- Butterworth order-K low-pass filter with cut-off freq. $= \omega_c$ : $|H(j\omega)|^2 = \dfrac{1}{1+(\omega/\omega_c)^{2K}}$

- Given normalized low-pass filter, $\begin{cases} lp2hp & s \to \dfrac{\omega_c}{s} \\ lp2bp & s \to \dfrac{s^2+\omega_0^2}{Bs} \\ lp2br & s \to \dfrac{Bs}{s^2+\omega_0^2} \end{cases}$

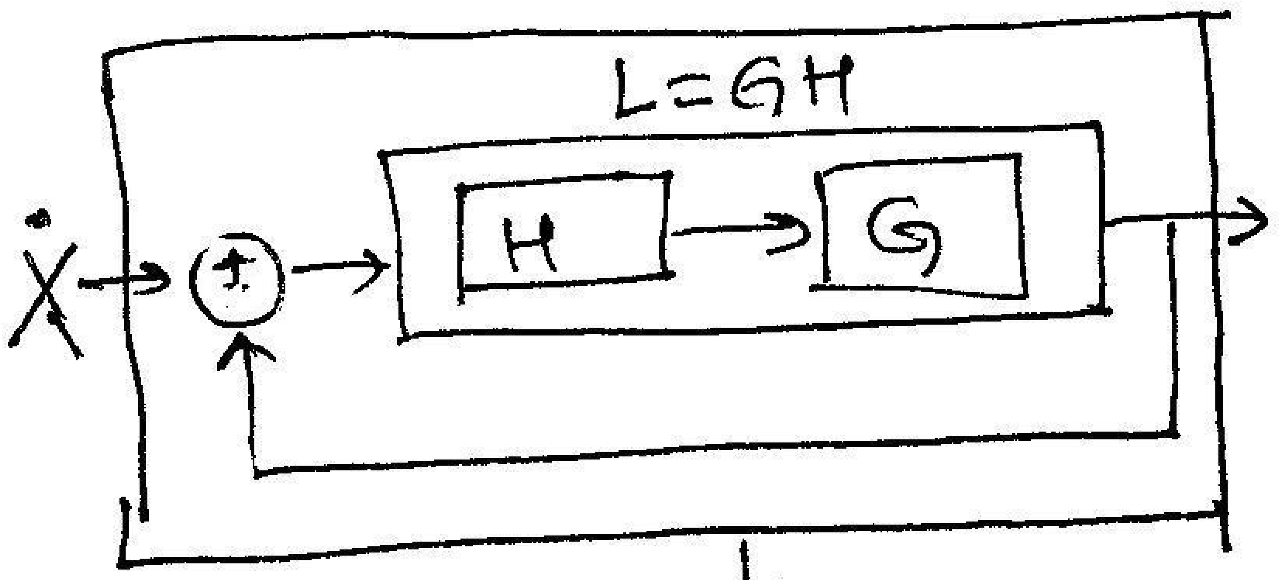
- FIR filter desired freq. response : $H_d(e^{j\Omega}) = H(e^{j\Omega})\, e^{-j\Omega M/2}$ (M= window size)

$$h_{FIR}[n] = h_d[n]\, w[n]$$

- IIR filter design: $H(j\omega) \xrightarrow[\text{filter design}]{\text{analog}} H(s) \longrightarrow H(z)$ $\quad s = \frac{2}{T}\left(\frac{z-1}{z+1}\right) \Rightarrow \underbrace{\omega = \frac{2}{T}\tan\left(\frac{\Omega}{2}\right)}_{\text{used for preserving critical freqs.}}$

- T is sampling period and is ~~found~~ based on freq. of interest, say $[\omega_0, \omega_0]$

$\Rightarrow T \leq \dfrac{2\pi}{2\omega_0} = \dfrac{\pi}{\omega_0}$ $\quad \left(= \dfrac{2\pi}{\text{Bandwidth}}\right)$




$$T = \frac{L}{1+L}$$

sensitivity, $S = \dfrac{\Delta T/T}{\Delta G/G} = \dfrac{1}{1+L}$

Effect of noise, $Y = TX + \dfrac{1}{1+L}\text{Noise}$

Analyze denominator of T to determine stability of T

- Routh-Hurwitz : Analyze denominator of T to determine stability of T as a function of loop-gain K of L(s).
- Root-locus : Plot of poles of T(s) as a function of loop-gain K of L(s). Starts at poles of L, ends at zeros of L

\# branches = max. num/den. degree; starts at poles of L, ends at zeros of L

\# asymptotes = \#poles − \# zeros; $\begin{cases} \text{angles} = \dfrac{\text{odd multiples of } \pi}{\#\text{ asymptotes}} \\ \text{meet at} : \dfrac{\text{sum(poles)} - \text{sum(zeros)}}{\#\text{ asymptotes}} \end{cases}$

breakaway point $\Rightarrow \left[\dfrac{d}{ds}\dfrac{1}{L(s)} = 0\right]$ (converse need not hold)

angle criteria : Sum (pole angles) − Sum(zero angles) = odd multiple of $\pi$

magnitude criteria : product (pole distances) / product (zero distances) = K

Use Routh-Hurwitz to find poles on imaginary axis, and gain value there

- Nyquist plot : Plot of $L(j\omega)$ as $\omega$ ranges from $+\infty$ to $-\infty$ (K is fixed and given)

\# RHP poles of T = \# RHP poles of L + \# Counterclock encirclement of (−1,0)

- Gain margin = $\dfrac{1}{|L(j\omega_p)|}$ $\quad (\angle L(j\omega_p) = \pi)$ $\quad$ phase margin = $\angle L(j\omega_g) - \pi$ $\quad (|L(j\omega_g)| = 1)$

## *z*-Transforms

| Signal | Transform | ROC |
|---|---|---|
| $x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$ | $X[z] = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$ | ROC |
| $\delta[n]$ | $1$ | All $z$ |
| $u[n]$ | $\frac{1}{1 - z^{-1}}$ | $\lvert z\rvert > 1$ |
| $\alpha^n u[n]$ | $\frac{1}{1 - \alpha z^{-1}}$ | $\lvert z\rvert > \lvert\alpha\rvert$ |
| $n\alpha^n u[n]$ | $\frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}$ | $\lvert z\rvert > \lvert\alpha\rvert$ |
| $[\cos(\Omega_1 n)]u[n]$ | $\frac{1 - z^{-1}\cos\Omega_1}{1 - z^{-1}2\cos\Omega_1 + z^{-2}}$ | $\lvert z\rvert > 1$ |
| $[\sin(\Omega_1 n)]u[n]$ | $\frac{z^{-1}\sin\Omega_1}{1 - z^{-1}2\cos\Omega_1 + z^{-2}}$ | $\lvert z\rvert > 1$ |
| $[r^n\cos(\Omega_1 n)]u[n]$ | $\frac{1 - z^{-1}r\cos\Omega_1}{1 - z^{-1}2r\cos\Omega_1 + r^2z^{-2}}$ | $\lvert z\rvert > r$ |
| $[r^n\sin(\Omega_1 n)]u[n]$ | $\frac{z^{-1}r\sin\Omega_1}{1 - z^{-1}2r\cos\Omega_1 + r^2z^{-2}}$ | $\lvert z\rvert > r$ |

### ■ E.1.1 Bilateral Transforms for Signals that Are Nonzero for $n < 0$

| Signal | Bilateral Transform | ROC |
|---|---|---|
| $u[-n-1]$ | $\frac{1}{1 - z^{-1}}$ | $\lvert z\rvert < 1$ |
| $-\alpha^n u[-n-1]$ | $\frac{1}{1 - \alpha z^{-1}}$ | $\lvert z\rvert < \lvert\alpha\rvert$ |
| $-n\alpha^n u[-n-1]$ | $\frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}$ | $\lvert z\rvert < \lvert\alpha\rvert$ |

## *z*-Transform Properties

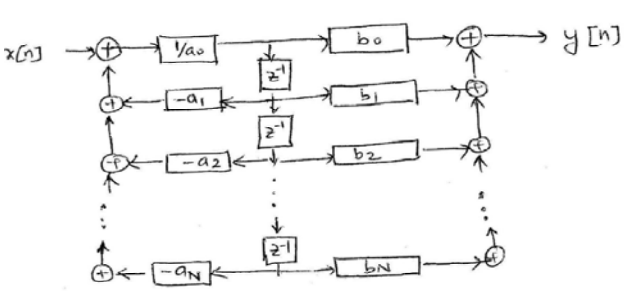
| Signal | Unilateral Transform $x[n] \xleftrightarrow{z_u} X(z)$ $y[n] \xleftrightarrow{z_u} Y(z)$ | Bilateral Transform $x[n] \xleftrightarrow{z} X(z)$ $y[n] \xleftrightarrow{z} Y(z)$ | ROC $z \in R_x$ $z \in R_y$ |
|---|---|---|---|
| $ax[n] + by[n]$ | $aX(z) + bY(z)$ | $aX(z) + bY(z)$ | At least $R_x \cap R_y$ |
| $x[n-k]$ | See below | $z^{-k}X(z)$ | $R_x$, except possibly $\lvert z\rvert = 0, \infty$ |
| $\alpha^n x[n]$ | $X\left(\frac{z}{\alpha}\right)$ | $X\left(\frac{z}{\alpha}\right)$ | $\lvert\alpha\rvert R_x$ |
| $x[-n]$ | — | $X\left(\frac{1}{z}\right)$ | $\frac{1}{R_x}$ |
| $x[n] * y[n]$ | $X(z)Y(z)$ if $x[n] = y[n] = 0$ for $n < 0$ | $X(z)Y(z)$ | At least $R_x \cap R_y$ |
| $nx[n]$ | $-z\frac{d}{dz}X(z)$ | $-z\frac{d}{dz}X(z)$ | $R_x$, except possibly addition or deletion of $z = 0$ |

### ■ E.2.1 Unilateral *z*-Transform Time-Shift Property

$$x[n-k] \xleftrightarrow{z_u} x[-k] + x[-k+1]z^{-1} + \cdots + x[-1]z^{-k+1} + z^{-k}X(z) \quad \text{for } k > 0$$

$$x[n+k] \xleftrightarrow{z_u} -x[0]z^{k} - x[1]z^{k-1} - \cdots - x[k-1]z + z^{k}X(z) \quad \text{for } k > 0$$

<u>diagram</u> of $a_0 y[n] + a_1 y[n-1] + \cdots + a_N y[n-N] = b_0 x[n] + \cdots + b_N x[n-N]$



## Laplace Transforms

| Signal $x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}\,ds$ | Transform $X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$ | ROC |
|---|---|---|
| $u(t)$ | $\frac{1}{s}$ | $\text{Re}\{s\} > 0$ |
| $tu(t)$ | $\frac{1}{s^2}$ | $\text{Re}\{s\} > 0$ |
| $\delta(t-\tau), \quad \tau \geq 0$ | $e^{-s\tau}$ | for all $s$ |
| $e^{-at}u(t)$ | $\frac{1}{s+a}$ | $\text{Re}\{s\} > -a$ |
| $te^{-at}u(t)$ | $\frac{1}{(s+a)^2}$ | $\text{Re}\{s\} > -a$ |
| $[\cos(\omega_1 t)]u(t)$ | $\frac{s}{s^2+\omega_1^2}$ | $\text{Re}\{s\} > 0$ |
| $[\sin(\omega_1 t)]u(t)$ | $\frac{\omega_1}{s^2+\omega_1^2}$ | $\text{Re}\{s\} > 0$ |
| $[e^{-at}\cos(\omega_1 t)]u(t)$ | $\frac{s+a}{(s+a)^2+\omega_1^2}$ | $\text{Re}\{s\} > -a$ |
| $[e^{-at}\sin(\omega_1 t)]u(t)$ | $\frac{\omega_1}{(s+a)^2+\omega_1^2}$ | $\text{Re}\{s\} > -a$ |

### ■ D.1.1 Bilateral Laplace Transforms for Signals That Are Nonzero for $t < 0$

| Signal | Bilateral Transform | ROC |
|---|---|---|
| $\delta(t-\tau), \tau < 0$ | $e^{-s\tau}$ | for all $s$ |
| $-u(-t)$ | $\frac{1}{s}$ | $\text{Re}\{s\} < 0$ |
| $-tu(-t)$ | $\frac{1}{s^2}$ | $\text{Re}\{s\} < 0$ |
| $-e^{-at}u(-t)$ | $\frac{1}{s+a}$ | $\text{Re}\{s\} < -a$ |
| $-te^{-at}u(-t)$ | $\frac{1}{(s+a)^2}$ | $\text{Re}\{s\} < -a$ |

| Signal | Unilateral Transform $x(t) \xleftrightarrow{\mathcal{L}_u} X(s)$, $y(t) \xleftrightarrow{\mathcal{L}_u} Y(s)$ | Bilateral Transform $x(t) \xleftrightarrow{\mathcal{L}} X(s)$, $y(t) \xleftrightarrow{\mathcal{L}} Y(s)$ | ROC $s \in R_x$, $s \in R_y$ |
|---|---|---|---|
| $ax(t) + by(t)$ | $aX(s) + bY(s)$ | $aX(s) + bY(s)$ | At least $R_x \cap R_y$ |
| $x(t-\tau)$ | $e^{-s\tau}X(s)$ if $x(t-\tau)u(t) = x(t-\tau)u(t-\tau)$ | $e^{-s\tau}X(s)$ | $R_x$ |
| $e^{s_o t}x(t)$ | $X(s-s_o)$ | $X(s-s_o)$ | $R_x + \text{Re}\{s_o\}$ |
| $x(at)$ | $\frac{1}{a}X\left(\frac{s}{a}\right), \quad a > 0$ | $\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$ | $\frac{R_x}{\lvert a\rvert}$ |
| $x(t) * y(t)$ | $X(s)Y(s)$ if $x(t) = y(t) = 0$ for $t < 0$ | $X(s)Y(s)$ | At least $R_x \cap R_y$ |
| $-tx(t)$ | $\frac{d}{ds}X(s)$ | $\frac{d}{ds}X(s)$ | $R_x$ |
| $\frac{d}{dt}x(t)$ | $sX(s) - x(0^-)$ | $sX(s)$ | At least $R_x$ |
| $\int_{-\infty}^{t} x(\tau)\,d\tau$ | $\frac{1}{s}\int_{-\infty}^{0^-} x(\tau)\,d\tau + \frac{X(s)}{s}$ | $\frac{X(s)}{s}$ | At least $R_x \cap \{\text{Re}\{s\} > 0\}$ |

### ■ D.2.1 Initial-Value Theorem

$$\lim_{s\to\infty} sX(s) = x(0^+)$$

This result does not apply to rational functions $X(s)$ in which the order of the numerator polynomial is equal to or greater than the order of the denominator polynomial. In that case, $X(s)$ would contain terms of the form $cs^k$, $k \geq 0$. Such terms correspond to the impulses and their derivatives located at time $t = 0$.

### ■ D.2.2 Final-Value Theorem

$$\lim_{s\to 0} sX(s) = \lim_{t\to\infty} x(t)$$

This result requires that all the poles of $sX(s)$ be in the left half of the $s$-plane.

### ■ D.2.3 Unilateral Differentiation Property, General Form

$$\frac{d^n}{dt^n}x(t) \xleftrightarrow{\mathcal{L}_u} s^n X(s) - \left.\frac{d^{n-1}}{dt^{n-1}}x(t)\right|_{t=0^-}$$

$$- s\left.\frac{d^{n-2}}{dt^{n-2}}x(t)\right|_{t=0^-} - \cdots - s^{n-2}\left.\frac{d}{dt}x(t)\right|_{t=0^-} - s^{n-1}x(0^-)$$