Online Robust PCA
or
Online Sparse + Low-Rank Matrix Recovery

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  - Chenlu Qiu and Brian Lois

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- Other collaborators: Han Guo (new student) and Prof. Leslie Hogben (Math, ISU)
Recovery from incomplete data: the question

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  - Question: can we recover the cross-section’s image from undersampled data?
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  \[ \text{image} = \text{background} + \text{foreground} \]

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  \[
  \text{image} = \text{background} + \text{foreground}
  \]
  - Question: can we recover two image sequences from one?
  - Yes: if exploit the low-rank structure of the background sequence and sparseness of the foreground
Outline

- Online Sparse Matrix Recovery
  (Recursive Recovery of Sparse Vector Sequences)
  - brief overview

- Online Sparse + Low-Rank Matrix Recovery
  (Online or Recursive Robust PCA)
  - most of this talk
Sparse recovery / Compressed sensing: Magnetic Resonance Imaging

▶ (a) Shepp-Logan phantom: 256 × 256 image
▶ (b) MR imaging pattern: 256-point DFT along 22 radial lines
▶ (c) Inverse-DFT
▶ (d) Basis Pursuit solution (uses sparsity: gives exact recovery!)

Example taken from [Candes, Romberg, Tao, T-IT, Feb 2006]
Sparse recovery / Compressive sensing [Mallat et al'93], [Feng,Bresler'96], [Gordinsky,Rao'97], [Chen,Donoho'98], [Candes,Romberg,Tao'05],[Donoho'05]

- Recover a sparse vector $x$, with support size at most $s$, from
  $$y := Ax + w$$
  when $A$ is a known fat matrix and $\|w\|_2 \leq \epsilon$ (small noise).
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- Applications: projection imaging - MRI, CT, astronomy, single-pixel camera
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- Solution by convex relaxation: $\ell_1$ minimization [Chen,Donoho’98]:

$$\min \|\tilde{x}\|_1 \text{ subject to } \|y - A\tilde{x}\|_2 \leq \epsilon$$

if $\delta_{2s}(A) < 0.4$, error bounded by $C\epsilon$ [Candes et al'05,'06,'08]

- restricted isometry constant (RIC) $\delta_s(A)$: smallest real $\#$ s.t.

$$(1 - \delta_s)\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1 + \delta_s)\|x\|_2^2$$

for all $s$-sparse vectors $x$ [Candes,Tao,T-IT’05]
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Recursive recovery of sparse seq’s: the problem \cite{Vaswani,ICIP’08}

Given measurements
\[ y_t := Ax_t + w_t, \quad \|w_t\|_2 \leq \epsilon, \quad t = 0, 1, 2, \ldots \]

- \( A = H\Phi \) (given): \( n \times m, \ n < m \)
  - \( H \): measurement matrix, \( \Phi \): sparsity basis matrix
  - e.g., in MRI: \( H = \) partial Fourier, \( \Phi = \) inverse wavelet

- \( y_t \): measurements (given)
- \( x_t \): sparsity basis vector
- \( N_t \): support set of \( x_t \)
- \( w_t \): small noise

Goal: recursively reconstruct \( x_t \) from \( y_0, y_1, \ldots y_t \),
- i.e. use only \( y_t \) and \( \hat{x}_{t-1} \) for recovering \( x_t \)

- Use slow support change: \( |N_t \setminus N_{t-1}| \approx |N_{t-1} \setminus N_t| \ll |N_t| \)
  - also use slow signal value change when valid

\[^1\text{N. Vaswani, Kalman Filtered Compressed Sensing, ICIP, 2008}\]
Recursive recovery of sparse seq’s: Solutions [KF-CS, ICIP’08], [LS-CS, T-SP, Aug10]

Introduced

- Kalman filtered CS (KF-CS) and Least Squares CS (LS-CS)
  - *first* set of recursive algorithms that needed fewer measurements for accurate recovery than simple $\ell_1$-min
  - able to obtain time-invariant error bounds on LS-CS error under reasonable assumptions [Vaswani, LS-CS, T-SP, Aug’2010]
  - limitation: exact recovery with fewer meas’s not possible

- Modified-CS and Regularized Modified-CS
  - achieved exact recovery using fewer measurements (weaker RIP assumptions) than simple $\ell_1$-min [Vaswani, Lu, ISIT’09, T-SP, Sept’10]
  - obtained time-invariant error bounds in the noisy case under realistic assumptions [Zhan, Vaswani, ISIT’13, T-IT, March’15]
Recursive recovery of sparse seq’s: Modified-CS [Modified-CS,ISIT’09,T-SP’10,T-IT’15]

- Idea: support at $t-1$, $\mathcal{N}_{t-1}$, is a good predictor of $\mathcal{N}_t$
- Reformulate: Sparse Recovery with Partial Support Knowledge $\mathcal{T}$
  - $\text{support}(x_t) = \mathcal{T} \cup \Delta \setminus \Delta_e$: $\Delta$, $\Delta_e$ unknown
Recursive recovery of sparse seq’s: Modified-CS \[\text{[Modified-CS, ISIT'09, T-SP'10, T-IT'15]}\]

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- **Modified-CS:** tries to find a vector $\tilde{x}$ that is sparsest outside $\mathcal{T}$ among all vectors satisfying the data constraint

\[
\min_{\tilde{x}} ||\tilde{x}_{\mathcal{T}^c}||_1 \text{ subject to } ||y - A\tilde{x}||_2 \leq \epsilon
\]
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- Exact recovery in noise-free case if $\delta_{s+|\Delta|+|\Delta_e|} < 0.4$ [Vaswani, Lu, ISIT’09, T-SP’10]
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$$\min_{\tilde{x}} \|\tilde{x}_{\mathcal{T}^c}\|_1 \text{ subject to } \|y - Ax\|_2 \leq \epsilon$$

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- For noisy case: time-invariant error bounds under a realistic signal change model and $\delta_{s+k_s} < 0.4$ [Zhan, Vaswani, ISIT’13, T-IT’15]
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- Significant advantage over existing work for dynamic MRI
Using slow signal value change

- **Modified-CS on residual (KF-ModCS)**

\[
\hat{x} = \mu + \left[ \arg \min_b \gamma \| b T^c \|_1 + 0.5 \| y - A(\mu + b) \|_2^2 \right]
\]

Here we can use \( \mu = \hat{x}_{t-1} \) (modified-CS-residual) or \( \mu = \) the Kalman filtered estimate of \( x_{t-1} \) on the set \( T \).

- **Regularized modified-CS**

\[
\hat{x} = \arg \min_b \gamma \| b T^c \|_1 + 0.5 \| y - Ab \|_2^2 + \lambda \| (b - \mu) T \|_2^2
\]
This problem:

\[ y_t = Ax_t + w_t \]

where \( x_t \) is sparse and \( w_t \) is small noise: \( \| w_t \|_2 \leq \epsilon \).

Rest of the talk:

\[ y_t = Ax_t + \ell_t \]

where \( x_t \) is sparse and \( \ell_t \) lies in a low-dimensional subspace that is either fixed or slowly-changing over time.

- no constraint on how large \( \ell_t \) can be: the case of (potentially) large but structured noise
- for this problem, even the case \( A = I \) (online robust PCA) is hard
Robust Principal Components’ Analysis (PCA): Background

- Many high-dimensional datasets approximately lie in much lower dimensional subspace
- PCA: estimate the low-dimensional subspace that best approximates a given dataset
  - SVD on data matrix, compute top left singular vectors
- Robust PCA: PCA in presence of outliers; many useful heuristics in older work, e.g., RSL [De la Torre et al, 2003]
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- Robust PCA: PCA in presence of outliers; many useful heuristics in older work, e.g., RSL [De la Torre et al, 2003]

- [Candes et al, 2009] posed robust PCA as: separate a low-rank matrix $L$ and a sparse matrix $X$ from

$$Y := X + L$$

- outliers occur occasionally; when they occur, their magnitude can be large: well modeled as sparse vectors/matrices
Robust PCA: Applications – 1

- Robust PCA: separate low-rank $L$ and sparse $X$ from

$$Y := X + L$$

or from a subset of entries of $(X + L)$

- if $L$ or range($L$) is the quantity of interest: robust PCA
- if $X$ is quantity of interest: robust sparse recovery

- Video analytics (e.g. for surveillance, tracking, mobile video chat, occlusion removal,...) [Candes et al, 2009]

$$X = [x_1, x_2, \ldots, x_t, \ldots x_{t_{\text{max}}}], \quad L = [\ell_1, \ell_2, \ldots \ell_t, \ldots \ell_{t_{\text{max}}}]$$

- $\ell_t$: bg - usually slow changing, global (dense) changes
- $x_t$: fg - sparse, consists of one or more moving objects (technically $x_t$: (fg-bg) on fg support)
Robust PCA: Applications – II

- **Recommendation systems design** [Candes et al'2009]
  (robust PCA with missing entries / robust matrix completion)
  - $\ell_t$: ratings of movies by user $t$
  - the matrix $L$ is low-rank: user preferences governed by only a few factors
  - $x_t$: some users may enter completely incorrect ratings due to laziness or malicious intent or just typos: outliers
  - missing entries: a given user will rate only a subset of all the movies;
  - goal: recover the matrix $L$ in order to recommend movies or other products

- **Detecting anomalous connectivity patterns in social networks or in computer networks**
  - $\ell_t$: vector of n/w link “strengths” at time $t$ when no anomalous behavior
Robust PCA: Applications – III

- \( x_t \): outliers or anomalies on a few links
- functional MRI based brain activity detection or other dynamic MRI based region-of-interest detection problems
  - only a sparse brain region activated in response to stimuli, everything else: very slow changes
A practical provably correct solution: PCP

- [Candes et al, 2009; Chandrasekharan et al, 2009; Hsu et al, 2011] introduced and studied a convex opt program called PCP:

\[
\min_{\tilde{X}, \tilde{L}} \|\tilde{L}\|_* + \lambda \|\tilde{X}\|_1 \quad \text{s.t.} \quad Y = \tilde{X} + \tilde{L}
\]

- If (a) left and right singular vectors of \( L \) are dense enough; (b) support of \( X \) is generated uniformly at random; (c) rank and sparsity are bounded, then PCP exactly recovers \( X \) and \( L \) from \( Y := X + L \) w.h.p. [Candes et al, 2009]

- [Chandrasekharan et al, 2009; Hsu et al, 2011]: similar flavor; replace ‘unif rand support’ by upper bound on \# of nonzeros in any row of \( X \).

- first set of guarantees for a practical robust PCA approach
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- Much later work on the batch robust PCA w/ guarantees
Need for an online method

- Disadvantages of batch methods:
  - slower especially for online applications;
  - memory intensive;
  - do not allow infrequent/slow support change of columns of $X$
    - reason: this can result in $X$ being rank deficient

- Video analytics: have occasionally static or slow moving foreground objects; often need online solution

- Functional MRI: the activated brain region does not change a lot from frame to frame

- Network anomaly detection: anomalous behavior continues for a period of time after begins; need an online solution
Online Sparse Matrix Recovery

Online Robust PCA

Background, Problem Formulation and Related Work
ReProCS Algorithm and Correctness Result
Proof Outline and Experiments

(a) Background recovery

(b) Foreground recovery

Figure: ReProCS: proposed. Background, $t = t_0 + 60, 120, 199, 475, 1148$.
Some definitions for rest of the talk

- $P$ is a basis matrix $\iff P'P = I$, i.e. $P$ is a tall matrix with mutually orthonormal columns

- Estimate $P \iff$ estimate range($P$): subspace spanned by col's of $P$

- $\hat{P}$ is an accurate estimate of $P \iff SE(\hat{P}, P) := \|(I - \hat{P}\hat{P}')P\|_2 \ll 1$
“Online” robust PCA: the problem \cite{Qiu,Vaswani,Allerton'10,'11} \cite{Guo,Qiu,Vaswani,T-SP'14} \footnote{C. Qiu and N. Vaswani, Real-time Robust Principal Components’ Pursuit, Allerton, 2010\newline H. Guo, C. Qiu, N. Vaswani, An Online Algorithm for Separating Sparse and Low-Dimensional Signal Sequences From Their Sum”, IEEE Trans.SP, Aug 2014}

- Given sequentially arriving \(n\)-length data vectors \(y_t\) satisfying

\[
y_t := \ell_t, \quad t = 1, 2, \ldots, t_0
\]

and

\[
y_t := x_t + \ell_t, \quad t = t_0 + 1, t_0 + 2, \ldots, t_{\text{max}}
\]

- \(x_t\)'s are sparse vectors with support sets, \(T_t\), of size at most \(s\);
- \(\ell_t\)'s lie in a slowly-changing low-dimensional subspace of \(\mathbb{R}^n\);
  - \(\Leftrightarrow \ell_t = P_t a_t \text{ w/ } \| (I - P_{t-1} P_{t-1}') \ell_t \|_2 \ll \| \ell_t \|_2 \) (\(P_t\): tall)
“Online” robust PCA: the problem [Qiu,Vaswani,Allerton’10,’11] [Guo,Qiu,Vaswani,T-SP’14] ²

- Given sequentially arriving $n$-length data vectors $y_t$ satisfying
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  and
  \[ y_t := x_t + \ell_t, \quad t = t_0 + 1, t_0 + 2, \ldots, t_{\max} \]
  - $x_t$'s are sparse vectors with support sets, $T_t$, of size at most $s$;
  - $\ell_t$'s lie in a slowly-changing low-dimensional subspace of $\mathbb{R}^n$;
    - $\leftrightarrow \ell_t = P_t a_t \text{ w/ } \| (I - P_{t-1} P_{t-1}') \ell_t \|_2 \ll \| \ell_t \|_2 \ (P_t: \text{ tall})$
  - support sets of $x_t$, $T_t$ have at least some changes over time
  - left singular vectors of the matrix $L_t := [\ell_1, \ell_2, \ldots, \ell_t]$ are dense

² C. Qiu and N. Vaswani, Real-time Robust Principal Components’ Pursuit, Allerton, 2010
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- support sets of \( x_t, T_t \) have at least some changes over time
- left singular vectors of the matrix \( L_t := [\ell_1, \ell_2, \ldots, \ell_t] \) are dense

Goal: recursively estimate \( x_t, \ell_t \) and range(\( L_t \)) at all \( t > t_0 \).
“Online” robust PCA: the problem [Qiu, Vaswani, Allerton’10,’11] [Guo, Qiu, Vaswani, T-SP’14] ³

- Initial outlier-free seq $y_t = \ell_t$ for first $t_0$ frames needed to estimate the initial subspace $P_{t_0}$: easy to obtain in many apps, e.g.,
  - in video surveillance, easy to get a short background-only training sequence before fg objects start appearing
  - for fMRI, this corresponds to acquiring a short sequence without any activation
  - alternative: use a batch method (e.g., PCP) for first $t_0$ frames

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- Note: extension of all our ideas to the undersampled case $y_t = Ax_t + B\ell_t$ is easy (relevant to MRI apps)

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3 C. Qiu and N. Vaswani, Real-time Robust Principal Components’ Pursuit, Allerton, 2010
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Related work

Batch robust PCA and performance guarantees

- Older work, e.g. RSL [de la Torre et al, IJCV'03]; PCP and much later work on provably correct robust PCA solutions

Recursive / incremental / online robust PCA algorithms

- Older work (before PCP): [Li et al, ICIP 2003] iRSL: doesn’t work
- [Qiu, Vaswani, Allerton'10, Allerton'11, T-SP’14]: ReProCS (Recursive Projected CS)
- [Balzano et al, CVPR 2012]: GRASTA
- [Mateos et al, JSTSP 2013]: batch, online; online: not enough info, no code

Online robust PCA performance guarantees: almost no work

- [Qiu, Vaswani, Lois, Hogben, ICASSP’13, ISIT’13, T-IT’14]: partial result;
- [Feng et al, NIPS’13 OR-PCA Stoch Opt]: partial result and only asymptotic
- [Lois, Vaswani, ICASSP’15, arXiv:1409.3959]: complete correctness result
Recall: for $t > t_0$, $y_t := x_t + \ell_t$, $\ell_t = P_t a_t$, $P_t$: tall $n \times r$ basis matrix
ReProCS algorithm [Qiu, Vaswani, Allerton’10, Allerton’11],[Guo, Qiu, Vaswani, T-SP’14]⁴

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**Initialize:** compute \( \hat{P}_0 = \) top left singular vectors of \([\ell_1, \ell_2, \ldots, \ell_{t_0}]\).

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**Initialize:** compute \( \hat{P}_0 = \) top left singular vectors of \([\ell_1, \ell_2, \ldots, \ell_{t_0}]\).

For \( t > t_0 \), do

- **Projection:** compute \( \tilde{y}_t := \Phi_t y_t \), where \( \Phi_t := I - \hat{P}_{t-1} \hat{P}_{t-1}' \)
  - then \( \tilde{y}_t = \Phi_t x_t + \beta_t \), \( \beta_t := \Phi_t \ell_t \) is small “noise” because of slow subspace change
ReProCS algorithm \cite{Qiu,Vaswani,Allerton'10,Allerton'11,Guo,Qiu,Vaswani,T-SP'14}\textsuperscript{4}

Recall: for $t > t_0$, $y_t := x_t + \ell_t$, $\ell_t = P_t a_t$, $P_t$: tall $n \times r$ basis matrix

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For $t > t_0$, do

- **Projection:** compute $\tilde{y}_t := \Phi_t y_t$, where $\Phi_t := I - \hat{P}_{t-1} \hat{P}'_{t-1}$
  - then $\tilde{y}_t = \Phi_t x_t + \beta_t$, $\beta_t := \Phi_t \ell_t$ is small “noise” because of slow subspace change

- **Noisy Sparse Recovery:** $\ell_1 \text{ min } + \text{support estimate } + \text{LS}: \text{get } \hat{x}_t$
  - denseness of $P_t$'s $\Rightarrow$ sparse $x_t$ recoverable from $\tilde{y}_t$

\textsuperscript{4}C. Qiu and N. Vaswani, Real-time Robust Principal Components' Pursuit, Allerton, 2010
H. Guo, C. Qiu, N. Vaswani, An Online Algorithm for Separating Sparse and Low-Dimensional Signal Sequences From Their Sum”, IEEE Trans.SP, Aug 2014
Recall: for $t > t_0$, $y_t := x_t + \ell_t$, $\ell_t = P_t a_t$, $P_t$: tall $n \times r$ basis matrix

**Initialize:** compute $\hat{P}_0 = \text{top left singular vectors of } [\ell_1, \ell_2, \ldots, \ell_{t_0}]$.

For $t > t_0$, do

- **Projection:** compute $\tilde{y}_t := \Phi_t y_t$, where $\Phi_t := I - \hat{P}_{t-1} \hat{P}_{t-1}^t$
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- **Recover $\ell_t$:** compute $\hat{\ell}_t = y_t - \hat{x}_t$
Recall: for $t > t_0$, $y_t := x_t + \ell_t$, $\ell_t = P_t a_t$, $P_t$: tall $n \times r$ basis matrix

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For $t > t_0$, do

1. **Projection:** compute $\tilde{y}_t := \Phi_t y_t$, where $\Phi_t := I - \hat{P}_{t-1} \hat{P}_{t-1}'$
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2. **Noisy Sparse Recovery:** $l_1$ min + support estimate + LS: get $\hat{x}_t$
   - denseness of $P_t$'s $\Rightarrow$ sparse $x_t$ recoverable from $\tilde{y}_t$

3. **Recover $\ell_t$:** compute $\hat{\ell}_t = y_t - \hat{x}_t$

4. **Subspace update:** update $\hat{P}_t$ every $\alpha$ frames by projection-PCA

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4 C. Qiu and N. Vaswani, Real-time Robust Principal Components' Pursuit, Allerton, 2010
H. Guo, C. Qiu, N. Vaswani, An Online Algorithm for Separating Sparse and Low-Dimensional Signal Sequences From Their Sum", IEEE Trans.SP, Aug 2014
Why ReProCS works [Qiu, Vaswani, Lois, Hogben, T-IT, 2014]  

- Slow subspace change: noise $\beta_t$ seen by sparse recovery step is small
- Denseness of columns of $P_t \Rightarrow$ RIC of $\Phi_t = I - \hat{P}_{t-1}\hat{P}^\prime_{t-1}$ is small
  - denseness assump: $(2s) \max_{t} \max_{i} \left\| (P_{t-1})_{i,:} \right\|_2^2 \leq 0.09$
  - easy to show [Qiu, Vaswani, Lois, Hogben, T-IT, 2014] :
    \[
    \delta_{2s}(\Phi_t) = \max_{|T| \leq 2s} \left\| I_T \hat{P}_{t-1} \right\|_2^2 \leq (2s) \max_i \left\| (\hat{P}_{t-1})_{i,:} \right\|_2^2 \leq 0.09 + 0.05
    \]
    
    (here: 0.05 is due to the small error b/w $\hat{P}_{t-1}$ and $P_{t-1}$)

- Above two facts + any result for $\ell_1$ min: $x_t$ is accurately recovered; and hence $\ell_t = y_t - x_t$ is accurately recovered

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5 C. Qiu, N. Vaswani, B. Lois and L. Hogben, Recursive Robust PCA or Recursive Sparse Recovery in Large but Structured Noise, IEEE Trans. IT, 2014
Why ReProCS works [Qiu, Vaswani, Lois, Hogben, T-IT, 2014] 5

- Slow subspace change: noise $\beta_t$ seen by sparse recovery step is small
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    (here: 0.05 is due to the small error b/w $\hat{P}_{t-1}$ and $P_{t-1}$)
- Above two facts + any result for $\ell_1$ min: $x_t$ is accurately recovered; and hence $\ell_t = y_t - x_t$ is accurately recovered
- Most of the work: show accurate subspace recovery $\hat{P}_t \approx P_t$
  - std PCA results not applicable: $e_t := \hat{\ell}_t - \ell_t = x_t - \hat{x}_t$ correlated w/ $\ell_t$

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5 C. Qiu, N. Vaswani, B. Lois and L. Hogben, Recursive Robust PCA or Recursive Sparse Recovery in Large but Structured Noise, IEEE Trans. IT, 2014
ReProCS algorithm: why std PCA not applicable?

- let \( e_t := \ell_t - \hat{\ell}_t = \hat{x}_t - x_t \)
- perturbation seen by standard PCA,
  \[
  \frac{1}{\alpha} \sum_t \hat{\ell}_t \hat{\ell}_t' - \frac{1}{\alpha} \sum_t \ell_t \ell'_t = \frac{1}{\alpha} \sum_t \ell_t e'_t + \left( \frac{1}{\alpha} \sum_t \ell_t e'_t \right)' + \frac{1}{\alpha} \sum_t e_t e'_t
  \]
- when \( e_t \) and \( \ell_t \) uncorrelated & \( e_t \) zero mean: first two terms are close to zero w.h.p.
ReProCS algorithm: why std PCA not applicable?

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ReProCS algorithm: why std PCA not applicable?

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ReProCS algorithm: why std PCA not applicable?

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- by sin \( \theta \) theorem \( [\text{Davis,Kahan, 1970}] \),
  \[
  \| (I - \hat{P} \hat{P}') P \|_2 \lesssim \frac{\| \text{perturbation} \|_2}{\lambda_{\min} \left( \frac{1}{\alpha} \sum_t \ell_t \ell'_t \right) - \| \text{perturbation} \|_2}
  \]
  \((P: \text{eigenvec's with nonzero eigenval's of } \frac{1}{\alpha} \sum_t \ell_t \ell'_t)\)
ReProCS algorithm: why std PCA not applicable?

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$$\| (I - \hat{P} \hat{P}') P \|_2 \lesssim \frac{\| \text{perturbation} \|_2}{\lambda_{\text{min}}(\frac{1}{\alpha} \sum_t \ell_t \ell_t') - \| \text{perturbation} \|_2}$$

($P$: eigenvec's with nonzero eigenval's of $\frac{1}{\alpha} \sum_t \ell_t \ell_t'$)
For most videos (i.e. w.p. at least $1 - n^{-10}$),

- the region occupied by the foreground objects (support of $x_t$) is exactly recovered at all times, and
For most videos (i.e. w.p. at least \(1 - n^{-10}\)),

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- foreground and background images are accurately recovered at all times (\(\|x_t - \hat{x}_t\|_2 = \|\ell_t - \hat{\ell}_t\|_2 \leq b\))
For most videos (i.e. w.p. at least $1 - n^{-10}$),

- the region occupied by the foreground objects (support of $x_t$) is exactly recovered at all times, and

- foreground and background images are accurately recovered at all times ($\|x_t - \hat{x}_t\|_2 = \|\ell_t - \hat{\ell}_t\|_2 \leq b$)

- the background subspace recovery error decays to a small value within a short delay of a subspace change time,

if

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6 B. Lois and N. Vaswani, A Correctness Result for Online Robust PCA, ICASSP 2015.
an initial background-only training sequence is available (to get an accurate initial subspace estimate)
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an initial background-only training sequence is available (to get an accurate initial subspace estimate)

the background images change slowly (\(\ell_t\) lies in a slowly changing low-dimensional subspace)

background changes (w.r.t. a mean background image) are dense,

there is some motion of the foreground objects at least once every so often (there is some change in the support of \(x_t\)’s)

Details follow in the next few slides . . .
ReProCS correctness result: Support change - examples

1. *(random motion)* all support sets mutually disjoint
   - this satisfies our model as long as \( s \in O\left(\frac{n}{\log n}\right) \)
ReProCS correctness result: Support change - examples

1. *(random motion)* all support sets mutually disjoint
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2. *(infrequent motion)* a 1D object of length \( s \) that moves at least once every \( \beta \) frames; and, when it moves, it moves down by at least \( \frac{s}{\varrho} \) pixels
   ▶ and by no more than \( b_2 s \) indices
   ▶ this satisfies our model as long as \( s \in O\left(\frac{n}{\log n}\right) \) and \( \varrho^2 \beta \leq 0.01 \alpha \)
ReProCS correctness result: Support change - examples

1. (random motion) all support sets mutually disjoint
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2. (infrequent motion) a 1D object of length $s$ that moves at least once every $\beta$ frames; and, when it moves, it moves down by at least $\frac{s}{\varrho}$ pixels
   ▶ and by no more than $b_2s$ indices
   ▶ this satisfies our model as long as $s \in O\left(\frac{n}{\log n}\right)$ and $\varrho^2\beta \leq 0.01\alpha$

3. (slow motion) an object of length $s$ moves down by at least one pixel in every frame
   ▶ this satisfies our model as long as $s \in O(\log n)$
ReProCS correctness result: Support change - examples

(a) disjoint supports  (b) infrequent motion  (c) slow moving

Figure: In any of these we could have randomly selected pixels (need not be a block) at a given time and also random ordering across time
ReProCS correctness result: Modeling data lying in a low-dimensional subspace

▶ Simplest and most commonly used model: $\ell_t \sim \mathcal{N}(0, \Sigma)$ with $\Sigma$ low-rank, but
  ▶ impractical – data statistics change with time, albeit “slowly”;
ReProCS correctness result: Modeling data lying in a low-dimensional subspace

- Simplest and most commonly used model: \( \ell_t \overset{i.i.d.}{\sim} \mathcal{N}(0, \Sigma) \) with \( \Sigma \) low-rank, but
  - impractical – data statistics change with time, albeit “slowly”;
- Most general model: \( \ell_t \) zero mean with covariance \( \Sigma_t \); with \( \Sigma_t \overset{EVD}{=} P_t \Lambda_t P_t' \), both \( P_t \) and \( \Lambda_t \) change at each \( t \), but
  - this has an identifiability problem: need at least \( r \) data points to identify \( \text{range}(P_t) \) for each time \( t \); and
ReProCS correctness result: Modeling data lying in a low-dimensional subspace

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- Most general model: \( \ell_t \) zero mean with covariance \( \Sigma_t \); with 
  \( \Sigma_t \) has EVD \( P_t \Lambda_t P_t' \), both \( P_t \) and \( \Lambda_t \) change at each \( t \), but
  - this has an identifiability problem: need at least \( r \) data points 
    to identify \( \text{range}(P_t) \) for each time \( t \); and 
  - the matrix \( L_t := [\ell_1, \ell_2, \ldots, \ell_t] \) may not be low rank
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  - the matrix $L_t := [\ell_1, \ell_2, \ldots, \ell_t]$ may not be low rank
- A general enough but tractable option:
  1. assume $P_t$ changes every so often, $\Lambda_t$ can change at each $t$;
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- Most general model: \( \ell_t \) zero mean with covariance \( \Sigma_t \); with
  - \( \Sigma_t \) EVD \( \equiv P_t \Lambda_t P_t' \), both \( P_t \) and \( \Lambda_t \) change at each \( t \), but
    - this has an identifiability problem: need at least \( r \) data points to identify range(\( P_t \)) for each time \( t \); and
    - the matrix \( L_t := [\ell_1, \ell_2, \ldots, \ell_t] \) may not be low rank
- A general enough but tractable option:
  1. assume \( P_t \) changes every so often, \( \Lambda_t \) can change at each \( t \);
  2. model “slow” change over time: eigenvalues along new subspace directions small for some time after a change in \( P_t \),
ReProCS correctness result: Modeling data lying in a low-dimensional subspace

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- A general enough but tractable option:
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  2. model “slow” change over time: eigenvalues along new subspace directions small for some time after a change in $P_t$,
  3. assume $\ell_t$ bounded: sensor data is usually bounded
ReProCS correctness result: Subspace change model

\( \ell_t \)'s are zero mean, bounded and mutually independent r.v.'s with covariance matrix \( \Sigma_t \) that is low-rank and “slowly changing”
ReProCS correctness result: Subspace change model

\( \ell_t \)'s are zero mean, bounded and mutually independent r.v.'s with covariance matrix \( \Sigma_t \) that is low-rank and “slowly changing”

- \( \Sigma_t \overset{EVD}{=} P_t \Lambda_t P_t' \) where \( P_t = P(j) \) for \( t \in [t_j, t_j+1 - 1] \), \( j = 1, 2, \ldots J \)

- \( P(j) \) is a tall \( n \times r_j \) basis matrix that changes as
  \[
P(j) = [P(j-1) \setminus P_{j,old}, P_{j,new}]
  \]

- “slow change”: \( \lambda^+_{new}(d) := \max_{t \in [t_j, t_j+d]} \lambda_{max}(\Lambda_t, \text{new}) \) is small and \( t_{j+1} - t_j \) is large
ReProCS correctness result: Subspace change model

\( \ell_t \)'s are zero mean, bounded and mutually independent r.v.'s with covariance matrix \( \Sigma_t \) that is low-rank and "slowly changing"

\[ \Sigma_t \overset{\text{EVD}}{=} P_t \Lambda_t P'_t \text{ where } P_t = P(j) \text{ for } t \in [t_j, t_{j+1} - 1], j = 1, 2, \ldots J \]

\[ P(j) \text{ is a tall } n \times r_j \text{ basis matrix that changes as} \]

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"slow change": \( \lambda^+_{\text{new}}(d) := \max_{t \in [t_j, t_{j+d}]} \lambda_{\max}(\Lambda_t, \text{new}) \) is small and \( t_{j+1} - t_j \) is large

Define

\[ c := \max_j \text{rank}(P_{j,\text{new}}), \gamma_{\text{new}}(d) := \max_{t \in [t_j, t_{j+d}]} \|a_{t,\text{new}}\|_{\infty} \]

\[ r := r_0 + Jc, \lambda^+ := \max_t \lambda_{\max}(\Lambda_t), \gamma := \max_t \|a_t\|_{\infty} \]
Theorem

Consider ReProCS. Pick a $\zeta \leq \min \left( \frac{10^{-4} \lambda^-_0}{(r_0 + Jc)^2 \lambda^+}, \frac{1}{(r_0 + Jc)^3 \gamma^2} \right)$. If ReProCS algorithm parameters $\alpha, K, \xi, \omega$ are set appropriately, and if
Theorem
Consider ReProCS. Pick a $\zeta \leq \min \left(\frac{10^{-4} \lambda_{0}^{-}}{(r_{0}+Jc)^{2} \lambda^{+}}, \frac{1}{(r_{0}+Jc)^{3} \gamma^{2}}\right)$. If ReProCS algorithm parameters $\alpha, K, \xi, \omega$ are set appropriately, and if

1. initial subspace accurately estimated: $\|(I - \hat{P}_{0} \hat{P}'_{0}) P_{0}\|_{2} \leq r_{0} \zeta$
2. “slow subspace change” holds:
   - projection of $\ell_{t}$ along new direc’s small for first $d$ frames after $t_{j}$: for a $d \geq (K + 2) \alpha$, $\lambda_{new}^{+}(d) \leq 3 \lambda_{0}^{-}$
   - and delay between change times is large: $(t_{j+1} - t_{j}) > d$, 

Namrata Vaswani
Theorem
Consider ReProCS. Pick a $\zeta \leq \min \left( \frac{10^{-4} \lambda_0^-}{(r_0 + Jc)^2 \lambda^+}, \frac{1}{(r_0 + Jc)^3 \gamma^2} \right)$. If ReProCS algorithm parameters $\alpha, K, \xi, \omega$ are set appropriately, and if

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   - projection of $\ell_t$ along new direc’s small for first $d$ frames after $t_j$: for a $d \geq (K + 2) \alpha$, $\lambda^+_{\text{new}}(d) \leq 3 \lambda_0^-$
   - and delay between change times is large: $(t_{j+1} - t_j) > d$,

3. subspace basis matrices are dense enough:
   $$(2s) \max_i \| (P_{j, \text{new}})_{i,:} \|_2^2 \leq 0.0004 \text{ and } (2s) \max_i \| (P_J)_{i,:} \|_2 \leq 0.09$$
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Consider ReProCS. Pick a $\zeta \leq \min \left( \frac{10^{-4} \lambda_0^-}{(r_0+Jc)^2 \lambda^+}, \frac{1}{(r_0+Jc)^3 \gamma^2} \right)$. If ReProCS algorithm parameters $\alpha, K, \xi, \omega$ are set appropriately, and if

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2. "slow subspace change" holds:
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   - and delay between change times is large: $(t_{j+1} - t_j) > d$,
3. subspace basis matrices are dense enough:
   \begin{align*}
   (2s) \max_i \| (P_{j,new})_{i,:} \|_2^2 \leq 0.0004 \text{ and } (2s) \max_i \| (P_{j})_{i,:} \|_2 \leq 0.09
   \end{align*}
4. support of $x_t$, $T_t$, has size smaller than $s$ and changes enough,
   - e.g., moves down by at least $s/10$ pixels at least once every $\alpha/500$ frames,
   \begin{align*}
   \text{and } \sqrt{c} \gamma_{new}(d) \leq 0.05 \min_t \min_{i \in T_t} |(x_t)_i|
   \end{align*}
then, with probability at least $1 - n^{-10}$,

1. $\text{support}(x_t)$ is exactly recovered at all times,

2. $SE_t := \| (I - \hat{P}_t \hat{P}'_t) P_t \|_2$ reduces to $(r + c) \zeta$ within $(K + 2) \alpha$ frames after $t_j$,

3. $\| \ell_t - \hat{\ell}_t \|_2 = \| x_t - \hat{x}_t \|_2 \leq b \ll \| x_t \|_2$

Notice: no bound needed on $\lambda^+$ or on $\gamma$: the result allows large but structured $\ell_t$
then, with probability at least $1 - n^{-10}$,

1. $\text{support}(x_t)$ is exactly recovered at all times,

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Notice: no bound needed on $\lambda^+$ or on $\gamma$: the result allows large but structured $\ell_t$

The condition "$\sqrt{c\gamma_{\text{new}}(d)} \leq 0.05 \min_t \min_{i \in T_t} |(x_t)_i|$" can be relaxed to "there exists a partition of $T_t$ s.t. $\max_t \| (x_t)_{T_t,\text{small}} \|_2 \leq 0.03\zeta\lambda_0^-$ and $0.05 \min_t \min_{i \in T_t,\text{large}} |(x_t)_i| > \max_t \| (x_t)_{T_t,\text{small}} \|_2 + \sqrt{c\gamma_{\text{new}}(d)}$"

Details:

- B. Lois and N. Vaswani, Online Robust PCA and Online Matrix Completion, ISIT 2015 and arXiv:1503.03525 [cs.IT].
- B. Lois and N. Vaswani, A Correctness Result for Online Robust PCA, ICASSP 2015.
Discussion: Contributions

- To our knowledge, first correctness result for online robust PCA
  - or online sparse + low-rank recovery / online sparse recovery in large but structured noise
  - online algorithm: faster; less storage needed: only $O(n \log n)$ instead of $O(nt_{\text{max}})$
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  - online algorithm: faster; less storage needed: only $O(n \log n)$ instead of $O(n t_{\text{max}})$
- Allows significantly more correlated support change than PCP
  - ReProCS allows the fraction of nonzeros per row of $X$ to be $O(1)$;
  - PCP only allows this to be $O\left(\frac{1}{\text{rank}(L)}\right)$ [Hsu et al’2011] or needs uniformly random support of $X$ [Candes et al]
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- New proof techniques needed: useful for various other problems
  - almost all existing robust PCA results are for batch approaches
  - previous PCA results require $e_t := \hat{\ell}_t - \ell_t$ uncorrelated w/ $\ell_t$
Discussion: Limitations

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  - let $s_{mat} := |\text{support}(X)|$ and $r_{mat} := \text{rank}(L)$
  - we allow $s_{mat} \in O\left(\frac{nt_{max}}{\log n}\right)$ and $r_{mat} \in O(\log n)$
  - PCP allows $s_{mat} \in O(nt_{max})$ and $r_{mat} \in O\left(\frac{n}{\log^2 n}\right)$
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- Needs
  - initial subspace knowledge and slow subspace change
    - both are usually practically valid
  - zero-mean & mutually independent assump. on $\ell_t$'s over $t$
    - models independent random variations around a fixed bg mean
    - can replace it by a more practical AR model (ongoing)
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    - can replace it by a more practical AR model (ongoing)
- Only ensures accurate recovery of $x_t, \ell_t$, not exact
Some Generalizations

- Direct application to online matrix completion
- Easy extension to $y_t = Ax_t + B\ell_t$
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- Direct application to online matrix completion
- Easy extension to \( y_t = Ax_t + B\ell_t \)
- Relax independence assumption on \( \ell_t \)'s, replace by AR model (ongoing) – almost exactly same result
- Result for ReProCS-deletion – ReProCS that also deletes direc’s (ongoing):
  - needs an extra clustering assumption on the eigenvalues for a certain period of time after subspace change has stabilized;
  - but relaxes denseness requirement and so allows \( r_{\text{mat}} \in O(n) \) instead of \( r_{\text{mat}} \in O(\log n) \)
Application to online matrix completion

- Can provide a provably accurate solution for online matrix completion; that also allows highly correlated set of unknown entries
  - but requires slow subspace change and initial subspace knowledge

- Low-rank matrix completion is a special case w/ known $T_t = \text{support}(x_t)$
  - in MC: $T_t$ is the set of unknown entries of $\ell_t$ at time $t$

- ReProCS for online matrix completion:
  - Assume: accurate initial subspace knowledge, $\hat{P}_0$.
  - Compute $\Phi_t := (I - \hat{P}_{t-1}\hat{P}'_{t-1})$
  - Given $T_t$, get an estimate of $\ell_t$ as
    \[
    \hat{\ell}_t = (I - IT_t(\Phi_t)T_t^\dagger\Phi_t)y_t
    \]
  - Use projection-PCA as before to update the subspace estimate
Complete ReProCS algorithm \[^7\]

Initialize: given \(\hat{P}_0\) with range(\(\hat{P}_0\)) \(\approx\) range([\(l_1, l_2, \ldots l_{t_0}\)])

For \(t > t_0\),

- **Projection**: compute \(\tilde{y}_t := \Phi_t y_t\), where \(\Phi_t := I - \hat{P}_{t-1} \hat{P}_{t-1}'\)
  - then \(\tilde{y}_t = \Phi_t x_t + \beta_t\), \(\beta_t := \Phi_t \ell_t\) is small “noise”

- **Noisy Sparse Recovery**: \(\ell_1\) min + support estimate + LS: get \(\hat{x}_t\)
  - \(\hat{x}_{t, cs} = \arg\min_x \|x\|_1\) s.t. \(\|\tilde{y}_t - \Phi_t x\|_2 \leq \xi\)
  - \(\hat{T}_t = \{i : |(\hat{x}_{t, cs})_i| > \omega\}\)
  - \(\hat{x}_t = l_{\hat{T}_t}(A_{\hat{T}_t}' A_{\hat{T}_t})^{-1} A_{\hat{T}_t}' y_t\)

- Get \(\hat{\ell}_t = y_t - \hat{x}_t\)

- **Subspace update**: update \(\hat{P}_t\) every \(\alpha\) frames by projection-PCA (next slide)

\[^7\] C. Qiu and N. Vaswani, Real-time Robust Principal Components’ Pursuit, Allerton, 2010
C. Qiu and N. Vaswani, Recursive Sparse Recovery in Large but Correlated Noise, Allerton 2011
ReProCS algorithm: projection PCA

Recall $t_{j+1} - t_j > (K + 2)\alpha$; $t_j$: subspace change times;

$P_t = P(j) = [P(j-1), P_{j,new}]$ for all $t_j \leq t < t_{j+1}$

Let $\hat{P}_{j-1} := \hat{P}_{j-1}$ be an (accurate) estimate of the previous subspace at $t = \hat{t}_j + k\alpha$, $k = 1, 2, \ldots K$,

- $\hat{P}_{t,new,k} \leftarrow \text{SVD} \left( (I - \hat{P}_{j,*}\hat{P}_{j,*}') [\hat{t}_{j+(k-1)\alpha+1}, \ldots, \hat{t}_{j+k\alpha}], \text{thresh} \right)$
- Update $\hat{P}_t = [\hat{P}_{j,*}, \hat{P}_{j,new,k}]$
Proof idea: Why projection PCA works?

- Before the first proj-PCA, i.e. for \( t \in [t_j, \hat{t}_j + \alpha] \),
  - \( P_t = [P_*, P_{new}], \hat{P}_{t-1} = [\hat{P}_*] \Rightarrow \beta_t \) (noise seen by sparse rec step) and hence \( e_t = \hat{x}_t - x_t = \ell_t - \hat{\ell}_t \) is largest
  - \( e_t \) still not too large due to slow subspace change; and \( e_t \) is sparse and supported on \( \mathcal{T}_t \)
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  - \( e_t \) still not too large due to slow subspace change; and \( e_t \) is sparse and supported on \( T_t \)
  - at \( t = \hat{t}_j + \alpha \), get \( \hat{P}_{\text{new},1} \): estimate is good because of above:
    \[
    \text{SE}(P_{\text{new}}, \hat{P}_{\text{new},1}) := \| (I - \hat{P}_{\text{new},1} \hat{P}_{\text{new},1}') P_{\text{new}} \|_2 < 0.6
    \]
Proof idea: Why projection PCA works?

- Before the first proj-PCA, i.e. for \( t \in [t_j, \hat{t}_j + \alpha] \),
  - \( P_t = [P_\ast, P_{\text{new}}], \hat{P}_{t-1} = [\hat{P}_\ast] \Rightarrow \beta_t \) (noise seen by sparse rec step) and hence \( e_t = \hat{x}_t - x_t = \ell_t - \hat{\ell}_t \) is largest
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    \]
- For \( t \in [\hat{t}_j + \alpha + 1, \hat{t}_j + 2\alpha] \),
  - \( P_t = [P_\ast, P_{\text{new}}], \hat{P}_{t-1} = [\hat{P}_\ast, \hat{P}_{\text{new},1}] \Rightarrow \beta_t \) and hence \( e_t \) smaller; and \( e_t \) is sparse and supported on \( T_t \)
  - at \( t = \hat{t}_j + 2\alpha \), get \( \hat{P}_{\text{new},2} \); estimate better because of above
- Continuing this way, show \( \text{SE}(P_{\text{new}}, \hat{P}_{\text{new},k}) < 0.6^k + 0.4c\zeta \); pick \( K \) so \( \text{SE}(P_{\text{new}}, \hat{P}_{\text{new},K}) < c\zeta \).
Proof Outline: $k$-th projection-PCA interval

Conditioned on accurate recovery so far,

- slow subspace change, denseness assumption, appropriate support threshold and LS ensure that $e_t := x_t - \hat{x}_t = \ell_t - \ell_t$ satisfies

$$e_t = l_{T_t} [\Phi_{T_t} \Phi_{T_t}^{-1} l_{T_t} \Phi_{T_t} \ell_t$$ where $\Phi := I - \hat{P}_{t-1} \hat{P}_{t-1}'$

and

$$\|[\Phi_{T_t} \Phi_{T_t}^{-1}]^{-1}\|_2 \leq 1.2$$

- by sin $\theta$ theorem [Davis,Kahan,1970],

$$\text{SE}(\hat{P}_{\text{new}, k}, P_{\text{new}}) \lesssim \frac{\|\text{perturbation}\|_2}{\lambda_{\text{new}} - \|\text{perturbation}\|_2}$$

$$\|\text{perturbation}\|_2 \lesssim 2 \| \frac{1}{\alpha} \sum_t (I - \hat{P}_* \hat{P}_*') \ell_t \ell_t' \|_2 + \| \frac{1}{\alpha} \sum_t e_t e_t' \|_2$$

- use matrix Hoeffding ineq [Tropp,2012] to bound these terms w.h.p.
Proof Outline: $k$-th projection-PCA interval – 2

Conditioned on accurate recovery so far,

- the dominant perturbation term

$$\text{dom} := \mathbb{E} \left[ \frac{1}{\alpha} \sum_{t=\hat{t}_j+(k-1)\alpha}^{\hat{t}_j+k\alpha} (I - \hat{\mathbf{P}}_* \hat{\mathbf{P}}'_*) e'_t \right] \approx \frac{1}{\alpha} \sum_t A_t B'_t$$

where $A_t := P_{\text{new}} \Lambda_{t, \text{new}} P'_{\text{new}}$ and $B_t := l_{t_f} [\Phi_{t_f} \Phi_T^{-1}] l_{t_f}'$

- use slow subspace change to get

$$\left\| \frac{1}{\alpha} \sum_t A_t A'_t \right\|_2 \leq \max_t \|A_t\|_2^2 \leq \lambda^+_{\text{new}}(d)^2 \leq 9\lambda_0^{-2}$$

- use model on $T_t$ to show that

$$\left\| \frac{1}{\alpha} \sum_t B_t B'_t \right\|_2 = \left\| \frac{1}{\alpha} \sum_t l_T [\Phi_T \Phi_T^{-1}] l_T' \right\|_2 \leq \frac{1}{\alpha} 1.2^2 \rho^2 \beta \leq 0.02$$

- use Cauchy-Schwartz to get $\|\text{dom}\|_2 \lesssim \sqrt{0.02 \cdot 3\lambda_0}$
Proof Outline: Overall idea

- Define subspace error, \( SE(P, \hat{P}) := \| (I - \hat{P}\hat{P}^{'})P \|_2 \).

- Start with \( SE(P_{j-1}, \hat{P}_{j-1}) \leq r_{j-1}\zeta \ll 1 \) at \( t = t_j - 1 \).
  1. First show that \( t_j \leq \hat{t}_j \leq t_j + 2\alpha \)
  2. Analyze projected sparse recovery for \( t \in [\hat{t}_j, \hat{t}_j + \alpha) \)
  3. Analyze proj-PCA at \( t = \hat{t}_j + \alpha : SE(P_{j,\text{new}}, \hat{P}_{j,\text{new},1}) \leq 0.6 \)
  4. Repeat for each of the \( K \) projection-PCA intervals: show that \( SE(P_{j,\text{new}}, \hat{P}_{j,\text{new},k}) \leq 0.6^k + 0.4c\zeta \)
  5. Pick \( K \) s.t. \( 0.6^K + 0.4c\zeta \leq c\zeta \). Set \( \hat{P}_j = [\hat{P}_{(j-1)}, \hat{P}_{j,\text{new},K}] \)
- Thus, at \( t = \hat{t}_j + K\alpha - 1 \),
  \[
  SE(P_j, \hat{P}_j) \leq SE(P_{j-1}, \hat{P}_{j-1}) + SE(P_{j,\text{new}}, \hat{P}_{j,\text{new},k}) \leq r_{j-1}\zeta + c\zeta = r_j\zeta
  \]
- \( t_{j+1} - t_j > (K + 2)\alpha \) implies \( SE(P_j, \hat{P}_j) \leq r_j\zeta \) at \( t = t_{j+1} - 1 \)
1. Real background simulated foreground: background of moving lake water video with a simulated moving rectangular object overlaid on it; object intensity similar to background intensity and object moving slowly (making it a difficult seq)

2. Real videos:
   http://www.ece.iastate.edu/~hanguo/PracReProCS.html
   http://www.ece.iastate.edu/~chenlu/ReProCS/ReProCS.htm
Figure: Recovery error (Monte Carlo over 100 realiz’s). Black: batch methods, Red: online methods, Red Circles: ReProCS
Online Sparse Matrix Recovery

Online Robust PCA

Background, Problem Formulation and Related Work
ReProCS Algorithm and Correctness Result
Proof Outline and Experiments

Figure: Online: ReProCS (proposed), GRASTA; Batch: PCP, RSL, MG.
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Figure: Online: ReProCS (proposed), GRASTA; Batch: PCP, RSL, MG.
Figure: Foreground layer recovery at \( t = t_{train} + 30, 80, 140 \).
Figure: Foreground layer recovery at $t = t_{\text{train}} + 30, 80, 140$. 
Figure: Foreground layer recovery at $t = t_{\text{train}} + 35, 500, 1300$. 
Figure: Background layer recovery at $t = t_{train} + 35, 500, 1300$. 
Algorithm parameters

Recall that $\zeta \leq \min\left(\frac{10^{-4}}{(r_0+Jc)^2f}, \frac{1}{(r_0+Jc)^3\gamma^2_*}\right)$.

- $\xi = \sqrt{c}\gamma_{\text{new}} + \sqrt{\zeta}(\sqrt{r_0} + Jc + \sqrt{c})$;
- $\omega$ satisfies $7\xi \leq \omega \leq \omega_{\text{min}} - 7\xi$;
- $K = \left\lceil \frac{\log(0.16c\zeta)}{\log(0.4)} \right\rceil$;
- $\alpha = C(\log(6KJ) + 11 \log(n))$, $C \geq C_{\text{add}} := 20^2 \cdot 8 \cdot 96^2 \frac{(1.2\xi)^4}{(c\zeta\lambda^-)^2}$
- If we assume that min and max eigenvalues are seen in the training data, then can estimate $\lambda^-, \lambda^+, \gamma_\ast$ from training data.
Summary

- To the best of our knowledge, this is the first correctness result for online sparse + low-rank recovery
  - equivalently also for online robust PCA / recursive sparse recovery in large but structured noise

- Advantages
  - online algorithm: faster; less storage needed; removes a key limitation of PCP: allows more correlated support change

- New proof techniques needed to obtain our results
  - almost all existing robust PCA results are for batch approaches
  - previous finite sample PCA results are not useful: assume $e_t := \hat{\ell}_t - \ell_t$ is uncorrelated with $\ell_t$
Ongoing and future work

- A key limitation of ReProCS: does not use the fact that $\beta_t = (I - \hat{P}_{t-1} \hat{P}_{t-1}') l_t$ approx lies in a $c$ dimensional subspace
  - the only way to use it is a piecewise batch approach: modified-PCP [Zhan, Vaswani, ISIT’14, T-SP’15]

$$\min_{L,X} \| (I - \hat{P}_{j-1} \hat{P}_{j-1}') L \|_* + \lambda \| X \|_1 \text{ s.t. } Y = L + X$$

- advantage: weaker rank-sparsity product assumption;
- disadvantage: does not handle correlated support change as well as ReProCS
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  - Advantage: weaker rank-sparsity product assumption;
  - Disadvantage: does not handle correlated support change as well as ReProCS

- Applications to understanding user preferences for recommendation system design

- Online robust PCA from moving sensors’ data, e.g. moving cameras

- Proof techniques applicable to more general problems involving “correlated-PCA” — correlated data and noise vectors
References


2. B. Lois and N. Vaswani, “A Correctness Result for Online Robust PCA”, ICASSP 2015


5. C. Qiu and N. Vaswani, Real-time Robust Principal Components’ Pursuit, Allerton, 2010