Make-up MidTerm, Fall 06 (Out of 30)

5 questions, 6 points for each question

1. X is a geometric random variable with parameter p. Let $Y = \min(a, X)$ and let $Z = a - Y$.

   (a) Compute PMF of Y

   (b) Compute PMF of Z.

   (c) Compute $E[Z]$ either directly or using the PMF.

   Solution:

   (a) $p_X(k) = (1-p)^{k-1}p$

   Y takes values from a to ∞

$$p_Y(k) = \begin{cases} 0 & k < a \\ \sum_{j=1}^{a} p_X(j) & k = a \\ (1-p)^{k-1}p & k > a \end{cases}$$

   (b) $Z = a - Y$, Z takes values from $-\infty$ to 0.

   $p_Z(z) = p_Y(a-z)$. Write out the detailed expression using the PMF of Y.

$$p_Z(k) = \begin{cases} (1-p)^{a-k-1}p & k < 0 \\ \sum_{j=1}^{a} p_X(j) & k = 0 \end{cases}$$

   (c) $E[Z] = a - E[Y]$ and $E[Y] = \displaystyle\sum_{a+1}^{\infty} k p_X(k)$

2. You drive $a = 250$ days in a year. Every day, there is a probability $p = 0.02$ that you will get a traffic ticket, independently of other days.

   (a) Compute the probability that you will get b tickets in the whole year.

   (b) What is the probability that you get more than *twice* the expected number of tickets in the whole year.

   (c) Any one of the tickets is 50, 100 or 200 dollars with respective probabilities 0.5, 0.3 and 0.2. Find the mean and variance of the amount of money you will pay in traffic tickets during the year.

Solution:

(a) Let X = number of traffic tickets. X is binomial(a,p).
P(b traffic tickets out of a days) $= p_X(b) = \binom{a}{b} p^b(1-p)^{a-b}$ for $0 \leq b \leq a$ and 0 otherwise.

(b) Expected value $= ap$.

P(get more than twice the expected value) $= P(X > 2ap) = \sum_{k=2ap+1}^{a} p_X(k)$

(c) Let Y = money spent on tickets.
Let Y_i = money spent on i^{th} day. Then

$$p_{Y_i}(y) = \begin{cases} 1-p & if\ y = 0 \\ 0.5p & if\ y = 50 \\ 0.3p & if\ y = 100 \\ 0.2p & if\ y = 200 \\ 0 & otherwise \end{cases}$$

Compute $E[Y_i]$ and $Var[Y_i]$.
$Y = \sum_{i=0}^{a} Y_i$, and Y_i independent.
So $E[Y] = aE[Y_i]$ and $Var[Y] = aVar[Y_i]$.

Alternative solution using Total Expectation Theorem (messy way)
Let Y = money spent on tickets.
Let Y_i = money spent on the i^{th} ticket.
$p_{Y_i}(50) = 0.5$, $p_{Y_i}(100) = 0.3$, $p_{Y_i}(200) = 0.2$,
$p_{Y_i}(y) = 0$ for all other values.
Thus $E[Y_i] = 50.0.5 + 100.0.3 + 200.0.2 = 25 + 30 + 40 = 95$
Given that you get X=k tickets, $Y = Y_1 + Y_2 + \ldots Y_k$, and thus $E[Y|X = k] = 95k$.

By total expectation theorem, $E[Y] = \sum_{k=0}^{a} p_X(k)E[Y|X = k] = \sum_{k=0}^{a} p_X(k)95k \triangleq \mu_Y$
Since the Y_i's are independent, $Var[Y|X = k] = kVar[Y_i]$ $Var[Y] = E[(Y - \mu_Y)^2] = \sum_{k=0}^{a} p_X(k)E[(Y - \mu_Y)^2|X = k] = \sum_{k=0}^{a} p_X(k)kE[(Y_i - \mu_Y)^2]$.
Note $E[(Y_i - \mu_Y)^2]$ is NOT EQUAL TO $Var[Y_i]$.

3. The median of a continuous random variable X is the value m such that $P(X \leq m) = 0.5$. The lifetime (time until it breaks down) of an air-conditioner is an exponentially distributed random variable with a **median** value of 20 years.

   (a) What is the probability that it breaks down in less than a day?

   (b) What is the probability that it lasts **at least** 30 years?

   Solution:

   (a) First compute λ.
   For an exponential r.v., m is the solution of:
   $P(X \leq m) = 0.5$, i.e.

   $$\int_{x=0}^{m} \lambda e^{-\lambda x} dx = 1 - e^{-\lambda m} = 0.5$$

   Thus

   $$\lambda = \frac{\log_e 2}{m}$$

   Then compute

   $$P(X \leq 1/365) = \int_{x=0}^{1/365} \lambda e^{-\lambda x} dx$$

   (b) $P(X \geq 30) = \int_{x=30}^{\infty} \lambda e^{-\lambda x} dx$

4. X is a random variable with PDF $f_X(x)$. Y is a random variable dependent on X with conditional PDF $f_{Y|X}(y|x)$ and conditional CDF $F_{Y|X}(y|x)$.

(a) Let $Z = XY$. Find the PDF of Z

(b) Let $Z = X^2 + Y$. Find the PDF of Z

Solution:

(a) First compute the conditional CDF $F_{Z|X}(z|x)$.

$$F_{Z|X}(z|x) = P(XY \leq z|X = x) = P(Y \leq z/x|X = x) = F_{Y|X}(z/x|x)$$

Differentiate w.r.t. z to get

$$f_{Z|X}(z|x) = \frac{dF_{Z|X}(z|x)}{dz} = \frac{dF_{Y|X}(z/x|x)}{dz} = (1/x)f_{Y|X}(z/x|x)$$

Now

$$f_Z(z) = \int_x f_{Z|X}(z|x)f_X(x)dx = \int_x (1/x)f_{Y|X}(z/x|x)f_X(x)dx$$

(b) First compute the conditional CDF $F_{Z|X}(z|x)$.

$$F_{Z|X}(z|x) = P(X^2 + Y \leq z|X = x) = P(Y \leq z - x^2|X = x) = F_{Y|X}(z - x^2|x)$$

Differentiate w.r.t. z to get

$$f_{Z|X}(z|x) = \frac{dF_{Z|X}(z|x)}{dz} = \frac{dF_{Y|X}(z - x^2|x)}{dz} = f_{Y|X}(z - x^2|x)$$

Now

$$f_Z(z) = \int_x f_{Z|X}(z|x)f_X(x)dx = \int_x f_{Y|X}(z - x^2|x)f_X(x)dx$$

5. Let R be uniformly distributed between 0 and 1. Given $R = r$, X is uniformly distributed between 0 and r. Find the conditional PDF of R given X.

Solution:

$$f_R(r) = \begin{cases} 1 & 0 \leq r < 1 \\ 0 & otherwise \end{cases}$$

$$f_{X|R}(x|r) = \begin{cases} 1/r & 0 \leq x < r < 1 \\ 0 & otherwise \end{cases}$$

So,

$$f_{R,X}(r, x) = \begin{cases} 1/r & 0 \leq x < r < 1 \\ 0 & otherwise \end{cases}$$

First compute marginal PDF of X: For $0 < x < 1$,

$$f_X(x) = \int_x^1 f_{R,X}(r, x) dr = \int_x^1 \frac{dr}{r} = -\log_e x$$

Then

$$f_{R|X}(r|x) = \begin{cases} \frac{1}{-r \log_e x} & 0 \leq x < r < 1 \\ 0 & otherwise \end{cases}$$