Analyzing Least Squares & Kalman Filtered Compressed Sensing

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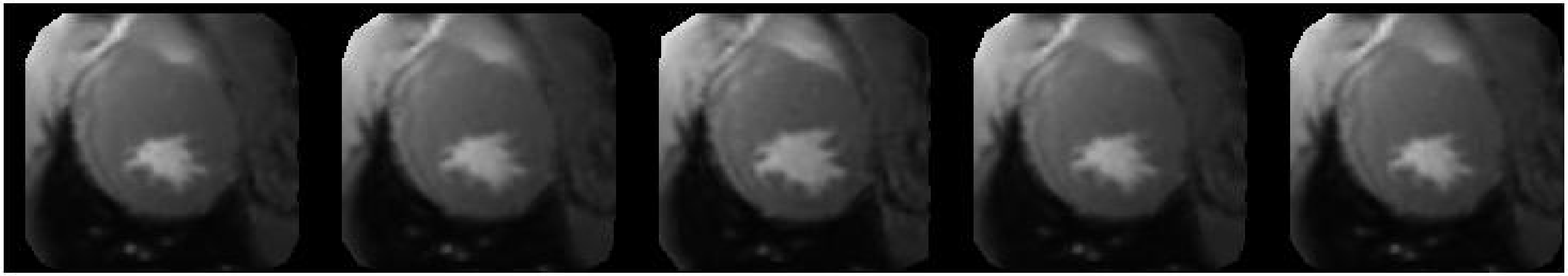
Goal

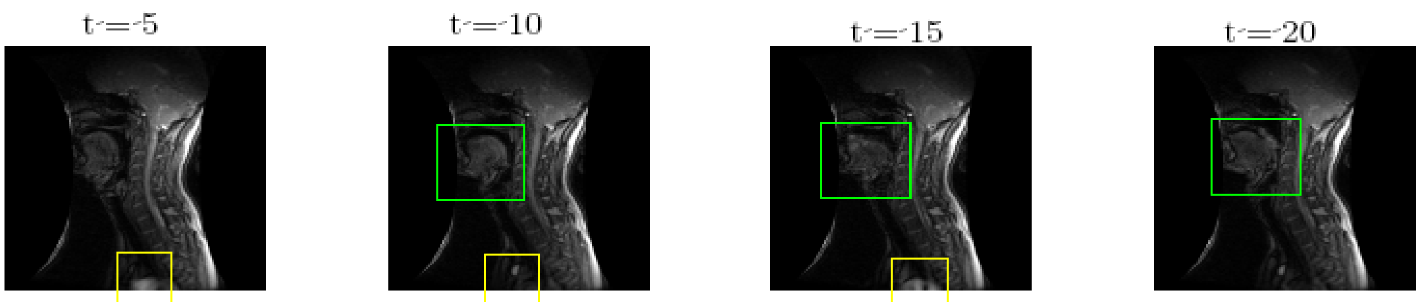
- **Goal:** Reconstruct
  - a time sequences of (spatially) sparse signals
  - with slowly changing sparsity patterns (support sets)
  - from a limited number of incoherent measurements
  - in real-time (recursively and causally)

- **Examples:** real-time dynamic MR imaging, video compression, single-pixel video, sensor nets for real-time estimation of time-varying fields

- **Key assumption:** sparsity pattern changes slowly over time

Slowly Changing Sparsity





- **Approx. Sparsity.** Size of 99%-energy support set: less than 7% for the larynx sequence and less than 9% for the cardiac sequence.
- **Slow Change in Sparsity Pattern.** Maximum size of change in support: less than 2% of minimum sparsity size in both cases

Outline

- Problem definition and background
- LS CS-residual (LS-CS) & KF CS-residual (KF-CS)
- Bounding LS CS error
- Conditions for stability of LS and KF CS
- Simulations

Problem Definition

Recursively reconstruct a sparse vector, x_t, from the current observation, $y_t := Ax_t + w_t$, & all past observations, $y_{1:t-1}$

- $dim(y_t) = n < dim(x_t) = m$
- x_t is S_t-sparse with support set, N_t
- the support, N_t, changes slowly over time
- A is S_*-"approximately orthonormal" ($\delta_{S_*} < 1/2$) and $S_* > S_t$
  - i.e. $||Ax||_2$ between 0.7 and 1.2 for S_* or less sparse vectors x

RIP and ROP constants [Candes,Tao]

- Restricted Isometry constant, δ_S: smallest real number satisfying

$$(1-\delta_S)||c||_2^2 \le ||A_T c||_2^2 \le (1+\delta_S)||c||_2^2$$

for all subsets T with $|T| \le S$ and for all c

  - **Easy to see:** $||(A_T{}' A_T)^{-1}||_2 \le 1/(1-\delta_{|T|})$

- Restricted Orthogonality constant, $\theta_{S,S'}$: smallest real number satisfying

$$|c_1{}' A_{T_1}{}' A_{T_2} c_2| \le \theta_{S,S'} ||c_1||_2 ||c_2||_2$$

for all disjoint sets T_1, T_2 with $|T_1| \le S$, $|T_2| \le S'$ and for all c_1, c_2

  - **Easy to see:** $||A_{T_1}{}' A_{T_2}||_2 \le \theta_{|T_1|,|T_2|}$

Compressed Sensing [Candes, Romberg, Tao] [Donoho]

- CS (noiseless) [Candes, Romberg, Tao '05] [Donoho'05]
- CS (noisy - BPDN) [Chen,Donoho] [Tropp'06]
- **CS (noisy - Dantzig Selector)** [Candes, Tao '06]

$$\min_{\beta} ||\beta||_1 \text{ s.t. } ||A'(y_t - A\beta)||_\infty < \lambda$$

We use $\hat{x}_t = \mathbf{CS}(y_t)$ to denote the solution of above

- If noise is bounded between $\pm\lambda/||A||_1$ in each dimension, and $\delta_{3S_t} < 1/2$,

$$\begin{aligned} ||x_t - \hat{x}_t||_2^2 &\leq C_1(S_t)\ S_t\ \lambda^2 \\ C_1(S) &:= 16/(1 - \delta_{2S} - \theta_{S,2S})^2 \end{aligned}$$

(simplification of Theorem 1.1 of [Dantzig Selector])

The Question

- Most existing work: Batch-CS on entire time sequence

  [Gamper et al '08 (dynamic MRI)], [Wakin et al (video)]

  – Offline and very slow, but uses few measurements

- Alternative: CS at each time separately (simple CS)

  – Causal and fast, but needs many more measurements

- The Question: How can we

  – improve simple CS by **using past observations**, and

  – how can we do it **recursively, i.e. by only using the previous signal estimate and the current observation?**

Finding a Recursive Solution

- Given $y_t := Ax_t + w_t$, x_t is sparse with support N_t, N_t changes slowly over time, A satisfies $\delta_{S_t} < 1/2$, $S_t := |N_t|$
- **If N_t known:** easy to compute a restricted-LS estimate

$$\hat{x}_t = \text{restrictedLS}(y_t, N_t) \quad := \quad (\hat{x}_t)_{N_t} = {A_{N_t}}^{\dagger} y_t, \; (\hat{x}_t)_{N_t^c} = 0$$

- **If N_t unknown:** an option is to estimate it by thresholding CS output

$$\hat{N}_t \quad = \quad \text{threshold}(\text{CS}(y_t)) \quad \text{threshold}(x) := \{i : x_i^2 > \alpha\}$$

  and then do the same thing

  - **But: not using past observations: large error**

CS-residual idea [Vaswani, ICIP'08, ICASSP'09]

- Let $T := \hat{N}_{t-1}$ (estimated support at $t-1$) and $\Delta := N_t \setminus \hat{N}_{t-1}$
- Assume that the undetected set, Δ, is small, i.e.
  - the support changes slowly, and
  - the support at $t-1$ is well estimated
- Use $T := \hat{N}_{t-1}$ to compute restricted LS estimate, & observation residual

$$\begin{aligned} \hat{x}_{t,\text{init}} &= \text{restrictedLS}(y_t, T) \\ y_{t,\text{res}} &= y_t - A\hat{x}_{t,\text{init}} \end{aligned}$$

- **CS-residual:** $\hat{x}_t = \hat{x}_{t,\text{init}} + \text{CS}(y_{t,\text{res}})$
  - **$y_{t,\mathbf{res}}$ is a noisy measurement of an approx. $|\Delta|$ sparse vector**

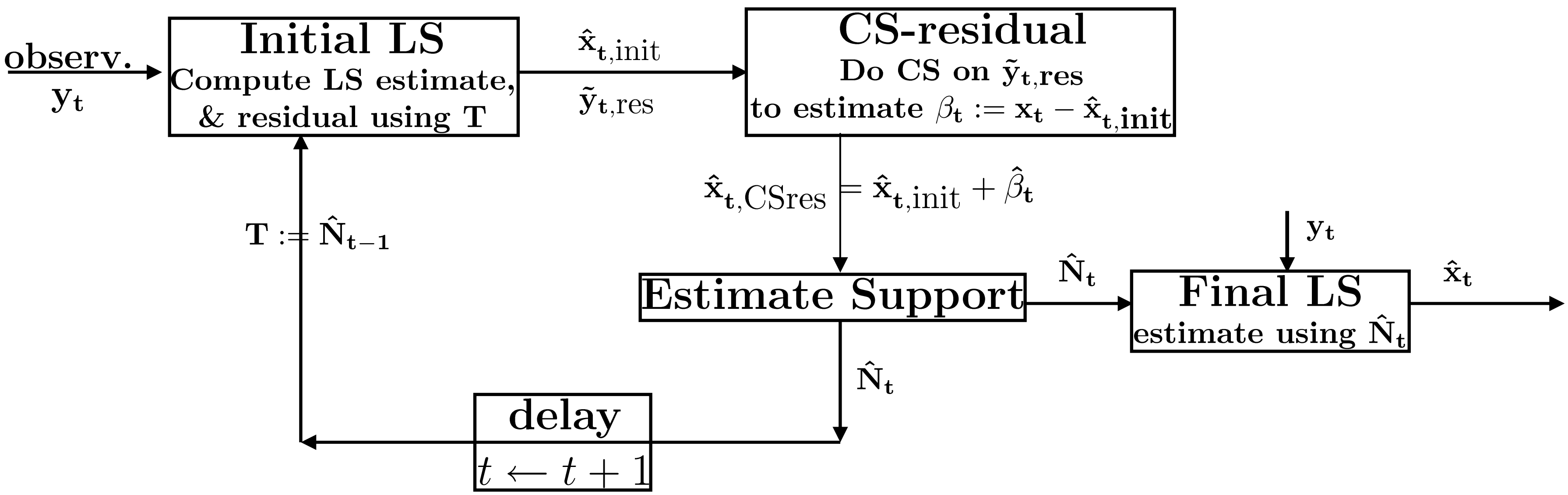
Why CS-residual works?

- Notice that $y_{t,\text{res}} = A\beta_t + w_t$ and $\beta_t := x_t - \hat{x}_{t,\text{init}}$ satisfies

$$\begin{aligned} (\beta_t)_\Delta &= (x_t)_\Delta \\ (\beta_t)_T &= -{A_T}^\dagger (A_\Delta (x_t)_\Delta + w_t) \\ (\beta_t)_{(T \cup \Delta)^c} &= 0 \end{aligned}$$

- If $|\Delta|$ small enough s.t. $||{A_T}' A_\Delta||_2 < \theta_{|T|,|\Delta|}$ small:
  - **β_t small along T, i.e. it is only $|\Delta|$-approx-sparse**
- CS error strongly depends on approx. sparsity size
  - **CS-residual: much smaller error** than CS on y_t (simple CS)

LS CS-residual (LS-CS) algorithm



Estimate Support. Either do

- $\hat{N}_t = \text{threshold}(\hat{x}_{t,CSres})$ (add and delete indices at the same time)
- Or $\hat{N}_t = T \cup \text{threshold}(\hat{x}_{t,CSres})$ (only add new indices) (easier to analyze)

Bounding LS CS-residual error [Vaswani, ICASSP'09]

- **Assume that**
  1. noise bounded b/w $\pm\lambda/||A||_1$ and has variance σ^2 in each dimension
  2. number of false additions, $|T \setminus N_t| \leq S_{fa}$ and $\delta_{S_t+S_{fa}} < 1/2$
  3. number of new plus undetected additions, $|\Delta| < S_t/3$
     - recall: $T := \hat{N}_{t-1}$, $\Delta := N_t \setminus T$
- **Expected CS-residual error given past $:= y_{1:t-1}$ is bounded by**

$$C_2(|\Delta|)\ |\Delta|\ \lambda^2\ +\ C_3(|\Delta|)\frac{|T|}{|\Delta|}\ [4\theta^2_{|T|,|\Delta|}E[||(x_t)_\Delta||^2|\text{past}] + 2|T|\sigma^2]$$

  - $C_2(S)$, $C_3(S)$: linear functions of $1/(1-\delta_{2S}-\theta_{S,2S})^2$
- Final LS usually further improves performance

Comparison with CS

- **If noise bounded and $\delta_{3S_t} < 1/2$, CS error bounded by**

$$C_1(S_t)\ |\Delta|\ \lambda^2\ +\ C_1(S_t)\ (S_t - |\Delta|)\ \lambda^2$$

  $C_1(S)$: linear function of $1/(1 - \delta_{2S} - \theta_{S,2S})^2$

  - **holds under much stronger assumption**
  - **the constant much larger and second term not much smaller**

- **If noise bounded and $\delta_{3S} < 1/2$ $(S < S_t)$, CS error bounded by**

$$\min_{S:\delta_{3S}<1/2}\ C_2(S)\ S\ \lambda^2\ +\ C_3(S)\ \frac{S_t - S}{S}\ E[||(x_t)_{\text{rest}}||^2|\text{past}]$$

  rest: $(S_t - S)$ nonzero elements of x_t which did not get estimated

  - **constants same, but second term much larger**

Comparison with CS: special case

- Let $S = |\Delta|$ is the optimal S for CS bound. Also, let $S_{fa} = 0$
- **CS error bound:**

$$C_2 \, |\Delta| \, \lambda^2 \quad + \quad C_3' \, E[||(x_t)_{\text{rest}}||^2|\text{past}]$$

- **CS-residual error bound:**

$$C_2 \, |\Delta| \, \lambda^2 \quad + \quad C_3' \, [4\theta^2 E[||(x_t)_{\Delta}||^2|\text{past}] + 2|T|\sigma^2]$$

  - $T := \hat{N}_{t-1}$, $\Delta := N_t \setminus T$, $\theta := \theta_{|T|,|\Delta|}$
- **CS-residual bound much smaller when**
  - $|\Delta|$ is small $\rightarrow$ much smaller than $|\text{rest}| = S_t - |\Delta|$
  - $|\Delta|$ small $\rightarrow \theta^2 = \theta^2_{|T|,|\Delta|}$ is small
  - SNR large (when SNR too small, nothing works) $\rightarrow \sigma^2$ small

Disclaimer

- We are only comparing upper bounds
- The upper bound on CS error being larger does not necessarily mean that CS-residual is better

Kalman filtered CS-residual (KF-CS)

[Vaswani, ICIP'08]

- So far only used $\hat{N}_{t-1}$ to improve accuracy of CS at t: did not use $\hat{x}_{t-1}$
- If a prior dynamic model for nonzero coefficients of x_t is available: do this by replacing initial LS by a KF for $(x_t)_T$
- A possible prior model: random-walk on $(x_t)_{N_t}$ starting with $x_0 = 0$

$$x_t = x_{t-1} + \nu_t, \ \nu_t \sim \mathcal{N}(0, Q_t), \ Q_t = \sigma_s^2 I_{N_t}$$

I_T: diagonal matrix, 1's at diagonal locations from set T, zero elsewhere

- **KF CS-residual:**
  - dimension-varying KF with current states' set being $T := \hat{N}_{t-1}$
  - compute $\hat{N}_t$ by thresholding output of CS on KF residual

KF CS-residual (KF-CS) algorithm

- **Initial KF.** Let $T = \hat{N}_{t-1}$. Run a Kalman prediction and update step using $\hat{Q}_t = \sigma_{sys}^2 I_T$ and compute the KF residual, $y_{t,\text{res}}$, i.e. compute

$$\begin{aligned} P_{t|t-1} &= P_{t-1} + \hat{Q}_t, \quad \text{where} \quad \hat{Q}_t := \sigma_s^2 I_T \\ K_t &= P_{t|t-1} A' (A P_{t|t-1} A' + \sigma^2 I)^{-1}, \quad P_t = (I - K_t A) P_{t|t-1} \\ \hat{x}_{t,\text{init}} &= (I - K_t A) \hat{x}_{t-1} + K_t y_t \\ y_{t,\text{res}} &= y_t - A \hat{x}_{t,\text{init}} \end{aligned}$$

- **CS-residual.** Compute $\hat{x}_{t,CSres} = \hat{x}_{t,\text{init}} + \text{CS}(y_{t,\text{res}})$
- **Estimate Support.** $\hat{N}_t = T \cup \text{Threshold}(\hat{x}_{t,CSres})$
- **Final LS.** If $\hat{N}_t$ is equal to $\hat{N}_{t-1}$, set $\hat{x}_t = \hat{x}_{t,\text{init}}$, else compute restricted LS estimate using $\hat{N}_t$ and update P_t, i.e.

$$\begin{aligned} \hat{x}_t &= \text{restrictedLS}(y_t, \hat{N}_t) \\ (P_t)_{\hat{N}_t, \hat{N}_t} &= (A_{\hat{N}_t}{}' A_{\hat{N}_t})^{-1} \sigma^2 \end{aligned}$$

Convergence to Genie KF/LS [Vaswani, ICASSP'09]

- **Assume that**

  1. the noise is bounded
  2. $(x_t)_{N_t}$ follows the Gaussian random walk model
  3. A is incoherent enough to ensure that $\delta_{3S_t} < 1/2$ for all t
  4. addition threshold, α, set large enough to prevent false additions
  5. all additions occur before a finite time

- **KF (LS) CS-residual estimate converges to the genie-aided KF (LS) estimate in probability**

Corollary: Stability [Vaswani, ICASSP'09]

- If replace "all additions before a finite time" by the following
  - number of additions at a given time less than $S_{a,max} << S_t$, and
  - delay between two addition times is large enough

  above: one way to quantify "slowly changing sparsity pattern"

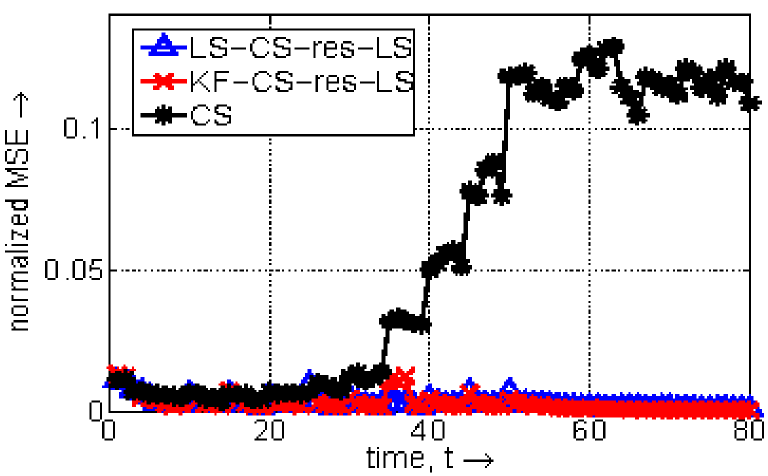
- **w.h.p. KF (LS) CS-residual gets to within a small error of the genie KF (LS), within a finite delay after a new addition time**
  - (and remains that way until next addition)

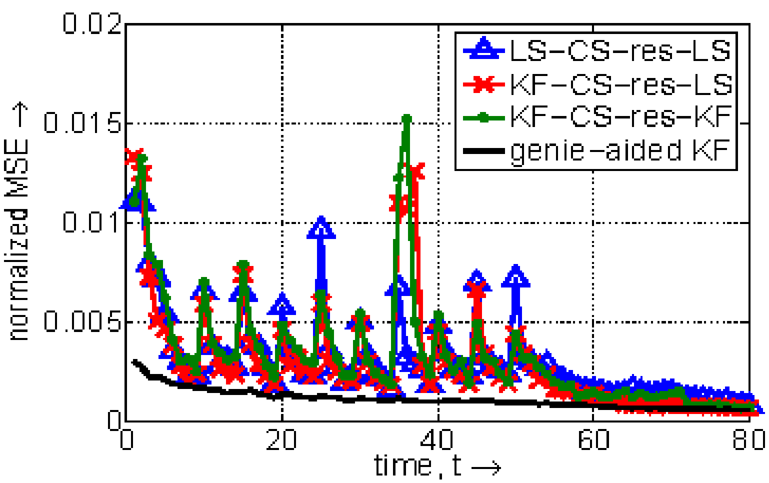
Main Idea of Proof

- Bounded noise and $\delta_{3S_t} < 1/2 \rightarrow$ CS-residual error bounded by a constant
- Addition threshold = error bound $\rightarrow$ ensures no false additions
- Gaussian random-walk model on $(x_t)_{N_t} \rightarrow$ expected value of the square of any nonzero coefficient keeps increasing linearly with t
  - w.h.p. it will exceed addition threshold plus error bound within a finite delay after being added, i.e. will get detected
- All additions before finite time $\rightarrow$ w.h.p. all detected before a finite time
  - LS CS-residual converges to genie LS in probability
- When all nonzero elements detected KF CS-residual runs a time-invariant KF with **correct model parameters**
  - will converge to genie KF in mean square and hence in probability

Simulations: random-Gaussian meas.

- Signal length: m=256
- Sparsity size: S_1 = 8, 2 new additions every 5 time units from t=10 to 50
- Observations: n=72, σ^2 = 16/9n
- **To answer a question raised during the talk**
  - The above is just one simulation example used to make it similar to the assumptions used by our theorem
  - **There could be new additions or deletions at every frame (this happens for real image sequences, e.g. MRI – see last page) and our algorithm still works**
  - **Even though the above simulation may appear to be solvable using MUSIC, the general scenario is not. Also MUSIC is NON-CAUSAL, we want a CASUAL algorithm.**
  - Also, to use MUSIC one would need to know how long there has been no addition (this may be possible to detect by thresholding norm of residual)





Related Work

- Our Kalman filtered CS work first appeared in ICIP'08
- **Works not using the current observation to compute the initial estimate that is then used to compute the observation residual**
  - k-t FOCUSS [Jung,Ye, ISBI'08]
  - Locally Competitive Algorithms for sparse coding [Rozell et al, ICIP'07]
- Very recent work
  - Recursive Lasso [Angelosante, Giannakis, ICASSP'09]
  - Dynamic l1 minimization [Asif, Romberg, CISS'09]
  - KF-CS for dynamic MRI [Qiu, Lu, Vaswani, ICASSP'09]
  - Modified-CS [Vaswani, Lu, ISIT'09]

Summary

- LS CS-residual and KF CS-residual
- **Bounded LS CS-residual error under mild assumptions**
  - bound much tighter than CS if sparsity pattern changes slowly enough
- **KF (LS) CS-residual gets to within a small error of the genie-aided KF (LS) within a finite delay after a new addition**
  - proved this under stronger assumptions

Ongoing/Future Work

- **Modified-CS** [Vaswani, Lu, ISIT'09]. $\hat{x}_t$ is the solution of

$$\min_{\beta} ||\beta_{T^c}||_1 \text{ s.t. } y_t = A\beta$$

  - an approach for provably exact reconstruction from noiseless measurements using partly known support, $T := \hat{N}_{t-1}$
  - **exact reconstruction if** $\delta_{|T|+2|\Delta|} < 1/5$ **(much weaker than CS)**

- Combine Modified-CS with CS-residual for noisy/compressible cases

- **Prove stability under weaker assumptions**

- KF CS-residual for dynamic MRI [Qiu, Lu, Vaswani, ICASSP'09]

- Combine Modified-CS with CS-residual for noisy/compressible cases

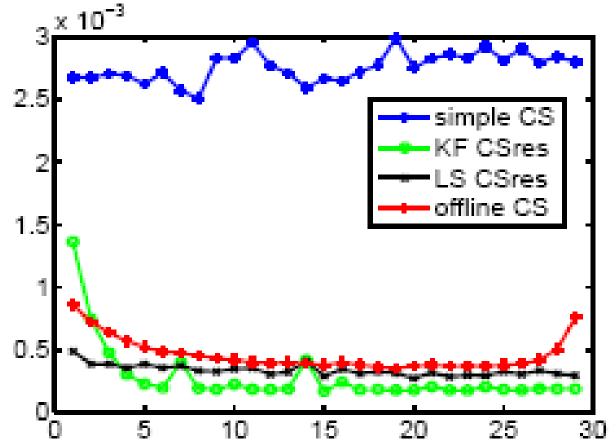
$$\min_{\beta} \gamma ||\beta_{T^c}||_1 + (1/2)||y_{t,res} - A\beta||_2^2$$

$$\min_{\beta} \gamma ||\beta_{T^c}||_1 + ||\beta_T - (\hat{x}_{t-1})_T||^2_{(P_{t|t-1})_{T,T}} + (1/2\sigma^2)||y_t - A\beta||_2^2$$

- A good way to delete zero coefficients

Simulated MRI [Qiu, Lu, Vaswani, ICASSP'09]

- Observations: n = m/2, m = 128 (one column at a time)
- Support size ~ 0.26m, change in support ~ 0.03m
- Variable density undersampling in ky, full resolution in kx
- Select γ using the error bound of [Tropp'06]



Original simple CS LS CSres KF CSres offline CS

Cardiac sequence: reconstructed last frame