Compressed Sensing

1) \( y = Ax \), \( A = \text{fat} \) \( n \times N \), \( n < N \), \( N = 256 \), \( K = 72 \).

\( x = \) \( \infty \) solutions.

A minimization of some kind,

Option 1: Minimum energy: \( x \) s.t. \( y = Ax \).

\( \hat{x} = A^T (A A^T)^{-1} y \).

Reason why JPEG works

Compressibility: Most natural signals are sparse. Approximately

\[ z = C x \nu \]

\[ y = \tilde{A} x \tilde{A} = A C \]

JPEG: DCT of Signal, zeros out small coeffs.

Given a lot of compression at small loss of quality, this assumption holds.

3) When can I do \( \min \| x \|_0 \) s.t. \( y = Ax \) ?

Hope to get \( \hat{x} = x \) true.

When \( K > 2S \) & any \( 2S \) columns of \( A \) are full rank, i.e. \( \text{rank}(A^T A) = \| T \|_1 \).

\( T = \{ t_1, t_2, \ldots, t_S \} \).
5) Above is a conditionally complex system, it's a contradiction.

\[ \text{A: small, } \text{RIP}(S) \]

\[ \text{If } S \leq A \text{ then } \]

\[ \text{so for any set } T \text{ with } \text{RIP}(S) = \text{RIP}(T) \leq S \text{, it's smaller than the set } \text{RIP}(S). \]

\[ \text{This is a contradiction.} \]

Why does this work? Is it sufficient for the given conditions? For example, if \( y = x^2 \) and \( x = y^2 \), then \( y = x^2 = y^4 \).

\[ \text{Therefore, } x \neq y. \]

\[ \forall x, y \in \mathbb{R}, x \neq y \text{ if } x = y^2 \text{ and } y = x^2. \]
The sum of \( \mathcal{P}_1 \) is equal to that \( \text{eq. } \mathcal{P}_1 \text{ gives the } \) correct solution if:

\[
\begin{align*}
\Rightarrow & \quad k > 3 \varepsilon \\
\Rightarrow & \quad A \text{ satisfies RIP}(3\varepsilon) \\
\Rightarrow & \quad \|x\|_1 \leq \frac{C_1}{\sqrt{k}} \|x\|_2 \\
\Rightarrow & \quad 3\varepsilon + 3\varepsilon + \frac{1}{2} \leq 1
\end{align*}
\]

\( \mathcal{P}_1 \)

\[
\begin{align*}
\hat{x} & = \arg \min_{x} \|x\|_1 \\
\hat{x} & = \mathcal{x}, \text{ true} \\
\text{st. } y = Ax
\end{align*}
\]

6) Why? \underline{Please convert to LP}.

\[
\begin{align*}
\min & \quad \sum_{i=1}^{N} u_i c_i \\
\text{s.t.} & \quad u_i \leq x_i \leq u_i \\
& \quad y_j - (Ax)_j = 0, \quad j = 1, \ldots, K.
\end{align*}
\]

7) Practical aspect: Where do we get such \( A \)'s?

\textbf{Thm 1:} If \( A \) is random, \( k \geq 3\varepsilon \), \( k = O(\log N) \) then \( A \) satisfies \( \# \mathcal{C}_8 \text{ w.h.p.} \)

\underline{Random Fourier}: you pick \( k \) rows of \( A_{N \times N} \).

\[
\text{id uniformly over } A_{N \times N} \text{ in some way.}
\]

\[
\text{where } \underline{MR} \text{ Cap. FT of cross-sectional imaging, one at a time.}
\]
\[ y = Ax \]

\( \mathbf{y} \) is sparse: Image of an \
egg:

\( T \) if only edge are of interest.

angiogram = blood vessel image.

\( b \) \( \implies \mathbf{x} = \mathbf{w} \theta \quad \mathbf{w} = \text{DWT} \)

\( \theta = \text{sparse} \).

\[ y = A \mathbf{w} \theta \]

\( A \mathbf{w} \) also satisfies Thm 1.

C) Thm 2. \( A = \text{random Bernoulli or random G.} \)

Application of sensor net.

channel error decoding

\( \rightarrow \) Compressive imaging

Thus D. If \( A = \text{random B} \).

then \( A \mathbf{B} \) \( \in \) any other basis.

\( \mathbf{B} \) also random Bernoulli.