Solving 1^{st}-order & 2^{nd}-order differential equations under constant inputs

1^{st}-order:

$$\dot{x} + \frac{x}{\tau} = f \qquad (\tau: \text{time constant},\ f: \text{constant input})$$

Solution: $x(t) = K_0 + K_1 e^{s_1 t}$,

where s_1 solution of $s + \frac{1}{\tau} = 0 \Rightarrow s_1 = -\frac{1}{\tau}$

To find K_0 and K_1 use $x(\infty)$ and $x(0)$ $\Rightarrow \begin{cases} x(\infty) = K_0 \\ x(0) = K_0 + K_1 \end{cases}$.

2^{nd}-order:

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = f \quad (\zeta: \text{damping ratio},\ \omega_0: \text{undamped freq.},\ f: \text{const. input})$$

Solution: $x(t) = \begin{cases} K_0 + K_1 e^{s_1 t} + K_2 e^{s_2 t} & \text{if } \zeta \neq 1 \\ K_0 + K_1 e^{s_1 t} + K_2 t e^{s_2 t} & \text{if } \zeta = 1 \end{cases}$, where

s_1 & s_2 solutions of $s^2 + 2\zeta\omega_0 s + \omega_0^2 = 0 \Rightarrow s_{1,2} = \omega_0\left[-\zeta \pm \sqrt{\zeta^2 - 1}\right]$.

Note when $\zeta = 1$, $s_1 = s_2 = -\omega_0$.

To find K_0, K_1, K_2 use $x(\infty), x(0), \dot{x}(0)$ $\Rightarrow \begin{cases} x(\infty) = K_0 \\ x(0) = \begin{cases} K_0 + K_1 + K_2 & \zeta \neq 1 \\ K_0 + K_1 & \zeta = 1 \end{cases} \\ \dot{x}(0) = \begin{cases} K_1 s_1 + K_2 s_2 & \zeta \neq 1 \\ K_1 s_1 + K_2 & \zeta = 1 \end{cases} \end{cases}$

inductor: $v(t) = L\frac{di}{dt}$, $w(t) = \frac{1}{2}L i^2(t)$, $L = \frac{\mu N^2 A}{l} = \frac{\phi}{i}$

$L_{series} = L_1 + L_2$; $L_{parallel} = \frac{L_1 L_2}{L_1 + L_2}$

Capacitor: $i(t) = C\frac{dv}{dt}$, $w(t) = \frac{1}{2}C v^2(t)$, $C = \frac{\varepsilon A}{d} = \frac{q}{V}$

$C_{parallel} = C_1 + C_2$; $C_{series} = \frac{C_1 C_2}{C_1 + C_2}$

Note: K_0 is constant only when input f is constant.

For example, $f = a e^{bt} \Rightarrow K_0 = c e^{bt}$

Mutual inductance: $V_1(t) = L\frac{di_1}{dt} \pm M\frac{di_2}{dt}$ (+ if aiding magnetic fields, − if opposing magnetic fields)