You will see that when X & W are Gaussian & independent the LSE, $E[X|Y]$ is equal to the linear least square estimate i.e. (a) & (c) have same answer

Quiz 7, Out of 10 points, 2 bonus points

1. A car is traveling at velocity X miles per hour and X is Gaussian distributed with mean 65 and variance 25. A police radar's measurement is $Y = X + W$, where W is Gaussian distributed with mean 2 and variance 1 and **is independent of** X.

   (a) Find the least square estimate of the car velocity based on the radar's measurement, Y.

   (b) Compute the least squares estimate of the car velocity when the radar's measurement is $Y = 65$ miles per hour

   (c) Find the formula for the linear least square estimate of the car velocity based on the radar's measurement, Y and compute it for $Y = 65$ miles per hour.

(10 points = 4 + 2 + 4)

1(a) Compute $E[X|Y]$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,a_x} e^{-\frac{(x-\mu_x)^2}{2a_x^2}} \qquad f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}\,a_w} e^{-\frac{(y-x-\mu_w)^2}{2a_w^2}}$$

$$f_{X,Y}(x,y) = f_X(x)\, f_{Y|X}(y|x)$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx, \qquad f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$E[X|Y=y] = \int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)\,dx$$

$$= \frac{\int_{-\infty}^{\infty} x\, f_{X,Y}(x,y)\,dx}{\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx}$$

$$= \frac{\frac{1}{\sqrt{2\pi}\,a_x a_w}\int e^{-\frac{(x^2+\mu_x^2-2x\mu_x)}{2a_x^2}}\, e^{-\frac{(x^2+(y-\mu_w)^2-2x(y-\mu_w))}{2a_w^2}}\, x\,dx}{\frac{1}{\sqrt{2\pi}\,a_x a_w}\int [\quad]\,dx}$$

Now the numerator ~~[illegible]~~

$$x \cdot e^{-\left(x^2\left(\frac{1}{\sigma_x^2}+\frac{1}{\sigma_w^2}\right) - 2x\left(\frac{\mu_x}{\sigma_x^2}+\frac{y-\mu_w}{\sigma_w^2}\right) + \frac{\mu_x^2}{\sigma_x^2} + \frac{(y-\mu_w)^2}{\sigma_w^2}\right)}$$

After a lot of messy algebra, you will end up with the following expression:

$$E[X|Y=y] = \int_{-\infty}^{\infty} x \underbrace{e^{-\frac{\left(x - \frac{\frac{\mu_x}{\sigma_x^2}+\frac{y-\mu_w}{\sigma_w^2}}{1/\sigma_x^2 + 1/\sigma_w^2}\right)^2 \cdot \left(\frac{1}{\sigma_x^2}+\frac{1}{\sigma_w^2}\right)}{2}}}_{\text{Gaussian pdf with mean } \frac{\frac{\mu_x}{\sigma_x^2}+\frac{y-\mu_w}{\sigma_w^2}}{1/\sigma_x^2+1/\sigma_w^2}} dx .$$

$$\therefore \quad E[X|Y=y] = \frac{\frac{\mu_x}{\sigma_x^2}+\frac{y-\mu_w}{\sigma_w^2}}{\frac{1}{\sigma_x^2}+\frac{1}{\sigma_w^2}}$$

can be rewritten as

$$\mu_x + \frac{\sigma_x^2}{\sigma_x^2+\sigma_w^2}(y-\mu_w-\mu_x)$$

which is equal to the linear least squares estimate

Plug in $\mu_x = 65$, $\sigma_x^2 = 25$, $\mu_w = 2$, $\sigma_w^2 = 1$.

~~[illegible]~~

b) Put $y = 65$.

(c) Linear least sq estimate. Use formula from book

$$\hat{X} = \mu_x + \frac{\text{cov}(X,Y)}{\sigma_y^2}(y-\mu_y)$$

$\sigma_y^2 = \sigma_x^2 + \sigma_w^2 = 26$.

$\mu_x = 65$ $\mu_y = \mu_x + \mu_w = 67$

$\text{cov}(X,Y) = E[(X-E[X])(X+W-EX-EW)] = E[(X-E[X])^2] = \sigma_x^2$

$\because E[XW] = EX\,EW$, $\because$ indepen

$\therefore \text{cov}(X,Y) = \sigma_x^2 = 25$.