithm is a constant $\frac{20}{9}$, independent of the data and the problem size, when the ratio of the maximum distance d_{max} to the minimum distance d_{min} satisfies: $\gamma := \frac{d_{max}}{d_{min}} < \frac{4}{3}$. Otherwise, i.e., when $\gamma \geq \frac{4}{3}$, then the worst case performance ratio is $\frac{11\gamma-8}{3}$, which is data dependent but independent of the problem size.

2 Notation and Preliminaries

Let $D_{n\times n}$ be a distance matrix for a TSP of size n. Then we define

$$d_{max} := \max_{i\neq j} d_{ij}; \quad d_{min} := \min_{i\neq j} d_{ij}.$$

D is said to be *positive* if $d_{min} > 0$. The set of all solutions of a TSP of size n is the set of all Hamiltonian tours, which can also be viewed as the set of all *cyclic* permutations of the set $\{1,\ldots,n\}$. We use $\mathcal{T}_D$ to denote the set of all Hamiltonian tours associated with the distance matrix D. Given a Hamiltonian tour $T \in \mathcal{T}_D$ of the form

$$T = i_1 \to i_2 \to \ldots \to i_n,$$

where $i_1 \neq i_2 \neq \ldots \neq i_n$, the *cost* of the tour T, denoted $C_D(T)$, is given by the sum of distances of all edges of the tour, i.e.,

$$C_D(T) := \sum_{k=1}^{n} d_{i_k i_{k+1}},$$

where $i_{n+1} := i_1$. $T^* \in \mathcal{T}_D$ is said to be an optimal Hamiltonian tour if

$$T^* \in \arg\left\{\min_T C_D(T)\right\}.$$

Thus $C_D(T^*)$ represents the cost of an optimal tour.

As discussed in introduction, an important issue is to develop heuristic algorithms that produce polynomially computable near-optimal Hamiltonian tours. One measure of performance of such a heuristic algorithm $\mathcal{A}$ is its *worst case performance ratio*, denoted $R(\mathcal{A})$, which is formally defined as:

$$R(\mathcal{A}) := \sup_D \left\{\frac{C_D(T_{\mathcal{A}})}{C_D(T^*)}\right\},$$

where $T_{\mathcal{A}}$ is the Hamiltonian tour computed by the heuristic algorithm $\mathcal{A}$. Clearly, $R(\mathcal{A}) \geq 1$ for any $\mathcal{A}$. Christofides algorithm that applies to the symmetric Eucledian case provides a worst case performance ratio of $\frac{3}{2}$. This is one of the best known ratios.

Given any subgraph G of the complete directed graph with distance matrix D, we use $C_D(G)$, called cost of G, to denote the sum of distances of all edges in G. An undirected subgraph G is said to be *Eulerian* if the degree, i.e., the total number of edges, of each of its nodes is an even number. Given a graph G, there exists a closed-path containing each edge of G if and only if G is Eulerian. A closed-path containing each edge of G is called an *Eulerian tour*. An Eulerian tour E can be used to obtain a Hamiltonian tour by applying *short-cuts*, i.e., by skipping the nodes that are visited more than once on E.

3 ATSP to STSP Transformation

Consider an asymmetric distance matrix $D = [d_{ij}]_{n\times n}$. We assume that

$$-\infty < d_{min} \leq d_{max} < \infty. \tag{1}$$

We first show that by adding a suitable constant to each entry of D, it is possible to obtain a matrix D' such that $\frac{d'_{max}}{d'_{min}} < \frac{4}{3}$. Define the distance matrix $D' = [d'_{ij}]_{n \times n}$ as follows:

$$\forall i,j : d'_{ij} := \begin{cases} 0 & \text{if } i = j \\ d_{ij} & \text{if } [4d_{min} - 3d_{max}] > 0, i \neq j \\ d_{ij} + [3d_{max} - 4d_{min} + \epsilon] & \text{otherwise,} \end{cases} \tag{2}$$

where $\epsilon > 0$ is a small positive number. It follows from the assumption of (1) and the definition of (2) that

$$-\infty < d'_{min} \leq d'_{max} < \infty.$$

We have the following straightforward lemma, which is key to the results obtained in this paper.

Lemma 1 For any distance matrix D satisfying (1), the matrix D' as defined by (2) is such that $\frac{d'_{max}}{d'_{min}} < \frac{4}{3}$.

Proof: By definition if i,j are such that $d'_{ij} = d'_{min}$ or $d'_{ij} = d'_{max}$, then $i \neq j$. So in order to prove the assertion of the lemma, it suffices to consider the last two cases of the definition given by (2). We show that $4d'_{min} - 3d'_{max} > 0$. First suppose that $[4d_{min} - 3d_{max}] > 0$, i.e., the second case of the definition given by (2) holds, then we have

$$4d'_{min} - 3d'_{max} = 4d_{min} - 3d_{max} > 0.$$

On the other hand, if $[4d_{min} - 3d_{max}] \leq 0$, i.e., the third case of the definition given by (2) holds, then we have

$$\begin{aligned} 4d'_{min} - 3d'_{max} &= [4(d_{min} + 3d_{max} - 4d_{min} + \epsilon)] - [3(d_{max} + 3d_{max} - 4d_{min} + \epsilon)] \\ &= [12d_{max} - 12d_{min} + 4\epsilon] - [12d_{max} - 12d_{min} + 3\epsilon] \\ &= \epsilon \\ &> 0, \end{aligned}$$

as desired. ■

Using the distance matrix D' we define a symmetric distance matrix $\overline{D} = [\overline{d}_{ij}]_{2n \times 2n}$ which we show below is a desired symmetric distance matrix:

$$\overline{D} := \left[\begin{array}{c|c} \infty & (D')^\top \\ \hline D' & \infty \end{array} \right] \tag{3}$$

For notational simplicity, given $i \leq n$, we use $[i]$ to denote $i + n$. Thus $[1] = 1 + n, [n] = 2n$, etc. We call i and $[i]$ to be a *complementary* pair of nodes. Furthermore, for each $i \leq n$, the node i is called a *real* node, whereas the node $[i]$ is called a *virtual* node. Note that for each i, j,

$$\overline{d}_{ij} = \overline{d}_{[i][j]} = \infty; \quad \overline{d}_{[i]j} = \overline{d}_{j[i]} = d'_{ij}; \quad \overline{d}_{[i]i} = \overline{d}_{i[i]} = d'_{ii} = 0.$$

In other words, the distance between a pair of real or virtual nodes is infinity, whereas the distance between a real and virtual node is finite and symmetric, and the distance between a complementary pair of nodes is zero.

Next we define the notion of a feasible Hamiltonian tour for the distance matrix $\overline{D}$. Feasible Hamiltonian tours induce a Hamiltonian tour for the original asymmetric distance matrix D. A Hamiltonian tour $\overline{T} \in \mathcal{T}_{\overline{D}}$ is said to be *feasible* if it visits real and virtual nodes alternately, and visits a complementary node immediately after visiting a node, i.e., it is of the form:

$$\overline{T} = i_1 \to [i_1] \to i_2 \to [i_2] \to \ldots \to i_n \to [i_n], \tag{4}$$

where $i_1 \neq i_2 \neq \ldots \neq i_n$. Clearly, $\overline{T}$ is a Hamiltonian tour, as it is a cyclic permutation of the set $\{1, \ldots, n, n+1, \ldots, 2n\} = \{1, \ldots, n, [1], \ldots, [n]\}$. Let $\mathcal{T}^f_{\overline{D}} \subseteq \mathcal{T}_{\overline{D}}$ denote the set of all feasible Hamiltonian tours. It is easy to see that a feasible Hamiltonian tour of the symmetric distance matrix $\overline{D}$ induces a Hamiltonian tour of the original asymmetric distance matrix D, and vice-versa.

Definition 1 Given a feasible Hamiltonian tour $\overline{T} \in \mathcal{T}^f_{\overline{D}}$ of the form (4), it induces a Hamiltonian tour $T \in \mathcal{T}_D$ defined as:

$$T := i_1 \to i_2 \to \ldots \to i_n. \tag{5}$$

Similarly given a Hamiltonian tour $T \in \mathcal{T}_D$ of the form (5), it induces a feasible Hamiltonian tour $\overline{T} \in \mathcal{T}^f_{\overline{D}}$ of the form (4).

It can also be easily seen that:

$$\forall T \in \mathcal{T}_D : C_{\overline{D}}(\overline{T}) = \begin{cases} C_D(T) & \text{if } [4d_{min} - 3d_{max}] > 0 \\ C_D(T) + n[3d_{max} - 4d_{min} + \epsilon] & \text{otherwise.} \end{cases} \tag{6}$$

Similarly,

$$\forall \overline{T} \in \mathcal{T}^f_{\overline{D}} : C_D(T) = \begin{cases} C_{\overline{D}}(\overline{T}) & \text{if } [4d_{min} - 3d_{max}] > 0 \\ C_{\overline{D}}(\overline{T}) - n[3d_{max} - 4d_{min} + \epsilon] & \text{otherwise.} \end{cases} \tag{7}$$

The following lemma proves that the set of optimal Hamiltonian tours of the symmetric distance matrix is contained in the set of its feasible Hamiltonian tours.

Lemma 2 Consider the distance matrix $\overline{D}$ as defined by (3). Then $\mathcal{T}^*_{\overline{D}} \subseteq \mathcal{T}^f_{\overline{D}}$.

Proof: We need to show that if $\overline{T} \in \mathcal{T}^*_{\overline{D}}$ is an optimal Hamiltonian tour, then $\overline{T}$ is a feasible Hamiltonian tour, i.e., it is of the form (4). First note that since $\overline{d}_{ij} = \overline{d}_{[i][j]} = \infty$ (for each $i \neq j$), given any $\overline{T} \in \mathcal{T}_{\overline{D}}$, $C_{\overline{D}}(\overline{T}) < \infty$ if and only if it visits real and virtual nodes alternately, i.e., $\overline{T}$ is of the form:

$$\overline{T} = i_1 \to [j_1] \to i_2 \to [j_2] \to \ldots \to i_n \to [j_n], \tag{8}$$

where $i_1 \neq i_2 \neq \ldots \neq i_n$, and $j_1 \neq j_2 \neq \ldots \neq j_n$. Hence if $\overline{T} \in \mathcal{T}^*_{\overline{D}}$, it must be of the form (8). It remains to show that for each k, $j_k = i_k$.

Suppose for contradiction that there exists k such that $j_k \neq i_k$. Let l be the smallest such number. Then

$$\overline{T} = i_1 \to [i_1] \to \ldots \to i_l \to [j_l] \to i_{l+1} \to [j_{l+1}] \to \ldots i_m \to [i_l] \to i_{m+1} \to \ldots \to i_n \to [j_n].$$

Consider a new Hamiltonian tour $\overline{T}'$:

$$\overline{T}' := i_1 \to [i_1] \to \ldots \to i_l \to [i_l] \to i_{l+1} \to [j_{l+1}] \to \ldots i_m \to [j_l] \to i_{m+1} \to \ldots \to i_n \to [j_n],$$

which is obtained by only exchanging $[j_l]$ with $[i_l]$ in $\overline{T}$. Then we obtain the following contradiction to the optimality of $\overline{T}$:

$$\begin{aligned} C_{\overline{D}}(\overline{T}) - C_{\overline{D}}(\overline{T}') &= [\overline{d}_{i_l[j_l]} + \overline{d}_{[j_l]i_{l+1}} + \overline{d}_{i_m[i_l]} + \overline{d}_{[i_l]i_{m+1}}] - [\overline{d}_{i_l[i_l]} + \overline{d}_{[i_l]i_{l+1}} + \overline{d}_{i_m[j_l]} + \overline{d}_{[j_l]i_{m+1}}] \\ &= [\overline{d}_{i_l[j_l]} + \overline{d}_{[j_l]i_{l+1}} + \overline{d}_{i_m[i_l]} + \overline{d}_{[i_l]i_{m+1}}] - [\overline{d}_{[i_l]i_{l+1}} + \overline{d}_{i_m[j_l]} + \overline{d}_{[j_l]i_{m+1}}] \\ &= [d'_{j_l i_l} + d'_{j_l i_{l+1}} + d'_{i_l i_m} + d'_{i_l i_{m+1}}] - [d'_{i_l i_{l+1}} + d'_{j_l i_m} + d'_{j_l i_{m+1}}] \\ &\geq 4d'_{min} - 3d'_{max} \\ &> 0, \end{aligned}$$

where the second equality follows from the fact that $\overline{d}_{i_l[i_l]} = 0$, and the last inequality follows from Lemma 1. ■

Remark 1 In the proof of Lemma 2, the fact that the distance between a pair of real nodes or a pair of virtual nodes is infinity was needed only to conclude that an optimal Hamiltonian tour must visit real and virtual nodes alternately. Clearly, this can be concluded even when the distance between a pair of real or a pair of virtual nodes is a sufficiently large finite value. Since each edge of an optimal Hamiltonian tour of $\overline{D}$ has a cost smaller than d'_{max}, the value $2n(d'_{max})$ will serve the purpose.

We are now ready to state the main result of this section:

Theorem 1 If $T \in \mathcal{T}^*_D$, then $\overline{T} \in \mathcal{T}^*_{\overline{D}}$. Conversely, if $\overline{T} \in \mathcal{T}^*_{\overline{D}}$, then $T \in \mathcal{T}^*_D$.

Proof: We only prove the first part; the second part can be proved analogously. Let $T \in \mathcal{T}^*_D$. Then by definition $\overline{T} \in \mathcal{T}^f_{\overline{D}}$. Suppose for contradiction that $\overline{T} \notin \mathcal{T}^*_{\overline{D}}$. Pick $\overline{T}' \in \mathcal{T}^*_{\overline{D}}$ which is clearly non-empty. Then it follows from Lemma 2 that $\overline{T}' \in \mathcal{T}^f_{\overline{D}}$; consequently, the induced Hamiltonian tour $T' \in \mathcal{T}_D$ can be defined. From the facts that $\overline{T} \notin \mathcal{T}^*_{\overline{D}}$ and $\overline{T}' \in \mathcal{T}^*_{\overline{D}}$, it follows $C_{\overline{D}}(\overline{T}) > C_{\overline{D}}(\overline{T}')$. This together with (7) implies that $C_D(T) > C_D(T')$. This is a contradiction to the fact that $T \in \mathcal{T}^*_D$. ■

Remark 2 It follows from Remark 1 that an optimal Hamiltonian tour of the asymmetric distance matrix $D_{n\times n}$ can be computed by computing an optimal Hamiltonian tour of the symmetric matrix $\overline{D}'_{2n\times 2n}$ defined as:

$$\overline{D}' := \left[\begin{array}{c|c} K & (D')^\top \\ \hline D' & K \end{array}\right],$$

where $K_{n\times n}$ is a matrix consisting of constant entries $2n(d'_{max})$. From a practical point of view, a computation based on the $\overline{D}'$ distance matrix is easier than that based on the $\overline{D}$ distance matrix.

4 Performance Bound for Asymmetric TSP

We proved in the previous section that the problem of finding an optimal Hamiltonian tour for any asymmetric distance matrix $D = [d_{ij}]_{n\times n}$ can be transformed into that of finding an optimal Hamiltonian tour for the corresponding symmetric distance matrix $\overline{D} = [\overline{d}_{ij}]_{2n\times 2n}$. We use this transformation to construct a polynomially computable solution tour of the given ATSP, and derive its worst case performance ratio. Our construction is based on the the Christofides algorithm which applies to the symmetric and Eucledian case, and provides the worst case performance ratio of $\frac{3}{2}$ [4].

Given a symmetric and Eucledian distance matrix $D_{n\times n}$, the Christofides algorithm constructs a polynomially computable Hamiltonian tour as follows:

Step 1. Obtain a minimal spanning tree S^* for the set of all nodes (an order $O(n^2)$ construction).

Step 2. Obtain a minimal matching M^* for the set of all odd-degree vertices in S^* (an order $O(n^3)$ construction).

Step 3. Obtain an *Eulerian* tour for the *Eulerian* graph obtained as the union of S^* and M^*, and convert it to a *Hamiltonian* tour T using *short-cuts* (an order $O(n)$ construction).

Since $C_D(S^*) \le C_D(T^*)$ and $C_D(M^*) \le \frac{1}{2}C_D(T^*)$, and since triangle inequality holds, it follows that $C_D(T) \le C_D(S^*) + C_D(M^*) \le \frac{3}{2}C_D(T^*)$, i.e., the Christofides algorithm provides the worst case performance ratio of $\frac{3}{2}$.

This algorithm can be modified for the general case as follows. Given a general distance matrix $D_{n\times n}$, we first transform it into an instance of STSP using the construction of the previous section. We know that the resulting matrix $\overline{D}_{2n\times 2n}$ is such that the entries in the sub-matrix $D'_{n\times n}$ satisfy properties of Lemma 1. This can be used to prove further properties of the distance matrix D'. We exploit some of these properties in developing the heuristic algorithm.

Corollary 1 Consider the distance matrix D' defined by (2). Then

1. D' is positive.
2. $d'_{ij} < \frac{4}{3}d'_{ji}$ for any $i \neq j$.
3. $d'_{ij} < d'_{kl} + d'_{k'l'}$ for any $i \neq j, k \neq l, k' \neq l'$.
4. D' is Eucledian.

Proof: For the first part, it suffices to show that $d'_{min} > 0$. Suppose for contradiction that $d'_{min} \le 0$, which implies $\frac{4}{3}d'_{min} \le d'_{min}$. Then from Lemma 1 we obtain the following contradiction:

$$d'_{max} < \frac{4}{3}d'_{min} \le d'_{min}.$$

The second part can be shown as follows:

$$d'_{ij} \le d'_{max} < \frac{4}{3}d'_{min} \le \frac{4}{3}d'_{ji},$$

where the second inequality follows from Lemma 1.

The third part can be shown as follows:

$$d'_{ij} \leq d'_{max} < \frac{4}{3}d'_{min} < 2d'_{min} \leq d'_{kl} + d'_{k'l'},$$

where the second inequality follows from Lemma 1, and the third inequality follows from the fact that $d'_{min} > 0$.

Finally, for the last part we need to show that $d'_{ij} \leq d'_{ik} + d'_{kj}$ for all $i \neq j \neq k$, which follows from the third part.

■

The first step in the algorithm is to construct a minimal spanning tree of $\overline{S}^*$ for the $2n$-node graph with symmetric distance matrix $\overline{D}$. Since the distance between a real node and a virtual node is infinity, such an edge does not belong to the constructed minimal spanning tree. Also, since (i) the distance between a complementary pair of nodes is zero, and (ii) from Corollary 1 the distance between a pair of real and virtual nodes is positive, we can assume without loss of generality that edges connecting complementary pair of nodes belong to the minimal spanning tree. Since a spanning tree can be obtained by removing an edge from an optimal Hamiltonian tour, and since all distances in an optimal tour are non-negative (Corollary 1), it follows that the cost of a minimal spanning tree is upper bounded by the cost of an optimal Hamiltonian tour, i.e.,

$$C_{\overline{D}}(\overline{S}^*) \leq C_{\overline{D}}(\overline{T}^*). \tag{9}$$

Refer to Figure 1(a) for an illustration of a minimal spanning tree.

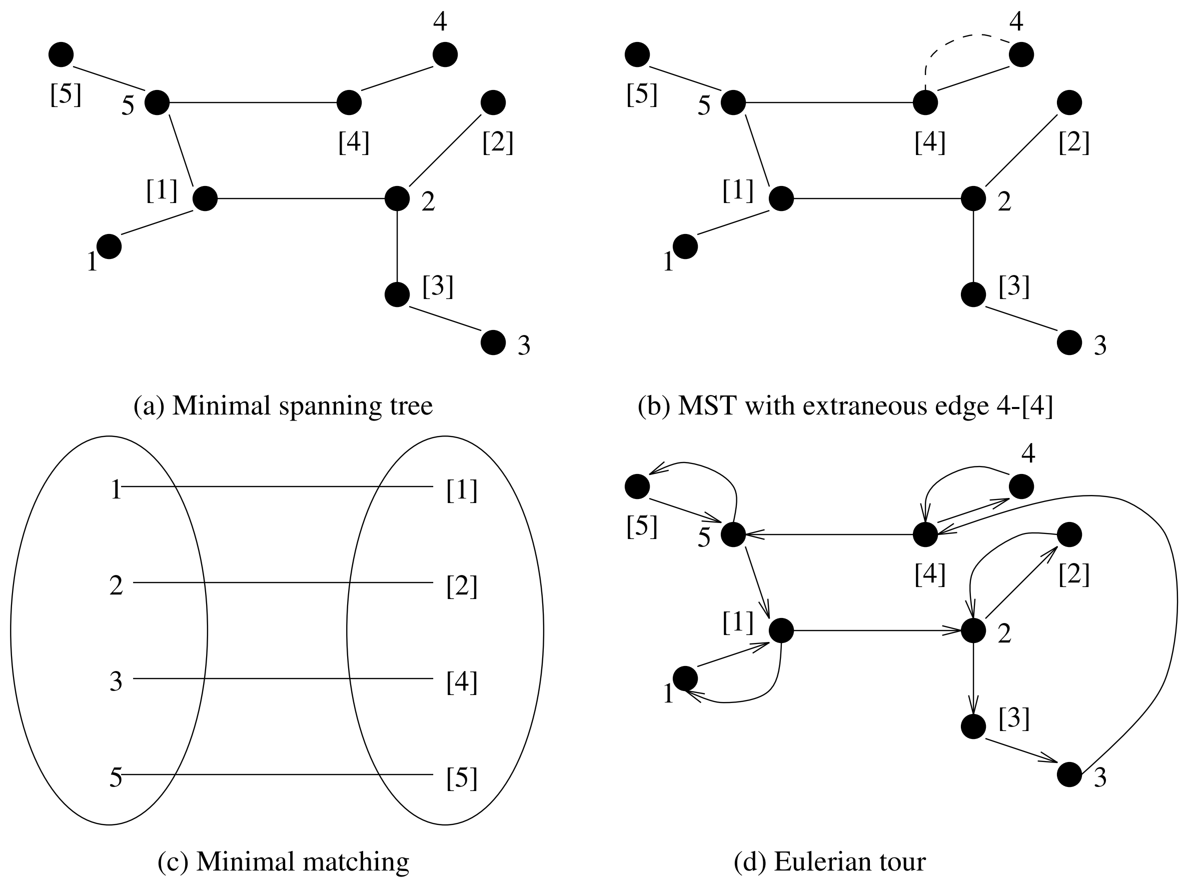

Figure 1: Tour construction in ATSP setting

The next step in the algorithm is to obtain a minimal matching of the set of odd-degree nodes of $\overline{S}^*$. The problem one may encounter is that the number of real odd-degree nodes and the number of virtual odd-degree node may not be same. For example in Figure 1(a) all five real nodes are of odd-degree, whereas only three virtual nodes [1], [2] and [5] are of odd-degree. Say there are more number of real odd-degree nodes. Then during the matching phase, a real node will be matched to another real node which will add an edge of infinite cost. (Recall that $\overline{d}_{ij} = \infty$ for any $i, j \leq n$.) So we add zero-cost "extraneous" edges to balance the number of real and virtual odd-degree nodes. If there are more number of real odd-degree nodes, then there exists i such that it is odd-degree, whereas the degree of $[i]$ is even. So we add a zero-cost extraneous edge between i and $[i]$. Then the degree of i becomes even, whereas the degree of $[i]$ becomes odd. For example in Figure 1(b) we add the extraneous edge between nodes 4 and [4].

By adding zero-cost extraneous edges between complementary pair of nodes as needed, we balance the number of real and virtual odd-degree nodes. The cost of the resulting graph remains the same as $C_{\overline{D}}(\overline{S}^*)$. A minimal matching $\overline{M}^*$ of the odd-degree nodes is then obtained. Refer to Figure 1(c) for an illustration. Then we have the following lemma:

Lemma 3 Let $\overline{M}^*$ be a minimal matching of the odd-degree nodes in the graph obtained by union of $\overline{S}^*$ and extraneous edges. Then $C_{\overline{D}}(\overline{M}^*) < \frac{2}{3} C_{\overline{D}}(\overline{T}^*)$.

Proof: From Lemma 2, $\mathcal{T}^*_{\overline{D}} \subseteq \mathcal{T}^f_{\overline{D}}$. So $\overline{T}^*$ is of the form:

$$\overline{T}^* = i_1 \rightarrow [i_1] \rightarrow i_2 \rightarrow [i_2] \rightarrow \ldots \rightarrow i_n \rightarrow [i_n].$$

Then a matching $\overline{M}_1$ (respectively, $\overline{M}_2$) of the odd-degree nodes can be obtained by matching a real odd-degree node i to the virtual odd-degree node $[j]$ which is the "nearest right (respectively, left) neighbor" of i on the Hamiltonian tour $\overline{T}^*$. Since the length of any edge of D' is smaller than the lenght of two or more of its edges (Corollary 1, part 3), and since any edge of D' is smaller than $\frac{4}{3}$ times the length of the corresponding reverse edge of D' (Corollary 1, part 2), it follows that

$$C_{\overline{D}}(\overline{M}_1) + C_{\overline{D}}(\overline{M}_2) < \frac{4}{3} C_{\overline{D}}(\overline{T}^*).$$

This together with the fact that $C_{\overline{D}}(\overline{M}^*) \leq \frac{1}{2}[C_{\overline{D}}(\overline{M}_1) + C_{\overline{D}}(\overline{M}_2)]$ establishes the desired inequality. ■

It follows from the above discussions that the cost of the Eulerian graph obtained by taking the union of $\overline{S}^*$, $\overline{M}^*$, and the extraneous edges is smaller than $C_{\overline{D}}(\overline{T}^*) + \frac{2}{3} C_{\overline{D}}(\overline{T}^*) = \frac{5}{3} C_{\overline{D}}(\overline{T}^*)$. So if $\overline{E}$ is the associated Eulerian tour, then

$$C_{\overline{D}}(\overline{E}) < \frac{5}{3} C_{\overline{D}}(\overline{T}^*).$$

We construct $\overline{E}$ such that whenever possible it visits a complementary node immediately after visiting a node. Since there is at least one and at most two edges between each pair of complementary nodes in the Eulerian graph, such motions occur at least once and at most twice. Refer to Figure 1(d) which contains the directed edges $[1] \rightarrow 1 \rightarrow [1]$, $2 \rightarrow [2] \rightarrow 2$, $[3] \rightarrow 3$, $[4] \rightarrow 4 \rightarrow [4]$, and $5 \rightarrow [5] \rightarrow 5$.

The final step in the algorithm is the construction of an Hamiltonian tour from the Eulerian tour $\overline{E}$ using short-cuts. While constructing a Hamiltonian tour using short-cuts, a node is skipped if it appears more than once on the Eulerian tour. However, care must be taken in performing short-cuts so as not to increase the tour cost. Consider the first node where a short-cut needs to be performed. Suppose it is a real node labeled i, and suppose the next unvisited node on the Eulerian tour is j, which is also real. Since j is unvisited, $[j]$ must be the node immediately after j on the Eulerian tour. So we perform the short-cut $i \rightarrow [j]$ for which the cost is $\overline{d}_{i[j]} = d'_{ji} < \infty$. If it turns out that the cost $\overline{d}_{i[j]}$ is not comparable to the cost of the Eulerian tour segment:

$$\ldots \rightarrow [i] \rightarrow i \rightarrow \ldots \rightarrow j \rightarrow [j] \rightarrow \ldots,$$

then instead of performing the shortcut

$$\ldots \rightarrow [i] \rightarrow i \rightarrow [j] \rightarrow \ldots$$

we perform the short-cut

$$\ldots \rightarrow [i] \rightarrow j \rightarrow [j] \rightarrow \ldots.$$

A similar analysis is needed for applying short-cuts if the Eulerian tour visits $[j]$ before j, i.e., if it is of the form:

$$\ldots \rightarrow [i] \rightarrow i \ldots \rightarrow [j] \rightarrow j \rightarrow \ldots.$$

This is clarified further in the example below.

Since the cost of an edge of D' is smaller than two or more of its edges (Corollary 1, parts 3 and 1), the cost of the resulting tour $\overline{T}$ is smaller than the cost of the Eulerian tour, i.e.,

$$C_{\overline{D}}(\overline{T}) < \frac{5}{3} C_{\overline{D}}(\overline{T}^*). \tag{10}$$

Note that there is a slight abuse of notation since $\overline{T}$ is not necessarily a Hamiltonian tour of the $2n$-node graph, as it may not visit a node as well as its complementary node. However, $\overline{T}$ visits at least one node from each complementary pair. So it does induce a Hamiltonian tour T of the original n-node graph.

Example 1 Consider for example the Eulerian tour of Figure 1(d):

$$1 \to [1] \to 2 \to [2] \to 2 \to [3] \to 3 \to [4] \to 4 \to [4] \to 5 \to [5] \to 5 \to [1] \to 1.$$

Then the first short-cut must be performed at [2] since the next node 2 has already been visited. Since [3] is the next unvisited node, according to our short-cut procedure, we first try the short-cut

$$\ldots \to 2 \to [2] \to 3 \to \ldots$$

for which the cost is $\overline{d}_{[2]3} = d'_{23}$. This however is not comparable to the cost of the Eulerian tour segment

$$\ldots \to 2 \to [2] \to 2 \to [3] \to 3 \to \ldots,$$

which is $\overline{d}_{2[3]} = d'_{32}$ and is different from d'_{23} due to asymmetry. So instead we perform the short-cut:

$$\ldots \to 2 \to [3] \to 3 \to \ldots$$

for which the cost is $\overline{d}_{2[3]} = d'_{32}$, as required. Proceeding as above, the "pseudo-Hamiltonian" tour $\overline{T}$ obtained by applying short-cuts is:

$$1 \to [1] \to 2 \to [3] \to 3 \to [4] \to 5 \to [1].$$

So the resulting Hamiltonian tour T of the original n-node graph is:

$$1 \to 2 \to 3 \to 4 \to 5 \to 1.$$

Our construction can be summarized in form of the following algorithm:

Algorithm 1 Given an arbitrary distance matrix $D_{n\times n}$ satisfying (1) construct a Hamiltonian tour T for the n-node graph as follows:

Step 1. Obtain the distance matrix $\overline{D}_{2n\times 2n}$ as defined by (3).

Step 2. Obtain a minimal spanning tree $\overline{S}^*$ for the $2n$-node graph.

Step 3. Add zero-cost extraneous edges between complementary pair of nodes so as to make the number of real and virtual odd-degree nodes equal.

Step 4. Obtain a minimal matching $\overline{M}^*$ for the set of all odd-degree vertices in the graph obtained as union of $\overline{S}^*$ and the extraneous edges.

Step 5. Obtain an *Eulerian* tour $\overline{E}$ for the *Eulerian* graph obtained as the union of $\overline{S}^*, \overline{M}^*$, and the extraneous edges such that whenever possible it visits a complementary node immediately after visiting a node.

Step 6. Obtain a "pseudo-Hamiltonian" tour $\overline{T}$ from $\overline{E}$ by carefully applying short-cuts so that there is no increase in tour cost, and finally obtain the corresponding Hamiltonian tour T of the n-node graph.

Since an edge of the tour $\overline{T}$ may get replaced by its reverse edge when the tour T is constructed, the costs of the two tours may be different. However, it follows from the second part of Corollary 1 that

$$C_{D'}(T) < \frac{4}{3} C_{\overline{D}}(\overline{T}).$$

This together with (10) implies that

$$C_{D'}(T) < \left(\frac{4}{3}\right)\left(\frac{5}{3}\right) C_{\overline{D}}(\overline{T}^*) = \frac{20}{9} C_{\overline{D}}(\overline{T}^*). \tag{11}$$

Then we have the following result for the worst case performance ratio of Algorithm 1:

Theorem 2 For any distance matrix $D_{n\times n}$ satisfying (1), the Hamiltonian tour T generated by Algorithm 1 is such that

$$C_D(T) \leq \begin{cases} \frac{20}{9} C_D(T^*) & \text{if } \gamma < \frac{4}{3} \\ \frac{11\gamma - 8}{3} C_D(T^*) & \text{otherwise,} \end{cases}$$

where $\gamma := \frac{d_{max}}{d_{min}}$.

Proof: First suppose $\gamma < \frac{4}{3}$. Then $C_D(T) = C_{D'}(T)$. Also from (6), $C_{\overline{D}}(\overline{T}^*) = C_D(T^*)$. So it follows from (11) that

$$C_D(T) = C_D(T') < \frac{20}{9} C_{\overline{D}}(\overline{T}^*) = \frac{20}{9} C_D(T^*),$$

as desired.

Next suppose $\gamma \geq \frac{4}{3}$, then $C_D(T) = C_{D'}(T) - n\lambda$, where $\lambda := [3d_{max} - 4d_{min} + \epsilon] = (3\gamma - 4 + \frac{\epsilon}{d_{min}})d_{min}$. Also from (6), we have $C_{\overline{D}}(\overline{T}^*) = C_D(T^*) + n\lambda$. So by using (11) we obtain:

$$C_D(T) + n\lambda = C_{D'}(T) < \frac{20}{9} C_{\overline{D}}(\overline{T}^*) = \frac{20}{9}(C_D(T^*) + n\lambda).$$

In other words,

$$C_D(T) < \frac{20}{9} C_D(T^*) + \frac{11}{9}(n\lambda).$$

By substituting $\lambda = (3\gamma - 4 + \frac{\epsilon}{d_{min}})d_{min}$, we obtain

$$C_D(T) < \frac{20}{9} C_D(T^*) + \frac{11}{9}(3\gamma - 4 + \frac{\epsilon}{d_{min}})nd_{min}.$$

Since ϵ can be chosen to be arbitrary small, the term $\frac{\epsilon}{d_{min}}$ may be dropped from the right hand side of the inequality, and the strict inequality may be replaced by a non-strict inequality:

$$C_D(T) \leq \frac{20}{9} C_D(T^*) + \frac{11}{9}(3\gamma - 4)nd_{min}.$$

Finally, since $nd_{min} \leq C_D(T^*)$, we obtain

$$C_D(T) \leq \left[\frac{20}{9} + \frac{11}{9}(3\gamma - 4)\right] C_D(T^*) = \left(\frac{11\gamma - 8}{3}\right) C_D(T^*),$$

as desired. ■

5 Discussion

We showed that an instance of ATSP can be transformed into an instance of STSP by doubling the problem size. This result is more of a theoretical interest since there exist several "tour construction" and "tour improvement" heuristics for the ATSP case which provide good solution tours. The practical usefulness of the result is that using it we are able to develop a heuristic algorithm for computing a solution tour in a general setting for which the worst case performance ratio can be computed. The worst case performance ratio is independent of the data as well as the problem size when the ratio $\gamma := \frac{d_{max}}{d_{min}}$ is smaller than $\frac{4}{3}$. This result is in contrast to the common belief that such an algorithm is unlikely to exist even for the Eucledian case [4].

Our results are based on the key Lemma 1 which states that for any distance matrix D the ratio $\gamma = \frac{d_{max}}{d_{min}}$ can be made smaller than $\frac{4}{3}$ by adding a suitable constant to each of the entries in the distance matrix. In fact we can add a suitable constant to each of the entries of the distance matrix so that this ratio is any value $\delta > 1$. For example one can add a constant zero if $\gamma < \delta$, and $\frac{\gamma - \delta}{(\delta - 1)d_{min}}$ otherwise. We need δ to be smaller than $\frac{4}{3}$ for Lemma 2 and other results that depend on this lemma to hold. A more detailed but similar analysis as above yields the following measure of performance of Algorithm 1 when such a constant is added:

$$C_D(T) \leq \begin{cases} \delta[1 + \frac{\delta}{2}] C_D(T^*) & \text{if } \gamma < \delta \\ \left[\delta(1 + \frac{\delta}{2}) + (\frac{\gamma - \delta}{\delta - 1})(\delta(1 + \frac{\delta}{2}) - 1)\right] C_D(T^*) & \text{otherwise} \end{cases}$$

This result reduces to the result of Theorem 2 when $\delta = \frac{4}{3}$, and provides tighter worst case performance ratio for smaller γ values.

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