Student Name:

Useful formulas and notation given on last page.

Midterm Exam 2, EE 527, Spring 2008 (Out of 40 points)

1. We have learnt that the Kalman gain, K_t, is given by

$$K_t \triangleq \Sigma_{t|t-1} H^T (H \Sigma_{t|t-1} H^T + R)^{-1}$$

and the filtering error covariance $\Sigma_{t|t} \triangleq [I - K_t H]\Sigma_{t|t-1}$.

(a) Show that K_t can also be rewritten as

$$K_t = (\Sigma_{t|t-1}^{-1} + H^T R^{-1} H)^{-1} H^T R^{-1}$$

(b) Use this to show that $\Sigma_{t|t}$ can be rewritten as $\Sigma_{t|t} = (\Sigma_{t|t-1}^{-1} + H^T R^{-1} H)^{-1}$.

8 + 2 points

2. Consider a state space model of the form

$$\begin{aligned} Y_t &= H_t X_t + r(Y_1, Y_2, \ldots Y_{t-1}) + V_t, \quad V_t \sim \mathcal{N}(0, R) \\ X_{t+1} &= F_t X_t + q(Y_1, Y_2, \ldots Y_t) + G U_t, \quad U_t \sim \mathcal{N}(0, Q) \end{aligned}$$

with $X_0 \sim \mathcal{N}(0, \Sigma_0)$. $X_0, \{U_t, t = 0, \ldots \infty\}, \{V_t, t = 0, \ldots \infty\}$ are mutually independent.

(a) The above is a Kalman model, but with a nonzero "feedback control" input in both equations. Derive the Kalman recursion for it, both prediction and update steps.

(b) Now consider the following nonlinear state space model

$$\begin{aligned} Y_t &= h(X_t) + V_t, \; V_t \sim \mathcal{N}(0, R) \\ X_{t+1} &= f(X_t) + G U_t, \; U_t \sim \mathcal{N}(0, Q) \end{aligned}$$

$X_0 \sim \mathcal{N}(0, \Sigma_0)$. $X_0, \{U_t, t = 0, \ldots \infty\}, \{V_t, t = 0, \ldots \infty\}$ are mutually independent. Linearize $h(X_t)$ about $\hat{X}_{t|t-1}$ and linearize $f(X_t)$ about $\hat{X}_{t|t}$. Assume the linearized model to be the true one and apply the Kalman filter derived above. This gives the *extended Kalman filter* for the nonlinear model.

5 + 5 = 10 points

3. Consider the Kalman model but with U_t, V_t correlated for the same t, i.e.

$$\begin{aligned} Y_t &= HX_t + V_t, \quad V_t \sim \mathcal{N}(0, R) \\ X_{t+1} &= FX_t + GU_t, \quad U_t \sim \mathcal{N}(0, Q) \end{aligned}$$

where $X_0 \sim \mathcal{N}(0, \Sigma_0)$ and $X_0, \{U_t, t = 0, \ldots \infty\}$ are mutually independent; $\{V_t, t = 0, \ldots \infty\}$ are mutually independent; and U_t is independent of $V_\tau, \tau = 1, \ldots t-1$ and of $V_\tau, \tau = t+1, \ldots \infty$ *But* U_t, V_t *are correlated with covariance,* $E[U_t V_t^T] = C$.

(a) Derive the expression for $\hat{X}_{t+1|t}$ in terms of $\hat{X}_{t|t}$, $\hat{X}_{t|t-1}$, $\Sigma_{t|t}$ and the known matrices.

(b) Derive the expression for $\Sigma_{t+1|t}$ in terms of $\Sigma_{t|t}$, $\Sigma_{t|t-1}$ and the the known matrices.

7 + 3 = 10 points

4. You are given a sequence of observations $Y_0, Y_2, \dots Y_{N-1}$ which satisfy

$$Y_n = Xr^n + w_n, \ w_n \sim \mathcal{N}(0, \sigma^2)$$

with $X \sim \mathcal{N}(0, \sigma_x^2)$. X, $\{w_n, n = 0, \dots N-1\}$ are mutually independent. r is a constant.

(a) Use the joint Gaussian formula to find the MMSE estimate of X given $Y_0, Y_1, \dots Y_{N-1}$.

(b) Can you find the MAP estimate?

9 + 1 = 10 points

Formulas

1. Matrix Inversion Identity

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1}$$

2. For jointly Gaussian X, Y with joint PDF

$$\begin{bmatrix} Y \\ X \end{bmatrix} \sim \mathcal{N}(\begin{bmatrix} \mu_Y \\ \mu_X \end{bmatrix}, \begin{bmatrix} \Sigma_Y & \Sigma_{YX} \\ \Sigma_{XY} & \Sigma_X \end{bmatrix})$$

$$\begin{aligned} E[X|Y] &= \mu_X + \Sigma_{XY}\Sigma_Y^{-1}(Y - \mu_Y) \\ Cov[X|Y] &= \Sigma_X - \Sigma_{XY}\Sigma_Y^{-1}\Sigma_X, \\ \text{Note } \; Cov[X|Y] &\triangleq E[(X - E[X|Y])(X - E[X|Y])^T|Y] \end{aligned}$$

3. Kalman filter equations. For the state space model,

$$\begin{aligned} Y_t &= H_t X_t + V_t, \; V_t \sim \mathcal{N}(0, R) \\ X_{t+1} &= F_t X_t + G_t U_t, \; U_t \sim \mathcal{N}(0, Q) \end{aligned}$$

where $X_0, \{U_t, t = 1, \ldots \infty\}, \{V_t, t = 1, \ldots \infty\}$ are mutually independent and $X_0 \sim \mathcal{N}(0, \Sigma_0)$, we have

$$\begin{aligned} K_t &= \Sigma_{t|t-1} H_t^T (H_t \Sigma_{t|t-1} H_t^T + R)^{-1} \\ \hat{X}_{t|t} &= \hat{X}_{t|t-1} + K_t(Y_t - H_t \hat{X}_{t|t-1}) \\ \Sigma_{t|t} &= [I - K_t H_t]\Sigma_{t|t-1} \\ \hat{X}_{t+1|t} &= F_t \hat{X}_{t|t} \\ \Sigma_{t+1|t} &= F_t \Sigma_{t|t} F_t^T + G_t Q G_t^T \end{aligned}$$

with initialization, $\hat{X}_{0|-1} = 0$, $\Sigma_{0|-1} = \Sigma_0$. Here

$$\begin{aligned} \hat{X}_{t|s} &\triangleq E[X_t|Y_{1:s}] \\ \Sigma_{t|s} &\triangleq Cov[X_t|Y_{1:s}] \end{aligned}$$