CONVEX POLYTOPES

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1. *Introduction*

The study of convex polytopes in Euclidean space of two and three dimensions is one of the oldest branches of mathematics. Yet many of the more interesting properties of polytopes have been discovered comparatively recently, and are still unknown to the majority of mathematicians. In this paper we shall survey the subject, mentioning some of the most recent results, and stating the more important unsolved problems. In order to make the exposition as self-contained as possible, in §2.1 and §3.1 we give a number of definitions with which the reader may not be familiar. We have separated our account of the combinatorial properties of polytopes in §2, from those of a metrical character in §3. As will be seen, these two aspects of the subject overlap, and distinguishing between them is, to a large extent, a matter of expository convenience only.

It is beyond the scope of the present survey to indicate proofs of the results mentioned. For these the reader is referred either to the original papers, or, for the earlier results, to the standard textbooks. In the notes[(1)] references are given to recent publications, and in these, precise references to the older literature may be found. It has, of course, been necessary to make a small selection from the vast amount of published material. To some extent the selection has been made according to the authors' personal interests, but we hope that it nevertheless represents a reasonably balanced account of our subject.

Before the beginning of this century, three events can be picked out as being of the utmost importance for the theory of convex polytopes. The first was the publication of Euclid's Elements which, as Sir D'Arcy Thompson once remarked,[(2)] was intended as a treatise on the five regular (Platonic) 3-polytopes, and not as an introduction to elementary geometry. The second was the discovery in the eighteenth century[(3)] of the celebrated Euler's Theorem (see §2.2) connecting the numbers of vertices, edges and polygonal faces of a convex polytope in E^3. Not only is this a result of great generality, but it initiated the combinatorial theory of polytopes. The third event occurred about a century later with the discovery of polytopes in $d \geqslant 4$ dimensions. This has been attributed to the Swiss mathematician Ludwig Schläfli;[(4)] it happened at a time when very few mathematicians (Cayley, Grassmann, Möbius) realised that geometry in more than three dimensions was possible.

During the latter half of the nineteenth century a large amount of work concerning polytopes was done, mostly extending the earlier metrical work to $d \geqslant 4$ dimensions. The symmetry groups of polytopes were extensively studied (see §3.4) and the

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"regular" polytopes were discovered and rediscovered many times.[5] Many famous mathematicians of this period made contributions to the subject, but very little of their work was of a combinatorial nature, probably because one of the central problems, the enumeration problem, had proved completely intractable (see §2.1 and §2.5).

In spite of Minkowski's important work[6] on convex sets, and on convex polytopes in particular, there was a rapid decline in interest early in the present century. Geometers began concentrating on other subjects, and the study of polytopes was neglected by all except a very few. For this reason, a number of important results, such as Steinitz' Theorem (§2.4) were unnoticed, and have become widely known only during the last few years.

The modern theory of convex polytopes began about 1950. The work of Gale, Motzkin, Klee and others caused a revival of interest especially in combinatorial problems, and the publication of "Convex Polytopes"[7] in 1967 led directly to a great deal of research. The influence of linear programming and other applications to practical problems must also be mentioned.

In the present paper we shall be mainly concerned with the developments that have taken place in this most recent period.

2. *The Combinatorial Theory of Polytopes*

2.1. *The facial structure*[8]

We work throughout in Euclidean space E^d ($d \geqslant 0$), using lower-case letters x, y, etc. for points or vectors (we do not distinguish between these) and upper-case letters for sets. All subspaces of E^d to be considered are affine subspaces, that is, are given by a system of (not necessarily homogeneous) linear equations. A set of points $X = \{x_1, \ldots, x_n\}$ is *affinely dependent* if there exist scalars (real numbers) $\lambda_1, \ldots, \lambda_n$, not all zero, such that

$$\lambda_1 x_1 + \ldots + \lambda_n x_n = o, \quad \lambda_1 + \ldots + \lambda_n = 0,$$

and X is *affinely independent* if no such scalars exist. An *affinity*, or non-singular affine transformation, is a mapping $A : E^d \to E^d$ which preserves affine dependence and independence; it is a non-singular linear transformation followed by a translation. Any set $S \subseteq E^d$ is said to have (affine) *dimension* r (written $\dim S = r$) if a maximal affinely independent subset of S contains exactly $r+1$ points. The (unique) affine subspace of smallest dimension containing S is called the *affine hull* of S and is denoted by aff S. Clearly $\dim S = \dim \operatorname{aff} S$. In E^d, affine subspaces of $d-1$ dimensions are called *hyperplanes*, those of 1 dimension are called *lines*, and we do not distinguish betweeh a 0-dimensional affine subspace and the single point which it contains.

A *projectivity*, or non-singular projective transformation, is a mapping of the form

$$x \to \frac{Lx + a}{\langle x, b \rangle + \kappa}$$

where L is a linear transformation, a and b are fixed vectors of E^d and κ is a constant such that

$$\det \begin{pmatrix} L & a \\ b^T & \kappa \end{pmatrix} \neq 0.$$

If we write

$$T = \{x \in E^d : \langle x, b\rangle + \kappa = 0\},$$

so that T is a hyperplane if $b \neq 0$ and is empty otherwise, then the projectivity maps $E^d \setminus T$ into E^d. (In the language of classical projective geometry, T is mapped by the projectivity into the "hyperplane at infinity".) Such a projectivity is said to be *permissible* for a set S if $S \cap T = \emptyset$.

A set $S \subseteq E^d$ is *convex* if it has the property that for any pair of points $x, y \in S$, the *line segment*

$$\{\lambda x + (1-\lambda) y : 0 \leqslant \lambda \leqslant 1\}$$

with end points x and y, lies entirely in S. For any set S, the smallest convex set containing S (the intersection of the family of all convex sets that contain S) is called the *convex hull* of S, and is denoted by conv S. Clearly $\dim S = \dim \operatorname{conv} S$.

A *convex polytope* P is defined to be the convex hull of any finite set of points in E^d. A d-dimensional convex polytope P will be referred to, for brevity, as a *d-polytope*. A 2-polytope is called a convex polygon, but we shall refrain from using the term "convex polyhedron" for a 3-polytope since the word "polyhedron" has recently acquired a different technical meaning. The set of all d-polytopes in E^d is denoted by $\mathscr{P}^d$.

A hyperplane H *supports* a closed bounded convex set S if $H \cap S \neq \emptyset$, and S lies in one of the two closed half-spaces bounded by H. If H supports S, then $H \cap S$ is called a *face* of S. Every point of bd S (the boundary of S) lies on some supporting hyperplane of S, and so belongs to some face of S. In the case of a d-polytope P, the following properties hold:

(i) The faces of P are polytopes.

(ii) P possesses faces of every dimension $0, 1, \ldots, d-1$. For brevity, a j-dimensional face is called a *j-face* of P. A 0-face is called a *vertex*, a 1-face is called an *edge*, and a $(d-1)$-face is called a *facet* of P. The set of vertices of P is denoted by vert P.

(iii) Every face of a face of P is also a face of P.

In certain applications it is convenient to introduce two *improper* faces, namely $\emptyset$ (whose dimension is conventionally taken to be -1) and P itself (a d-face of P). If this is done then we have the following additional properties:

(iv) For every two faces F_1, F_2 of P, $F_1 \cap F_2$ is also a face of P. We write $F_1 \wedge F_2$ for $F_1 \cap F_2$.

(v) For every two faces F_1, F_2 of P there exists a uniquely defined face $F_1 \vee F_2$ of P, namely the "smallest" face of P (in the obvious sense) which contains both F_1 and F_2.

The combinatorial theory of polytopes is concerned with their facial structure, and in this connection it is worth mentioning that properties (i), (ii), and (iii), on which much of the theory depends, do not hold for convex sets in general. Let $\mathscr{F}(P)$ denote the set of all faces of P (both proper and improper). Then, with the operations $\wedge$ and $\vee$ defined in (iv) and (v) above, $\mathscr{F}(P)$ is a lattice, called the *face-lattice* of P. Two polytopes P_1 and P_2 are said to be *combinatorially equivalent* (written $P_1 \approx P_2$) if their face-lattices $\mathscr{F}(P_1)$ and $\mathscr{F}(P_2)$ are isomorphic. Equivalently, $P_1 \approx P_2$ if there exists a one-to-one inclusion-preserving mapping from $\mathscr{F}(P_1)$ onto $\mathscr{F}(P_2)$. Roughly speaking, combinatorially equivalent polytopes have the same number of faces for each dimension arranged in the same way, but possibly of different shapes. For example, every 3-polytope with 6 quadrilateral 2-faces is combinatorially equivalent to the 3-cube. The combinatorial theory of polytopes may be regarded as a study of the face-lattices $\mathscr{F}(P)$; it is concerned with combinatorial equivalence classes of polytopes rather than with polytopes themselves.

It is clear that the image of a polytope P under a permissible projective transformation is a polytope P' combinatorially equivalent to P. For certain d-polytopes (for example those with $d+1$ or $d+2$ vertices) the converse holds, that is, every polytope combinatorially equivalent to P is the image of P under a permissible projectivity. These are called *projectively stable* polytopes, an interesting class of polytopes about which very little is known.[(9)]

The convex hull of $d+1$ affinely independent points is a d-polytope known as a *d-simplex* (a 3-simplex is often called a tetrahedron). If all the proper faces of a d-polytope P are simplexes, then P is called a *simplicial* polytope, and the set of all simplicial d-polytopes in E^d is denoted by $\mathscr{P}_s{}^d$. A d-polytope conv $\{(\pm 1, \pm 1, \ldots, \pm 1)\}$ is called a *d-cube*, and a d-polytope all of whose proper faces are combinatorially equivalent to cubes is called a *cubical* polytope. A number of other subsets of $\mathscr{P}^d$, defined by the combinatorial character of their proper faces have been studied, but these two are the most important.

For each d-polytope P it is easy to establish by polarity that there exists a d-polytope P^*, called a *dual* of P, with the property that $\mathscr{F}(P)$ and $\mathscr{F}(P^*)$ are anti-isomorphic, that is, there exists a one-to-one inclusion-reversing mapping from $\mathscr{F}(P)$ onto $\mathscr{F}(P^*)$. A dual of a simplicial polytope is called *simple*. A simplex is an example of a *self-dual* polytope; many other such polytopes are known. A dual of a d-cube is called a d-crosspolytope, the analogue of the familiar octahedron in E^3.

If the j-faces of a d-polytope P are simplexes for all $j \leqslant r$, and the k-faces of a dual P^* of P are simplexes for all $k \leqslant s$, then we say that P is of type (r, s). In this notation a simplicial d-polytope is of type $(d-1, 1)$ and a simple d-polytope is of type $(1, d-1)$. It is easy to see that if $r+s > d$ then P must be a simplex. It is curious that no $(r+s)$-polytopes of type (r, s) different from the simplex are known except when r or s is small.[(10)]

Let $c(v, d)$ represent the number of distinct combinatorial types of d-polytopes with v vertices. (Clearly, only the case $v > d \geqslant 3$ is interesting.) The classical enumeration problem is the determination of $c(v, d)$ for all $v \geqslant d+1$. Apart from the values $c(d+1, d) = 1$, and $c(d+2, d) = [\frac{1}{4}d^2]$, only seven other values are known and one of these has not been independently checked and so is open to doubt.[11] [*Added in proof*: Recently an explicit expression for $c(d+3, d)$ has been found. See note (62a).] Somewhat more is known about the number $c_s(v, d)$ of distinct combinatorial types of simplicial d-polytopes with v vertices. We have $c_s(d+1, d) = 1$, $c_s(d+2, d) = [\frac{1}{2}d]$, and Perles' remarkable formula

$$c_s(d+3, d) = 2^{[\frac{1}{2}d]} - \left[\frac{d+4}{2}\right] + \frac{1}{4(d+3)} \Sigma\phi(h)\, 2^{(d+3)/h},$$

where ϕ is the Euler function, and the summation is over all odd divisors h of $d+3$. This was determined using Gale diagrams (see §2.5). The only other values of $c_s(v, d)$ known[12] are $c_s(v, 3)$ for $7 \leqslant v \leqslant 12$ and[13] $c_s(8, 4) = 37$.

The central problem of the combinatorial theory may be stated as follows: *find necessary and sufficient conditions for a given lattice $\mathscr{L}$ to be isomorphic to the face-lattice $\mathscr{F}(P)$ of some polytope P.* In this generality the problem seems completely intractable except for the fact that the lattices of 3-polytopes may be characterised by Steinitz' Theorem.[13a] As this result is more naturally stated in terms of graphs, we defer details until §2.4.

It is, of course, easy to find necessary conditions on $\mathscr{L}$—to do this we need only investigate the special properties of face-lattices. One such property is the following. Let $F_1, F_2 \in \mathscr{F}(P)$ and $F_1 \subset F_2$. To avoid triviality assume that $\dim F_1 \leqslant \dim F_2 - 2$ and consider the set of faces $F \in \mathscr{F}(P)$ such that $F_1 \subseteq F \subseteq F_2$. These faces form a sublattice of $\mathscr{F}(P)$ and it is not difficult to prove that this sublattice is isomorphic to the face-lattice of a $(\dim F_2 - \dim F_1 - 1)$-polytope.[14] We may denote this polytope (or, rather, its combinatorial type) by F_2/F_1. For any face F, we have $F/\emptyset \approx F$. The polytopes P/F^0, where F^0 is a vertex of P, have received considerable attention in the classical literature where they are known as the *vertex figures* of P. The most perspicuous proof of the above statements depends upon the fact that the polytopes F_2/F_1 can be exhibited as sections of P by affine subspaces of suitable dimension. One might hope that the above property of the sublattices of $\mathscr{F}(P)$ could be used as an inductive characterization of the face-lattices of polytopes. This is, in fact, not the case.[15]

By a *maximal tower* of faces of a d-polytope P we mean a sequence of faces

$$\emptyset \subset F^0 \subset F^1 \subset \ldots \subset F^{d-1} \subset P$$

where each inclusion is proper and such that F^j is of dimension j. The study of such towers may turn out to be a fruitful way of investigating $\mathscr{F}(P)$, although so far, only one application of this idea is known.[16]

2.2. *f-vectors of polytopes*

Let P be any d-polytope and write $f_j(P)$ for the number of j-faces of P. Then we define the vector

$$f(P) = (f_0(P), \ldots, f_{d-1}(P))$$

in E^d to be the *f-vector* of P. It is convenient also to use the notation $f(\mathscr{P})$, where $\mathscr{P}$ is any family of polytopes, for the set $\{f(P) : P \in \mathscr{P}\}$.

Since the problem of characterising the face-lattices of polytopes has proved too difficult, it is natural to attempt the apparently simpler one of characterising the f-vectors of polytopes. Although this problem is not completely solved, at least some progress has been made, which we shall describe in this section.

The most basic result is:

THEOREM 1 (Euler's Theorem). *For any* $P \in \mathscr{P}^d$, *the relation*

$$f_0(P) - f_1(P) + \ldots + (-1)^{d-1} f_{d-1}(P) = 1 + (-1)^{d-1}$$

holds.

In other words, the f-vectors of all polytopes $P \in \mathscr{P}^d$ lie on a certain hyperplane (the *Euler hyperplane*) in E^d. In a sense this is the best possible result for it is easy to show that the set $f(\mathscr{P}^d)$ lies on no affine subspace of smaller dimension, that is, $\dim \operatorname{aff} f(\mathscr{P}^d) = d-1$. Many proofs of Theorem 1 are known; the most elementary avoid all topological considerations.[17]

For certain subsets $\mathscr{P} \subset \mathscr{P}^d$, it may happen that $\dim \operatorname{aff} f(\mathscr{P}) < d-1$. The case of simplicial polytopes is especially important:

THEOREM 2. $\dim \operatorname{aff} f(\mathscr{P}_s^d) = [\frac{1}{2}d]$, *the* $[\frac{1}{2}d]$*-dimensional subspace* $\operatorname{aff} f(\mathscr{P}_s^d)$ *being defined by the equations*

$$\sum_{j=k}^{d-1} (-1)^j \binom{j+1}{k+1} f_j(P) = (-1)^{d-1} f_k(P)$$

for $k = -1, 0, 1, \ldots, d-1$. (We adopt the convention $f_{-1}(P) = 1$ and note that $k = -1$ leads again to the Euler equation.)

The equations of Theorem 2 are known as the Dehn–Sommerville equations.[18] The values $k = d-2$, $d-4$, $d-6, \ldots,$ lead to a linearly independent system which can be solved to give about half of the numbers $f_j(P)$ in terms of the others. These solutions are important in connection with the Upper Bound Conjecture stated later in this section.

Although Theorems 1 and 2 give the exact dimension of $\operatorname{aff} f(\mathscr{P}^d)$ and $\operatorname{aff} f(\mathscr{P}_s^d)$ respectively, they do not characterise the sets $f(\mathscr{P}^d)$ and $f(\mathscr{P}_s^d)$. That is to say, not every vector with positive integral components belonging to the subspace defined by Theorem 1 (or Theorem 2) is the f-vector of some d-polytope (or simplicial d-polytope). In fact, the complete characterisation is known only in the case $d = 3$, where we have the following theorem:[19]

THEOREM 3. (i) $(f_0, f_1, f_2) \in f(\mathscr{P}^3)$ *if and only if* f_0, f_1 *and* f_2 *are integers satisfying the relations* $f_0 - f_1 + f_2 = 2$, $4 \leqslant f_0 \leqslant 2f_2 - 4$, *and* $4 \leqslant f_2 \leqslant 2f_0 - 4$.

(ii) $(f_0, f_1, f_2) \in f(\mathscr{P}_s^3)$ *if and only if* f_0, f_1 *and* f_2 *are integers satisfying* $f_0 \geqslant 4$, $f_1 = 3f_0 - 6$, *and* $f_2 = 2f_0 - 4$.

The necessity of the condition (ii) is an immediate consequence of Theorem 2. One surprising consequence of Theorem 3 is that no 3-polytope can have seven edges. For $d = 4$ some partial results are known:[(20)]

THEOREM 4. (i) *There exists a* 4*-polytope* P *with* $f_0(P) = f_0$ *and* $f_3(P) = f_3$ *if and only if the integers* f_0 *and* f_3 *satisfy the inequalities* $5 \leqslant f_0 \leqslant \frac{1}{2}f_3(f_3 - 3)$ *and* $5 \leqslant f_3 \leqslant \frac{1}{2}f_0(f_0 - 3)$.

(ii) *There exists a* 4*-polytope* P *with* $f_0(P) = f_0$ *and* $f_1(P) = f_1$ *if and only if the integers* f_0 *and* f_1 *satisfy* $10 \leqslant 2f_0 \leqslant f_1 \leqslant \frac{1}{2}f_0(f_0 - 1)$, *and* (f_0, f_1) *is not equal to* (6, 12), (7, 14), (8, 17) *or* (10, 20).

Here, a feature of interest is the existence of the four exceptional pairs (f_0, f_1), the last two of which seem "accidental" in that they are excluded by no known general condition.[(20a)]

Since the precise characterisation of the sets $f(\mathscr{P}^d)$ and $f(\mathscr{P}_s^d)$ is unknown for $d \geqslant 4$, an apparently simpler question regarding the numbers $f_j(P)$ has been extensively studied, though again we are far from a complete solution. Given d and $v \geqslant d+1$, consider the set of all d-polytopes with v vertices. Then it is clear that for such polytopes P and for any j satisfying $1 \leqslant j \leqslant d-1$, the set of integers $f_j(P)$ must be bounded. The problem is to determine the least upper, and greatest lower, bounds for this set, in terms of j, v and d.

Before stating the known results and conjectures, it is convenient to make a digression by describing a special type of polytope known as a *cyclic polytope*. Let us consider any $v \geqslant d+1$ distinct points on the moment curve

$$\{(\tau, \tau^2, \ldots, \tau^d) : -\infty < \tau < \infty\}$$

(or on any other rational normal curve) in E^d. Then the convex hull of these v points is a d-polytope, denoted by $C(v, d)$ and called a cyclic polytope. The combinatorial type of $C(v, d)$ is uniquely determined in the sense that it depends only on v and d and not on the choice of curve or of the v points on it. These cyclic polytopes have many interesting properties.[(21)] They are simplicial and $[\frac{1}{2}d]$-neighbourly. (A polytope is said to be k-neighbourly if every subset of vert P containing k points is the set of vertices of some face of P.) From this we deduce that, for $0 \leqslant j \leqslant [\frac{1}{2}d] - 1$,

$$f_j(C(v, d)) = \binom{v}{j+1}.$$

For $[\frac{1}{2}d] \leqslant j \leqslant d-1$, slightly more complicated expressions for $f_j(C(v, d))$ in terms of binomial coefficients are known. We quote only

$$f_{d-1}(C(v, d)) = \binom{v - [\frac{1}{2}(d+1)]}{v-d} + \binom{v - [\frac{1}{2}(d+2)]}{v-d}.$$

Returning now to the problem of bounds for $f_j(P)$ we state the following conjecture:

THE UPPER BOUND CONJECTURE. *For all* $P \in \mathscr{P}^d$ *with* $f_0(P) = v$, *and for* $j = 1, 2, \ldots, d-1$,

$$f_j(P) \leqslant f_j(C(v,d)).$$

In other words, for each dimension j, the simplicial polytope $C(v,d)$ has, for given v and d, the maximum possible number of j-faces. The fact that the maximum is attained for a simplicial polytope is easily proved, but that $f_j(C(v,d))$ is the least upper bound has been proved only if v is either "sufficiently small", or else "sufficiently large", compared to d. More precisely, we have:

THEOREM 5. *The upper bound conjecture holds for the following values of j, v and d:*

(i) *For every* j, $1 \leqslant j \leqslant [\frac{1}{2}d]$.

(ii) *For every* $j = n+p$, *and*

(a) $d = 2n$ *and* $v \geqslant n-p-2+\dfrac{p+1}{p+2}n(n+1)$,

(b) $d = 2n+1$ *and* $v \geqslant n-p-2+\dfrac{p+1}{p+2}(n+1)(n+2)$.

(iii) *For each* j, $1 \leqslant j < d$ *and* $v \leqslant d+3$.

(iv) *For each* j, $1 \leqslant j < d$ *and* $v \leqslant 8$.

(v) *For* $j = d-1$ *and*

(a) $d = 2n$ *and* $v \geqslant n^2-2$,

(b) $d = 2n+1$ *and* $v \geqslant (n+1)^2-3$.

(vi) (a) *For* $d = 2n$, $j = n$, *and* $v \geqslant \frac{1}{2}(n^2+3n-6)$.

(b) *For* $d = 2n+1$, $j = n$, *and* $v \geqslant \frac{1}{2}(n^2+5n-4)$.

(vii) *For* $j = [\frac{1}{2}d]$, $v = d+4$.

(viii) (a) *For* $d = 9$, $j = 4$.

(b) *For* $d = 10$, $j = 5$.

If will be observed that, in the case of facets ($j = d-1$), for each dimension d, the conjecture has been proved for almost all (that is, all except a finite number of) values of v.

The proof[(22)] of Theorem 5 (some parts of which are very recent) is too technical for us to describe here. It depends heavily on the Dehn–Sommerville equations and their solutions (Theorem 2). It also makes use of a number of lemmas implying linear inequalities between the numbers $f_j(P)$. Opinions seem to be divided as to whether the conjecture will ever be completely proved by an extension of these methods. The authors make the guess that a complete proof will become possible—if ever—only when some completely new method is devised. Even the possibility that the Upper Bound Conjecture fails for some large values of v and d should be considered; pairs of the type $v = 2d$ seem to be good candidates.

Even less progress has been made in attempts to solve the other problem, that of determining greatest lower bounds for the numbers $f_j(P)$. We have the following:[23]

THEOREM 6. *For all* $P \in \mathscr{P}^d$, $1 \leqslant j \leqslant d-1$ *and* $5 \leqslant d+1 < v \leqslant d+4$,

$$f_j(P) \geqslant \binom{d+1}{j+1} + \binom{d}{j+1} - \binom{2d+1-v}{j+1}.$$

It has been conjectured that this result holds for all v with $d+1 \leqslant v \leqslant 2d$. For $v > 2d$ no one has even ventured to guess the minimum possible value of $f_j(P)$ nor the type of polytope for which the minimum is attained.

In the case of simplicial polytopes, rather more is known.

THE LOWER BOUND CONJECTURE. *For all* $P \in \mathscr{P}_s^d$ *with* $f_0(P) = v$,

$$f_j(P) \geqslant \binom{d}{j} v - \binom{d+1}{j+1} j \quad \textit{for} \quad 1 \leqslant j \leqslant d-2,$$

and

$$f_{d-1}(P) \geqslant (d-1)\,v-(d+1)(d-2).$$

These lower bounds are attained for polytopes obtained from the simplex by (repeatedly) adjoining suitably low pyramids to its facets.

The extent of our present knowledge is summarised in the following theorem.[24]

THEOREM 7. *The lower bound conjecture is true if* $d \leqslant 5$ *or if* $v \leqslant d+11$.

Strange as it may seem, though the restriction of our attention to simplicial polytopes simplifies the extremal problems just described, restrictions to other classes of polytopes seem, if anything, to make the problems more difficult. For example let us consider the class $\mathscr{P}_0^d$ of d-polytopes that are centrally-symmetric. (We say that P has a *centre* c if, for each $x \in \operatorname{vert} P$, the point $2c-x \in \operatorname{vert} P$ also. As far as the combinatorial theory is concerned, *centrally-symmetric* means combinatorially equivalent to some polytope with centre.) Then one can ask for the maximum (minimum) number of j-faces of a polytope $P \in \mathscr{P}_0^d$ with some specified number of vertices. In this case no one has even conjectured the bounds, nor guessed for which polytopes they are attained. In the case of the upper bound, one reason for this is that centrally-symmetric polytopes cannot have neighbourliness properties analogous to those of the cyclic polytopes. The following has been conjectured:

CONJECTURE.[25] *Let* P *be a* d*-polytope with centre and with* $2(d+n)$ *vertices* $(n \geqslant 1)$. *Then* P *is at most* $[(d+n-1)(n+1)^{-1}]$*-neighbourly.*

(Here k-neighbourly means that every subset of $\operatorname{vert} P$ containing k points, but not containing any two vertices whose join passes through the centre of P, is the set of vertices of a face of P.)

In the case of cubical polytopes, nothing is known about the bounds for $f_j(P)$, and the same applies to, for example, self-dual polytopes. Many unsolved problems remain in this area, none of which would appear to be easy.

It has recently been remarked[26] that there are a number of quantities related to complexes, besides the Euler characteristic, that are invariant under subdivision of the cells. These are defined in terms of the numbers of incidences between the cells of $\mathscr{C}$ of different dimensions. So far, it appears that these invariants have not been used at all in the theory of polytopes.

2.3. *Complexes*

A completely different approach to the problem of characterising the face-lattices of polytopes is through the theory of complexes, well-known (in the simplicial case at least) in the investigation of topological spaces. Here we summarise the relevant results.[27]

A *topological polytope* P' is the image of a convex polytope P under a homeomorphism $\phi : P \to E^n$. The faces of P' are the images of the faces of P under ϕ, and the dimension of P' is defined to be the dimension of P. Sometimes we shall use the term *geometric polytope* for a convex polytope when we wish to emphasise the difference from a topological polytope. A geometric (topological) cell complex $\mathscr{C} = \{C_i : i \in I\}$ is a finite family of geometric (topological) polytopes (*cells*) in Euclidean space E^n such that

(i) Every face of a cell C_i of $\mathscr{C}$ is itself a cell of $\mathscr{C}$.

(ii) The intersection $C_1 \wedge C_2$ of any two cells of $\mathscr{C}$ is a face (proper or improper) of each of them.

Extending the notation for polytopes in the obvious way, the number of j-cells in $\mathscr{C}$ is denoted by $f_j(\mathscr{C})$. $\mathscr{C}$ does not, in general, form a lattice since $C_1 \vee C_2$ may not be defined, but its cells are partially ordered by inclusion. An *abstract cell complex* is any set of polytopes on which an operation $\wedge$ is defined, and which satisfies (i) and (ii). We say that two complexes $\mathscr{C}_1$ and $\mathscr{C}_2$ are *isomorphic* (combinatorially equivalent) if there is a one-to-one correspondence between them which preserves the relation of inclusion. Similarly, dual complexes may be defined.

The *dimension* of $\mathscr{C}$ is defined to be k if it possesses cells of dimension k, but no cells of dimension $k+1$. For brevity, a k-dimensional complex is called a *k-complex*, to which we shall usually attach the adjective geometric or topological. A 1-complex is called a *graph*, the properties of which will be investigated in more detail in §2.4.

Let d be given. Then a geometric or topological k-complex $\mathscr{C}$ with $k \leqslant d$ is said to be geometrically (topologically) embeddable in E^d if there exists a geometric (topological) k-complex in E^d which is isomorphic to $\mathscr{C}$. If $d = 2$, then we shall refer to an embeddable 1-complex (graph) as a *planar* graph. It is a consequence of Steinitz' Theorem (Theorem 11 of §2.4) that in this particular case we do not need to distinguish between geometric and topological embeddability.

We now summarise the known results about embeddability.[28]

THEOREM 8. (i) *If an n-complex is simplicial it is always geometrically embeddable in* E^{2n+1}, *but some simplicial n-complexes are not topologically embeddable in* E^{2n}. (An example, for $n = 1$, is provided by the well-known non-planar graphs.)

(ii) *If an abstract n-complex is not simplicial then it may not be topologically embeddable in any Euclidean space.* (An example is provided by the 2-complex containing 6 pentagons as 2-cells formed by identifying opposite vertices of a regular dodecahedron.)

(iii) *If a topological n-complex is geometrically embeddable in some Euclidean space, then it is necessarily geometrically embeddable in* E^{2n+1}.

(iv) *Complexes exist which are geometrically embeddable in* E^d, *and topologically (but not geometrically) embeddable in* E^n *for some* $n < d$.

A number of questions on embeddability are suggested by Theorem 8, of which we mention only one:[(28a)] *If a simplicial k-complex is topologically embeddable in* E^{2k}, *is it necessarily geometrically embeddable in* E^{2k}?

The relevance of the above considerations to polytopes is as follows. Naturally associated with each d-polytope P is a family of $d+1$ geometric complexes, namely for $0 \leqslant k \leqslant d$, the set of all j-faces of P with $j \leqslant k$. The k-complex so formed is called the k-skeleton of P and is denoted by $\text{skel}_k P$. The 1-skeleton of P is called its *graph* and the $(d-1)$-skeleton of P is called its *boundary complex.* A k-complex is called *d-polytopal* if it is isomorphic to the k-skeleton of some d-polytope.

Although attempts to characterise polytopal complexes have proved just as fruitless as the attempts to characterise face-lattices of polytopes (the problems are clearly closely related), investigations have produced a number of interesting results which are partial in the sense that they provide necessary (but not sufficient) conditions for a given complex to be polytopal. Most of these concern graphs. We state them in a theorem, following which is an explanation of the terms used.

THEOREM 9. (i) *The graph of a d-polytope is d-connected.*[(29)]

(ii) *The graph of a d-polytope contains a subgraph which is a refinement of the graph of a d-simplex.*[(30)]

(iii) *Given any* $m = [\frac{1}{3}(d+1)]$ *pairs* (a_i, b_i) $(i = 1, \ldots, m)$ *of vertices of a d-polytope P, there exist, in the graph of P, m disjoint edge-paths, the i-th path joining* a_i *to* b_i.[(31)]

The term *edge-path* is self-explanatory. A graph is d-connected if every pair of its vertices (nodes) can be joined by at least d disjoint edge-paths. Equivalently,[(32)] it is d-connected if it has at least $d+1$ nodes and if every graph formed by erasing $d-1$ or fewer nodes is connected. A graph $\mathscr{G}_1$ is a *refinement* of a graph $\mathscr{G}_2$ if there exists a homeomorphism $\phi : |\mathscr{G}_1| \to |\mathscr{G}_2|$ such that for each cell (edge or node) $C \in \mathscr{G}_2$, $\phi^{-1}(C) = |\mathscr{G}'|$ for some subcomplex $\mathscr{G}'$ of $\mathscr{G}_1$. (Here, as usual, $|\mathscr{G}|$ means the set of all points belonging to the cells of $\mathscr{G}$.)

Various extensions of Theorem 8 are known. For example (i) and (ii) have been generalised[(33)] to skeletons of dimension $d > 1$. It has been conjectured[(33a)] that in (iii) the result remains true if $m = [\frac{1}{2}d]$; it clearly cannot hold for any larger value of m.

Theorem 8 (i) implies that the graph of any polytope can be embedded in E^3. This leads naturally to consideration of the following problem: *What is the minimal dimension of a Euclidean space in which the k-skeleton of a d-polytope may be geometrically (topologically) embedded?* The answer to this question is given completely by the following:[34]

THEOREM 10. *Let P be a d-polytope. Then the smallest value of n such that* $\text{skel}_k P$ *is geometrically embeddable in* E^n *is equal to the smallest value of n such that* $\text{skel}_k P$ *is topologically embeddable in* E^n, *and*

$$n = \begin{cases} d & \text{if } k \leqslant d \leqslant k+1, \\ d-1 & \text{if } k+2 \leqslant d \leqslant 2k+2, \\ 2k+1 & \text{if } 2k+2 \leqslant d. \end{cases}$$

The problem of characterising the skeletons of polytopes is complicated by the phenomenon of *dimensional ambiguity*. This means that it is possible for $\text{skel}_k P_1$ to be isomorphic to $\text{skel}_k P_2$, even when P_1 and P_2 are polytopes of different dimensions. The simplest example of this is given by the isomorphism between the graph of the 5-simplex and the graph of the cyclic polytope $C(6, 4)$ (defined in §2.2). Dimensional ambiguity cannot arise[35] if $k \geqslant [\frac{1}{2}d]$, but it is necessary to take $k = d-2$ for the isomorphism of $\text{skel}_k P_1$ and $\text{skel}_k P_2$ to imply the combinatorial equivalence[36] of P_1 and P_2.

Theorem 10 implies that the boundary complex $\text{skel}_{d-1} P$ of a d-polytope P can be embedded in E^d but in no space of lower dimension. However, if we delete from $\text{skel}_{d-1} P$ any one open $(d-1)$-cell (corresponding to a facet ${F_1}^{d-1}$ of P), the resulting $(d-1)$-complex is geometrically embeddable in E^{d-1}. This embedding may be carried out in the following way. Choose any point $x \in E^d$ which is on the same side of $\text{aff}\, {F_i}^{d-1}$ as P for each facet ${F_i}^{d-1}$ of P with $i \neq 1$, and on the opposite side of $\text{aff}\, {F_1}^{d-1}$ from P. Then projecting $\text{bd}\, P \setminus {F_1}^{d-1}$ by rays through x on to $\text{aff}\, {F_1}^{d-1}$ yields the required complex $\mathscr{C}$. Notice that $|\mathscr{C}| = {F_1}^{d-1}$. This particular complex is known as a *Schlegel diagram*, and occurs frequently in the classical literature in connection with problems on polytopes. Unfortunately, necessary and sufficient conditions for a $(d-1)$-complex to be the Schlegel diagram of some polytope are not known, a fact that has led to a number of errors in the past.[37]

2.4. *3-polytopes*

We know a great deal more about 3-polytopes than about polytopes of higher dimension. Even so, many questions of an elementary nature remain unanswered.[38]

Continuing the treatment of §2.3, we shall be concerned mainly with the graphs of 3-polytopes. The fundamental result, providing a converse to Theorem 9 (i), is the following:

THEOREM 11 (Steinitz' Theorem). *A graph is 3-polytopal if and only if it is planar and 3-connected.*

This theorem has an interesting history, including the publication of a number of incomplete proofs. Its importance lies in the fact that it enables us to deduce properties of 3-polytopes from graph-theoretic results, and vice-versa. It is a curious fact[39] that several authors have used this technique without even noticing that Theorem 11, or something equivalent to it, is necessary for the method to be valid. Several proofs of Theorem 11 are known,[40] none of which is easy, and except for the partial result of Theorem 9 (i), no extension to higher dimensions is known.

Theorem 11 has several immediate consequences. It shows that every graph $\mathscr{G}$ that can be topologically embedded in the plane can also be geometrically embedded, that is, with straight edges. (If $\mathscr{G}$ is 3-connected, then it is isomorphic to $\text{skel}_1 P$ for some 3-polytope P, and so to the 1-dimensional skeleton of a Schlegel diagram of P, see §2.3. This is a geometric complex.) The same argument also shows that it can be embedded in such a way that each " country " (region of the plane bounded by edges) is a convex polygon. From Theorem 11 we can also deduce that the isomorphism of the graphs of two 3-polytopes is sufficient to ensure their combinatorial equivalence.

Theorem 11 may be proved by induction on the number of edges in the graph $\mathscr{G}$. Roughly speaking, we can define a number of elementary transformations that reduce $\mathscr{G}$ to a graph with fewer edges, whilst preserving the property that $\mathscr{G}$ is 3-polytopal. Eventually we can reduce the graph to one with 6 edges, and since it is 3-connected, it is the graph of the 3-simplex and so is 3-polytopal. The same technique has been applied to the proof of several other properties of 3-polytopes, such as the following, the second of which is metrical.

THEOREM 12. *Every 3-polytope is combinatorially equivalent to one whose vertices have rational co-ordinates in a given Cartesian co-ordinate system.*[41]

THEOREM 13. *If one face of a 3-polytope is an n-gon, then there exists a polytope P' combinatorially equivalent to P, of which the corresponding face is any arbitrarily prescribed n-gon.*[42]

These results are interesting in that the analogues in higher dimensions do not hold (see §2.5).

The next few results can be formulated either in terms of graphs or in terms of 3-polytopes. We prefer the latter.

An *edge-path* in a polytope (corresponding to an edge-path in its graph) is said to be *simple* if it contains no vertex more than once. If the final vertex of an edge-path coincides with the first vertex, then it is called an edge-circuit, and an edge-circuit is simple if it visits no vertex more than once. A simple edge-path or edge-circuit on P is called Hamiltonian[43] if it passes through every vertex of P. It is easy to show that not every 3-polytope has a Hamiltonian circuit, or even a Hamiltonian path. An example of a 3-polytope with these properties is illustrated in Figure 1. It consists of a regular octahedron with a low triangular pyramid erected on each of its eight triangular 2-faces. The fourteen vertices are of two types: six

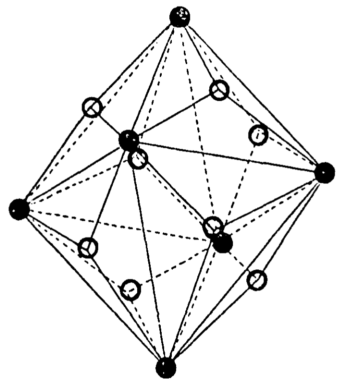

Fig. 1

0-stage vertices which belonged to the original octahedron (shown as ●) and eight 1-stage vertices (shown as o) which are the apexes of the pyramids on the faces of the octahedron. It is apparent that if we list the vertices of P in the order in which an edge-path visits them, then no two 1-stage vertices can occur consecutively in the list (for no two vertices of this type are joined by an edge). Since there are only six 0-stage vertices available, not more than 7 1-stage vertices can occur in any simple edge-path, and not more than 6 1-stage vertices can occur in any simple edge-circuit. We deduce that this polytope possesses no Hamiltonian path or circuit.[44]

In 1880 Tait[45] conjectured that every simple 3-polytope possesses a Hamiltonian circuit. This was proved false in 1946 by W. R. Tutte. If Tait's conjecture had been true, then a very simple proof of the famous 4-colour theorem would have resulted. We now have the following powerful result.[46]

THEOREM 14. *There exist constants $\alpha < 1$ and c such that for each positive integer $n \geqslant 4$ there exists a simple 3-polytope P_n with n vertices such that the longest simple edge-path on P_n contains less than $[cn^{\alpha}]$ vertices of P_n.*

The best values of α and c, known at present,[47] are $\alpha = 1-2^{-11}$ and $c = 3/2$. Another result of interest[48] is:

THEOREM 15. *Every 4-connected planar graph admits a Hamiltonian circuit.*

On the other hand there are many unsolved problems, such as the following conjecture of Barnette:

CONJECTURE. *If every 2-face of a simple 3-polytope is a polygon with an even number of edges, then P admits a Hamiltonian circuit.*

The corresponding result with "even" replaced by "divisible by 3" is known to be false.

Extensions of these results to $d \geqslant 4$ dimensions are of two types. Firstly we have the extensive investigations of Klee into edge-paths on d-polytopes, of which an excellent account already exists.[49] Secondly we have the idea of Hamiltonian $(d-2)$-manifolds in a d-polytope. This seems to be a completely unexplored field except for a single application in the case $d = 4$ by Barnette.[50]

The remaining properties of 3-polytopes to be described in this section are of a numerical nature. Write $p_k(P)$ (or p_k if there can be no confusion) for the number of k-gonal faces of a 3-polytope P ($k \geqslant 3$). Then it is easy to prove from Theorem 1 (Euler's Theorem) that

$$\sum_{k \geqslant 3} (6-k)p_k \geqslant 12. \tag{1}$$

If the polytope is simple then

$$\sum_{k \geqslant 3} (6-k)p_k = 12, \tag{2}$$

and if P is 4-valent, that is, four edges meet at each of its vertices, then

$$\sum_{k \geqslant 3} (4-k)p_k = 8. \tag{3}$$

We shall say that the finite sequence $(p_3(P), p_4(P), \ldots)$ is *associated* with the polytope P. Consider the following problem: *Given any finite sequence* $(p_3, p_4, \ldots)$ *of non-negative integers, find necessary and sufficient conditions for it to be associated with some* 3-*polytope* P, that is $p_k(P) = p_k$ for all $k \geqslant 3$.

Relations (1), (2) and (3) give necessary conditions in the case of 3-polytopes, simple 3-polytopes and 4-valent 3-polytopes respectively, but it is clear that they are not sufficient. One reason is that p_6 is absent from (1) and (2) (and p_4 from (3)) yet the value of p_6 (or p_4) is of importance.

Sufficient conditions are known in a number of special cases. To begin with, let us suppose that p_k is zero for all except one particular value of k. Then (1) implies that $k = 3$, 4 or 5 and we have the following results of Hawkins, Hill, Reeve and Tyrrell:[50a] *If* $p_i = 0$ *for* $i \neq 3$, *then* 3-*polytopes exist for which* p_3 *is any even integer satisfying* $p_3 \geqslant 4$. *If* $p_i = 0$ *for* $i \neq 4$, *then* 3-*polytopes exist for which* $p_4 = 6$ *or is any integer satisfying* $p_4 \geqslant 8$. *If* $p_i = 0$ *for* $i \neq 5$, *then* 3-*polytopes exist for which* $p_5 = 12$ *or is any even integer satisfying* $p_5 \geqslant 16$.

If p_k is non-zero for exactly two values of k, then some similar results are known. For example,[51] $(p_3, p_4, p_5, p_6) = (4, 0, 0, r)$ is associated with a 3-polytope if and only if r is a non-negative even integer, and $(p_3, p_4, p_5, p_6) = (0, 0, 12, r)$ is associated with a 3-polytope if and only if $r = 0$ or $r \geqslant 2$. Other results of this type [51a] apply only to simple or 4-valent polytopes.

More generally, we have the following theorem:[52]

THEOREM 16. (i) (Eberhard's Theorem). *If* $p_3, p_4, p_5, p_7, p_8, \ldots$ *is a finite set of non-negative integers which satisfies* (2), *then there exists a value of* p_6 *such that the sequence* $(p_3, p_4, p_5, p_6, p_7, \ldots)$ *is associated with a simple* 3-*polytope.*

(ii) *If* $p_3, p_5, p_6, p_7, \ldots$ *is a finite set of non-negative integers which satisfies* (3), *then there exists a value of* p_4 *such that the sequence* $(p_3, p_4, p_5, p_6, \ldots)$ *is associated with some* 4-*valent* 3-*polytope.*

The proofs of both parts of the theorem are similar, and that of (i) is far from easy. In neither case does the proof yield any information about the permissible values of p_6 (in (i)) or p_4 (in (ii)), for they depend on constructions in which immense numbers of hexagon (or quadrilaterals) are introduced. Nothing is known about the minimum permissible value of p_6 in the case of simple polytopes, except when $p_3 = p_4 = 0$, that is to say, when there are no triangular or quadrilateral 2-faces:[53]

THEOREM 17. *If* $p_5, p_7, p_8, \ldots$ *are non-negative integers satisfying* (2) *with* $p_3 = p_4 = 0$ and $p_6 \geqslant 8$, *then* $(p_5, p_6, p_7, \ldots)$ *is associated with a simple* 3*-polytope.*

A similar result cannot hold without the condition $p_3 = p_4 = 0$ since sets of integers $p_3, p_4, p_5, p_7, p_8, \ldots$ are known for which the permissible values of p_6 omit infinitely many integer values.[54] A very recent result, which gives a lower bound for p_6, is the following:[55]

THEOREM 18. *If P is a simple* 3*-polytope and* $\sum_{k \geqslant 6} p_k(P) \geqslant 3$, *then*

$$p_6(P) \geqslant 8 - \sum_{k=3}^{5} p_k(P) + \tfrac{1}{2} \sum_{k \geqslant 7} (k-8) p_k(P).$$

By duality, exactly equivalent problems can be formulated for the valencies of the vertices of a 3-polytope. Other relations are known connecting both the numbers $v_k(P)$ of vertices of P with valency k, and the number $p_k(P)$. For example,

$$\sum_{k \geqslant 3} (6-k) p_k(P) + 2 \sum_{k \geqslant 3} (3-k) v_k(P) = 12, \tag{4}$$

and

$$\sum_{k \geqslant 3} (4-k)\big(p_k(P) + v_k(P)\big) = 8. \tag{5}$$

Each such equation leads to problems analogous to those following Theorem 16, most of which are at present unsolved.[56] Other generalisations of these results to non-polytopal graphs are known, but to discuss these would lead us too far from the main topic of this paper.[57]

2.5. *Gale diagrams and c.s. diagrams*[58]

Some of the more important recent advances in the theory of convex polytopes have used the idea of representing a polytope by a "diagram". From a diagram of P we can read off, in a simple manner, those subsets of vert P which are vertices of faces of P, and in this way the whole combinatorial structure of P becomes apparent. Further, if the number of vertices of P is not too large, then the dimension of the diagram is small compared with that of P.

The idea originated in 1956 in a paper by David Gale,[59] though its importance was only realised as recently as 1966 by M. A. Perles. In 1968 another form of diagram, called a c.s. diagram, was introduced. Here we shall describe the construction of both types of diagram, and explain some of their applications.

First let us consider Gale diagrams, as they are now called. If P is a d-polytope with n vertices, then a Gale diagram of P consists of a set of n points in E^{n-d-1} in one-to-one correspondence with the vertices of P. We first describe how to construct a *Gale transform.* As before, write vert P for the set of vertices of P, and let vert $P = \{x_1, \ldots, x_n\}$. Consider the set $D(\text{vert } P)$ of affine dependences of vert P, that is, the set of all vectors $(\lambda_1, \ldots, \lambda_n) \in E^n$ such that

$$\left.\begin{aligned} \lambda_1 x_1 + \ldots + \lambda_n x_n = o, \\ \lambda_1 + \ldots + \lambda_n = 0. \end{aligned}\right\} \tag{1}$$

It is trivial to verify that $D(\text{vert } P)$ is a vector space of dimension $n-d-1$, and we choose a basis for it, say $\{a_1, \ldots, a_{n-d-1}\}$. Write $a_j = (\alpha_{j1}, \ldots, \alpha_{jn})$ for

$$j = 1, \ldots, n-d-1$$

and let

$$A(\text{vert } P) = \begin{pmatrix} a_1 \\ \vdots \\ a_{n-d-1} \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \vdots & \vdots \\ \alpha_{n-d-1,1} & \cdots & \alpha_{n-d-1,n} \end{pmatrix}$$

be the $(n-d-1) \times n$ matrix with rows $a_1, \ldots, a_{n-d-1}$. Then for each $i = 1, \ldots, n$ let

$$\bar{x}_i = (\alpha_{1i}, \ldots, \alpha_{n-d-1,i}) \in E^{n-d-1}$$

be the i-th column of $A(\text{vert } P)$. The set $\bar{V} = \{\bar{x}_1, \ldots, \bar{x}_n\} \in E^{n-d-1}$ is called a Gale transform of $V = \text{vert } P$. Notice that in $\bar{V}$ several points may coincide, even though V consists of distinct points. In the transform we must therefore label such points with their multiplicities.

There is clearly much arbitrariness in the construction of a Gale transform: for example there is freedom of choice of the basis of $D(\text{vert } P)$. To see how $\bar{V}$ determines the combinatorial properties of $P = \text{conv } V$, it is convenient to make the following definition. A subset $Z \subseteq V$ is called a *coface* of P if $\text{conv}(V \setminus Z)$ is a face of P. Also, for any subset $Z \subseteq V$ we write $\bar{Z} \subseteq \bar{V}$ for the set of transforms $\bar{x}_i$ of all the vertices x_i of P in Z.

THEOREM 19. *For any d-polytope P, with* $V = \text{vert } P$, *a subset* $Z \subseteq V$ *is a coface of P if and only if, in a Gale transform* $\bar{V}$ *of* V,

$$o \in \text{relint conv } \bar{Z}.$$

(o is the origin and "relint" means "relative interior of".)

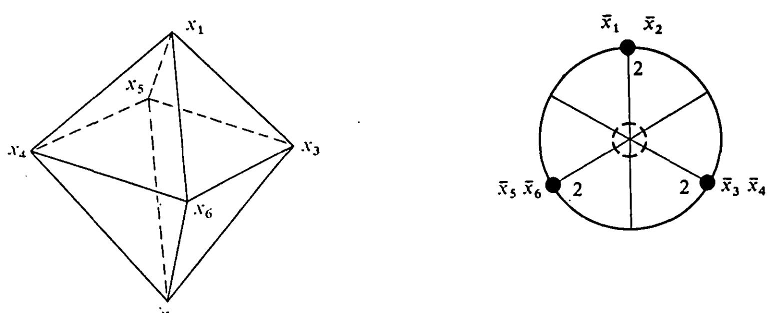

Fig. 2

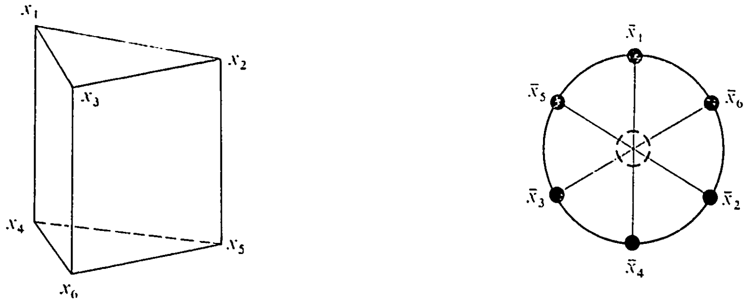

Fig. 3

Given a Gale transform of vert P, Theorem 19 enables us to determine by inspection, those subsets of vert P that are vertices of a face of P. As examples, in Figures 2 and 3 we reproduce Gale transforms of two familiar 3-polytopes, the regular octahedron and the triangular prism. In each case we have labelled the vertices of P and the corresponding points of the 2-dimensional Gale transform of P.

For any Gale transform $\bar{V}$ of V, it is clear from the construction that o is the centroid of $\bar{V}$. Further, since each vertex of P is a face of P, by Theorem 19, $o \in \operatorname{relint} \operatorname{conv} \bar{W}$ where $W \subset \operatorname{vert} P$ is any subset formed by deleting exactly one vertex of P. Equivalently, in any Gale transform $\bar{V}$ of V, at least two points of $\bar{V}$ lie in every open half-space bounded by a hyperplane through o. If both these conditions hold for an arbitrary set of n points in E^{n-d-1}, then it can be shown that this set is a Gale transform of vert P where P is some d-polytope with n vertices. In fact, in a certain sense, the relationship between V and $\bar{V}$ is symmetrical.

Intuitively the algebraic construction of $\bar{V}$ described above is difficult to comprehend. Recently a geometrical construction for Gale transforms has been described. In E^{n-1} let L_d and L_{n-d-1} be orthogonal affine subspaces of dimensions d and $n-d-1$, and write π_d and π_{n-d-1} for orthogonal projections on to L_d and L_{n-d-1} respectively. Let $T^{n-1} \subset E^{n-1}$ be a regular $(n-1)$-simplex, that is one whose edges have equal length, with centroid at $o = L_d \cap L_{n-d-1}$. Then

$$\pi_{n-d-1} (\operatorname{vert} T^{n-1}) \subset L_{n-d-1}$$

(with $\pi_{n-d-1} L_d$ as origin) is a Gale transform of the set of points $\pi_d (\operatorname{vert} T^{n-1}) \subset L_d$. The one-to-one correspondence between these two sets of points arises in the obvious way: If vert $T^{n-1} = \{y_1, \ldots, y_n\}$ and $x_i = \pi_d y_i$, then $\bar{x}_i = \pi_{n-d-1} y_i$. Since it can be shown that every d-polytope with n vertices is affinely equivalent to the image of T^{n-1} under orthogonal projection on to some suitably chosen d-dimensional subspace L_d, the above construction defines a Gale transform of the set of vertices of any polytope, at least within an affinity. The geometrical construction makes apparent the symmetry between V and $\bar{V}$ mentioned above.

We now generalise the idea of a Gale transform. Let $Y = \{y_1, \ldots, y_n\}$ and $Z = \{z_1, \ldots, z_n\}$ be two sets of n points in some Euclidean space. Then we say that

Y and Z are *isomorphic* if, for every subset $\{i_1, ..., i_j\}$ of $\{1, ..., n\}$ $(1 \leqslant j \leqslant n)$ either both the relations

$$o \in \operatorname{relint} \operatorname{conv} \{y_{i_1}, ..., y_{i_j}\}$$

and

$$o \in \operatorname{relint} \operatorname{conv} \{z_{i_1}, ..., z_{i_j}\}$$

hold, or neither does. Any set of n points in E^{n-d-1} which is isomorphic to a Gale transform of vert P is called a *Gale diagram*(60) of P. Clearly every Gale diagram of a polytope P determines the combinatorial type of P, and, in fact, Gale diagrams are much more useful than Gale transforms. The reason for this is that it is possible to find a Gale diagram for a polytope in canonical form. We shall illustrate this statement in the case where $n = d+3$, so that each Gale transform and Gale diagram of P is 2-dimensional. Here there are two possible canonical forms, both of which are essentially unique.

To begin with, if $\bar{x} \in \bar{V}$ and $\bar{x} \neq 0$, we may replace $\bar{x}$ by any point on the same ray from the origin (that is, on the open half-line containing $\bar{x}$ with o as end-point). Hence there is no loss of generality in restricting attention to Gale diagrams in which every point lies either at o, or on a unit circle S^1 centred at o. Secondly, if points of the diagram lie only at corresponding ends of "adjacent" diameters of S^1, then they may be moved into coincidence, or split into more such diameters, without altering the isomorphism type of the diagram (see Figure 4). Finally we may also assume, for convenience, that all the diameters of S^1 containing points of the diagram are at equal angles to their neighbours. In this way we obtain the *contracted standard Gale diagram* of P, which contains points on the minimum possible number of diameters, and the *distended standard Gale diagram* which contains points on the maximum possible number of diameters. These are the two canonical forms mentioned above. In Figure 4 the contracted and distended forms of the given Gale diagram are shown. In Figure 3, the Gale transform illustrated is both a contracted and distended Gale diagram; in Figure 2, the Gale transform is a contracted Gale diagram and the corresponding distended diagram is illustrated in Figure 5.

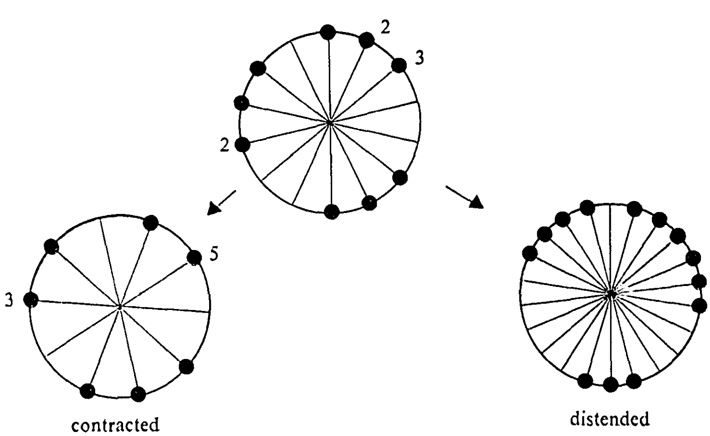

Fig. 4

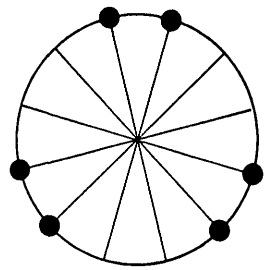

Fig. 5

We now summarise some of the properties of Gale diagrams:

THEOREM 20. (i) *A set* $\bar{V}$ *of n points in* E^{n-d-1} *is a Gale diagram of a d-polytope P with n vertices if and only if every open half-space in* E^{n-d-1} *bounded by a hyperplane through o contains at least two points of* $\bar{V}$ *(or, alternatively, all the points of* $\bar{V}$ *coincide with o and then P is a simplex).*

(ii) *If* F^{d-1} *is a facet of P, and Z is the corresponding coface, then in any Gale diagram* $\bar{V}$ *of P,* $\bar{Z}$ *is the set of vertices of a (non-degenerate) simplex with o in its relative interior.*

(iii) *A polytope P is simplicial if and only if, for every hyperplane H containing* $o \in E^{n-d-1}$,

$$o \notin \operatorname{relint} \operatorname{conv} (\bar{V} \cap H).$$

(iv) *A polytope P is a pyramid if and only if at least one point of* $\bar{V}$ *coincides with the origin* $o \in E^{n-d-1}$.

Numerous other results are known, for which the reader is referred to the original publications. For example, it is possible to read off, from a Gale diagram of *P*, Gale diagrams of the various faces of *P*, the vertex figures of *P*, and so on. It is even possible to determine the symmetries of *P* and certain other metrical properties, see §3.4.

The applications of Gale diagrams will be illustrated by four examples. All of these, except II, lead to results which are inaccessible by other known methods.

I. Bearing in mind Theorem 20 (i) and (iii) we see that every simplicial *d*-polytope corresponds to exactly one of the contracted standard Gale diagrams of the sequence illustrated in Figure 6. The permissible multiplicities of the various points are indicated.

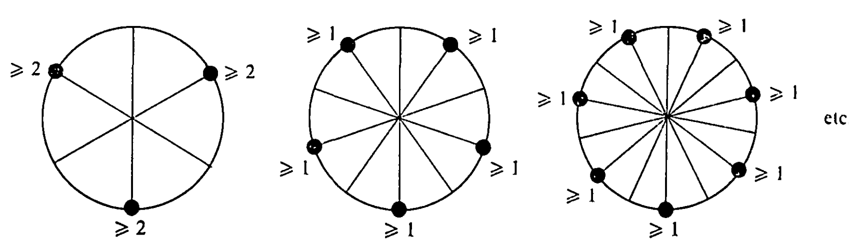

Fig. 6

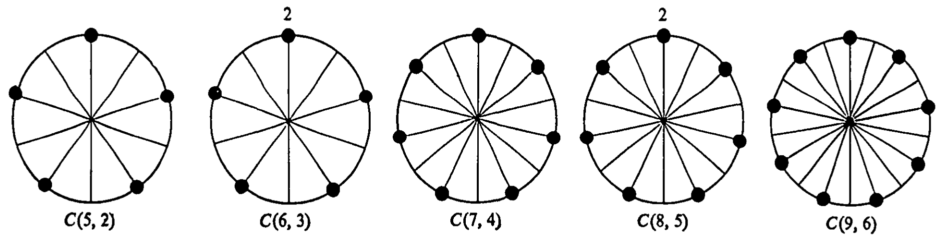

Fig. 7

It is possible to enumerate the number of diagrams with a given number of points using Polya's Theorem,[61] and hence we arrive at Perles' formula[62] for $c_s(d+3, d)$ quoted in §2.1. This result is the only major advance in the enumeration problem for convex polytopes in over half a century. It seems possible[62a] that similar considerations may lead to a determination of $c(d+3, d)$.

II. By considering Gale diagrams of the cyclic polytopes $C(d+3, d)$ (see §2.2) it is easy to establish[63] the Upper Bound Conjecture for d-polytopes with $d+3$ vertices. The Gale diagrams are illustrated in Figure 7.

III. Certain configurations in the projective plane cannot be rationally embedded. To find such a configuration C we need only choose a set of lines and points in such a way that the incidences imply that one set of four points in C has irrational cross-ratio. Then it is impossible for the co-ordinates of all the points of C to have rational co-ordinates. For any such configuration one can construct a set of points in E^3 satisfying the conditions of Theorem 18 (i) that cannot be rationally embedded. This is a Gale diagram of a d-polytope P that cannot be rationally embedded in E^d. The "smallest" example known of a polytope with this property is an 8-polytope with 12 vertices;[64] in this way we have established the surprising fact that the property of 3-polytopes stated in Theorem 12 does not hold for d-polytopes[65] with $d \geqslant 8$. Whether or not this property holds also for $d = 4$, 5, 6 or 7 is an open question.

IV. It can be shown by means of Gale diagrams that the 8-polytope mentioned in III has the property that one cannot prescribe the shape of one of its facets, thus showing that the analogue of Theorem 13 does not hold in $d \geqslant 8$ dimensions.[66]

These four applications illustrate the usefulness of Gale diagrams in solving combinatorial problems, especially when the number of vertices n is not much bigger than d. If $n > 2d$ then the dimension of a Gale diagram is larger than that of the polytope, and so there is not much advantage in using it. Since every centrally-symmetric d-polytope has at least $2d$ vertices, it will be seen that Gale diagrams are virtually useless in this case. For this reason c.s. diagrams were introduced. As in the case of Gale diagrams, these can be formulated in both an algebraic and a geometric manner.

Let P be a centrally-symmetric d-polytope with $2n$ vertices. Taking the centre of P as origin, we may write vert $P = X = \{\pm x_1, \ldots, \pm x_n\}$. Write $X_+ = \{x_1, \ldots, x_n\}$

and let $L(X_+)$ be the set of all linear dependences of X_+, that is, the set of all vectors $(\lambda_1, \ldots, \lambda_n) \in E^n$ such that

$$\lambda_1 x_1 + \ldots + \lambda_n x_n = o.$$

It is clear that $L(X_+)$ is a vector space of dimension $n-d$. Let $a_j = (\alpha_{j1}, \ldots, \alpha_{jn})$ for $j = 1, \ldots, n-d$ be a basis of $L(X_+)$ and let $B(X_+)$ be the $(n-d) \times n$ matrix whose rows are $a_1, \ldots, a_{n-d}$. If we write $\bar{x}_i = (\alpha_{1i}, \ldots, \alpha_{n-d,\, i})$ for the i-th column of $B(X_+)$ then the set of points $\{\pm\bar{x}_1, \ldots, \pm\bar{x}_n\} \in E^{n-d}$ is called a c.s. transform of X. The result corresponding to Theorem 17 is as follows:

THEOREM 21. *Let $\varepsilon_1 x_{i_1}, \ldots, \varepsilon_r x_{i_r}$ $(\varepsilon_s = \pm)$ be any r points of X with distinct suffixes $i_1, \ldots, i_r$. Then*

$$\operatorname{conv}\{\varepsilon_1 x_{i_1}, \ldots, \varepsilon_r x_{i_r}\}$$

is a face of $P = \operatorname{conv} X$ if and only if

$$\varepsilon_1 \bar{x}_{i_1}, + \ldots + \varepsilon_r \bar{x}_{i_r} \in \operatorname{relint} \operatorname{conv}\{(\pm\bar{x}_{j_1} \pm \ldots \pm \bar{x}_{j_{n-r}})\} \qquad (1)$$

where $\{j_1, \ldots, j_{n-r}\} = \{1, \ldots, n\} \setminus \{i_1, \ldots, i_r\}$, and the expression on the right of (1) *is to be interpreted as o if $n = r$.*

The geometrical construction for c.s. transforms is analogous to that described above for Gale transforms—the only difference is that we use orthogonal projections of the regular crosspolytope X^n instead of projections of the regular simplex T^{n-1}. (X^n, which is a dual of the n-cube, is defined by

$$X^n = \operatorname{conv}\{\pm e_1, \ldots, \pm e_n\}$$

where $e_1, \ldots, e_n$ are mutually orthogonal unit vectors.)

Isomorphic c.s. transforms are defined in a manner analogous to isomorphic Gale diagrams (using Theorem 21 instead of Theorem 19) and, similarly, c.s. diagrams are defined. Due to the comparatively complicated statement of Theorem 21 (as opposed to Theorem 19), the use of c.s. diagrams does not immediately lead to the solution of any enumeration problems. In fact, the simplest nontrivial case, that of determining the number of combinatorial types of centrally-symmetric d-polytopes with $2(d+1)$ vertices, is still unsolved. On the other hand, c.s. diagrams provided the method of establishing the conjecture at the end of §2.2 on the neighbourliness properties of centrally-symmetric polytopes in the case $n = 1$ and $n = 2$. They have also been applied to the study of polytopes with an axis of symmetry, but as these properties are metrical, their consideration will be postponed to §3.4.

3. *Metrical Properties of Polytopes*

3.1. *General remarks*

The most famous problem in the metrical theory of convex sets is the classical isoperimetric problem; one asks for the convex set of given volume that has the smallest surface area. Consideration of this problem led Minkowski and other mathematicians into a detailed investigation of the volumes and mixed volumes of convex sets. We shall not describe these here; for one thing, polytopes only play an auxiliary part, and for another, excellent accounts already exist.[67] If $K_1, \ldots, K_n$ are n convex sets in E^d, and $\lambda_1, \ldots, \lambda_n$ are real numbers, then the linear combination $\lambda_1 K_1 + \ldots + \lambda_n K_n$ is defined to be the convex set

$$\{\lambda_1 x_1 + \ldots + \lambda_n x_n : x_i \in K_1, \quad i = 1, \ldots, n\}.$$

If we write $v_d(K)$ for the volume of a convex set K in E^d in the Peano–Jordan sense (it is easy to show that this always exists) then it can be proved that, for $\lambda_i \geqslant 0$ $(i = 1, \ldots, n)$.

$$v_d(\lambda_1 K_1 + \ldots + \lambda_n K_n) = \Sigma \lambda_{i_1} \lambda_{i_2} \ldots \lambda_{i_d} v(K_{i_1}, \ldots, K_{i_d}),$$

where summation on the right is over all suffixes $1 \leqslant i_j \leqslant n$, $1 \leqslant j \leqslant d$. In other words, $v_d(\lambda_1 K_1 + \ldots + \lambda_n K_n)$ is a homogeneous polynomial of degree d in the λ_i. If we arrange that the coefficients $v(K_{i_1}, \ldots, K_{i_d})$ are invariant under permutation of the K_{i_j}, then these coefficients are called the *mixed volumes* of the given sets. Many properties of mixed volumes are known, though there are still a number of unsolved problems.[68] The above definition of mixed volumes will be required in §3.3.

Although polytopes play no part in the classical isoperimetric problem or its solution, there are a number of similar problems to which they are relevant.[68a] For example, we may ask which 3-polytope of given volume has the shortest possible total edge-length. Or, we may ask which packable 3-polytope of given volume has the smallest possible surface area. Here *packable* means that E^3 can be covered by translates of the given polytope in such a way that these translates have intersections of zero volume. Solutions to these problems, and many more of a similar nature, are unknown.[69]

The idea of packable polytopes leads us immediately to the mention of their relevance to the study of point sets in E^d. If, for example, p is one point of a lattice of points in E^d, then the set of points x such that $\|x-p\| \leqslant \|x-p'\|$ for all lattice points p' distinct from p, is a convex polytope traditionally known as the *Voronoi polyhedron* for the lattice. The relevance of these ideas both to the geometry of numbers, and also to the difficult problems of finding closest packings of convex sets will be apparent, and we refer the reader to the surveys of this topic that already exist.[70]

We must also mention the many problems that have been considered concerning the relationship between polytopes and spheres, quite apart, that is, from those that arise in connection with sphere packings. For example there is Steiner's problem as to whether there is a 3-polytope of every combinatorial type which is *inscribable* in a sphere (that is, such that all its vertices belong to a sphere). The answer is known to be negative, in fact, a counter-example is provided by the 3-cube truncated at one vertex (so that it has one triangular, three quadrilateral, and three pentagonal 2-faces).[71] Every non-inscribable 3-polytope is known to have at least 7 2-faces,[72] and the dual combinatorial type is *non-circumscribable*, that is, no polytope of this type has 2-faces which all touch a given sphere. A number of extremal problems concerning 3-polytopes circumscribed to spheres (or whose edges touch spheres) have been solved during the last few years.[(73)]

It is well known that the set $\mathscr{K}^d$ of all closed bounded convex sets in E^d forms a metrisable topological space. Several metrics have been defined,[(74)] the most commonly used being the Hausdorff metric ρ for which

$$\rho(K_1, K_2) = \max\{\sup_{x_2 \in K_2} \inf_{x_1 \in K_1} \|x_1 - x_2\|, \sup_{x_1 \in K_1} \inf_{x_2 \in K_2} \|x_2 - x_1\|\}$$

where $K_1, K_2 \in \mathscr{K}^d$. Alternatively, if, for any vector $u \in E^d$ and $K \in \mathscr{K}^d$, we define the *supporting function h* by

$$h(K, u) = \sup_{x \in K} \langle x, u \rangle$$

then

$$\rho(K_1, K_2) = \sup |h(K_1, u) - h(K_2, u)|,$$

the supremum being taken over all unit vectors $u \in E^d$. The name supporting function arises from the fact that for any $u \neq o$ and $K \in \mathscr{K}^d$,

$$H = \{x \in E^d : \langle x, u \rangle = h(K, u)\}$$

is a supporting hyperplane of K.

With the topology induced by ρ, $\mathscr{P}^d$ is dense in $\mathscr{K}^d$, and this leads us to the idea of an *approximation problem* which is concerned essentially with discovering subsets of $\mathscr{P}^d$ (for example $\mathscr{P}_s{}^d$) which are also dense in $\mathscr{K}^d$ (or in some specified subset of $\mathscr{K}^d$). There are a few recent results in this direction,[(75)] but the following fundamental problem seems to be unsolved.[(76)] Given any d-polytope P, find necessary and sufficient conditions on the combinatorial type of a d-polytope Q (in terms of the combinatorial type of P) for P to be approximable, arbitrarily closely in the Hausdorff metric, by polytopes combinatorially equivalent to Q.

In the following three sections we have selected those metrical topics in which recent advances have been made, and which seem to be of sufficient interest to describe in more detail. Some of this interest arises from the fact that they are relevant to, or closely analogous to, combinatorial problems.

3.2. *Angles of polytopes*

At each vertex of a polygon, a (plane) angle is defined, and in the case of a 3-polytope there are two sorts of angles: solid angles at its vertices and dihedral angles at its edges. More generally, for a d-polytope P, at every proper face F of P, an angle $\phi(P, F)$ may be defined as follows. Let z be a relatively interior point of F and consider a small ball $B(z, \rho)$ with centre z and radius $\rho > 0$. If ρ is so small that $B(z, \rho)$ does not meet any proper face of P other than those faces which are incident with int F, then

$$\phi(P, F) = v_d(B(z, \rho) \cap P)/v_d(B(z, \rho)).$$

In other words, it is the fraction of the volume of $B(z, \rho)$ that lies in P. The value of $\phi(P, F)$ does not depend on the choice of z. It will be noticed that $\phi(P, F)$, as defined, is an absolute measure of angle, being a real number in the interval [0, 1]. This turns out to be more convenient than the more usual angle measures such as radians.

A classical theorem is the following:

THEOREM 22 (The Gram-Euler Theorem). *For any d-polytope P,*

$$\sum_{j=0}^{d-1} (-1)^j \phi_j(P) = (-1)^{d-1}, \tag{1}$$

where $\phi_j(P) = \Sigma\phi(P, F^j)$, *the sum being over all the j-faces* F^j *of* P.

Several proofs of this theorem are known.[77] The quantity $\phi_j(P)$ is known as the j-th *angle-sum* of P, and it is convenient to define the *angle-sum vector*

$$\phi(P) = (\phi_0(P), \ldots, \phi_{d-1}(P))$$

in E^d; this vector plays an important part in the theory. Theorem 22 shows that $\phi(P)$ has a similar property to $f(P)$ (see Theorem 1) namely that it lies on a certain hyperplane in E^d. Further it can be shown that no affine subspace of dimension less than $d-1$ contains the angle-sum vectors of all d-polytopes.

The approach to Theorem 22, and to similar properties of polytopes, by integral geometry depends upon the following idea. Let $z \in \operatorname{relint} F$, where F is a face of P, and consider the set of all lines through z. $\phi(P, F)$ may be regarded as a measure of the set of these lines which have non-empty intersection with $\operatorname{int} P$. By considering orthogonal projections of P on to hyperplanes, these measures can be ascertained, and so a proof of Theorem 22 follows without difficulty. In fact, equation (1) may be regarded as essentially an averaged form of Euler's Theorem applied to all the orthogonal projections of P.

An immediate generalisation arises by considering k-dimensional subspaces (instead of lines) through z. These k-dimensional subspaces correspond to points on a Grassmann manifold $G_k{}^d$ on which a measure can be defined, invariant under the transformations induced by congruence transformations of E^d. The measure of the subset of $G_k{}^d$ corresponding to those k-dimensional subspaces which intersect $\operatorname{int} P$ is called the k-th *Grassmann angle*[78] of P at F and is denoted by $\phi^{(k)}(P, F)$. The generalisation of Theorem 22 is as follows:

THEOREM 23. *For any d-polytope P, and* $1 \leqslant k \leqslant d-1$,

$$\sum_{j=0}^{d-k-1} (-1)^j \phi_j^{(k)}(P) = 1-(-1)^{d-k}$$

where

$$\phi_j^{(k)}(P) = \Sigma\phi^{(k)}(P, F^j),$$

the summation being over all the j-faces F^j *of* P.

These Grassmann angles give information about the "shape" of the polytope near each of its faces. For certain values of k they have a simple geometrical interpretation. The case $k = 1$ is clear, and for $k = 2$,

$$\delta(P, F) = 1-\phi^{(2)}(P, F)$$

is the quantity known as the *angle deficiency* of P at the face F. In fact

$$\phi^{(2)}(P, F) = \Sigma\phi(F^{d-1}, F)$$

where summation is over all the facets F^{d-1} of P that are incident with F. It can be shown,[79] that if $0 \leqslant \dim F \leqslant d-3$, then $\delta(P, F) > 0$.

For $k = d-1$ we consider the set of hyperplanes through $z \in \operatorname{relint} F$ which do not meet int P, in other words support P, and we readily verify that $\phi^{(d-1)}(P, F)$ is the quantity known as the *exterior angle* of P at F. In the case $d = 3$, therefore, the exterior angle at a vertex F^0 is equal to $1-\delta(P, F)$, a result that is equal to the well-known formula for the area of a spherical triangle in terms of its angles.

In §2.2 we showed that when the faces of a d-polytope P are of certain specified combinatorial types, then the f-vector of P satisfied a number of additional linear relations (for example, the Dehn–Sommerville equations (Theorem 2) for simplicial polytopes). Exactly analogous properties hold for Grassmann angles when the faces of P, of certain dimensions, have prescribed combinatorial types. Instead of quoting these, we shall discuss additional results in a more general context, but, for simplicity, state these for $k = 1$ (ordinary angles) only.[80]

By a *regular projection* of a d-polytope P we mean the image of P under orthogonal projection on to a hyperplane H where the direction of projection (the normal to H) is not parallel to any face of P. For regular projection, the image of each j-face of P is a j-polytope in H $(0 \leqslant j \leqslant d-1)$. The regular projections of P are clearly of a finite number, say q, of combinatorial types, and we write $P_1, \ldots, P_q$ for $(d-1)$-polytopes representative of these types. For $i = 1, \ldots, q$, let

$$f(P_i) = (f_0(P_i), \ldots, f_{d-1}(P_i), 0) \in E^d,$$

where, as usual, $f_j(P_i)$ is the number of j-faces of P_i and $f_{d-1}(P_i) = 1$.

THEOREM 24.

$$f(P)-2\phi(P) = \sum_{i=1}^{q} \mu_i f(P_i)$$

where $\Sigma\mu_i = 1$ *and* $\mu_i > 0$ *for each* i.

Equivalently,

$$\phi(P) \in \operatorname{relint} \operatorname{conv} \{\tfrac{1}{2}(f(P)-f(P_i)) : i = 1, \ldots, q\}.$$

Although very little is known about the set $\{\phi(P) : P \in \mathscr{P}^d\}$ apart from this theorem, we have the following result which shows that it is, in a sense, the best possible. Let us write $\phi(\mathscr{A}(P))$ for the set of all angle-sum vectors $\phi(Q)$ when Q runs through all polytopes which are affinely equivalent to P.

THEOREM 25.

$$\operatorname{conv} \phi(\mathscr{A}(P)) = \operatorname{relint} \operatorname{conv} \{\tfrac{1}{2}(f(P)-f(P_i)) : i = 1, \ldots, q\}.$$

Of particular interest is the case $q = 1$. (This occurs, for example, in the case of a 3-cube, all of whose regular projections are hexagons. Thus $f(P) = (8, 12, 6)$ and $f(P_1) = (6, 6, 0)$. It also occurs for zonotopes and for certain other classes of polytopes.[81]) In this case $\phi(\mathscr{A}(P))$ consists of just one vector, so that its angle-sums are invariant under affinities. (Hence all parallelepipeds in E^3, being affinely equivalent to the 3-cube, have angle sums $\phi_0(P) = 1$, $\phi_1(P) = 3$, $\phi_2(P) = 3$.)

Theorem 24 enables us to deduce many numerical properties of angle-sums. Clearly we can write down bounds for the j-th angle sum $\phi_j(P)$ in terms of the numbers $f_j(P)$ and $f_j(P_i)$ $(i = 1, \ldots, q)$. Also, we see that the components of the vector $f(P)-2\phi(P)$ satisfy any linear equation that holds for all the components of the vectors $f(P_i)$. For example, if P is quasi-simplicial (that is, all its $(d-2)$-faces are simplexes) then the P_i are simplicial and the numbers $f_j(P_i)$ satisfy the Dehn–Sommerville equations (in $d-1$ dimensions). Hence for all quasi-simplicial P, the vectors $f(P)-2\phi(P)$ lie on a $[\frac{1}{2}(d-1)]$-dimensional affine subspace. Similar considerations apply to quasi-cubical polytopes (that is, polytopes whose $(d-2)$-faces are combinatorially equivalent to cubes).

It seems possible that further relations (such as non-linear inequalities) between the angles and Grassmann angles of polytopes remain to be discovered.[81a] No such relations are known except those that follow as trivial consequences of the properties stated above.

We explained how, in the combinatorial theory of polytopes, we were concerned with the properties of combinatorial equivalence classes of polytopes rather than with polytopes themselves. Similarly, in the metrical theory, the classes of congruent polytopes are the fundamental objects of study. When we are interested in the angles of polytopes only, it is convenient to consider rather larger classes of polytopes. Extending the idea of similarity (or homotheticity) we write $P_1 \, \sigma \, P_2$ if and only if $P_1 \approx P_2$ and, if the face F_1 of P_1 corresponds to the face F_2 of P_2 in the combinatorial equivalence, then $\phi(P_1, F_1) = \phi(P_2, F_2)$. The study of these σ-classes is a comparatively unexplored field. For example we may mention the following weak version of a problem of Stoker: *If the corresponding dihedral angles of two combinatorially equivalent* 3-*polytopes* P_1, P_2 *are equal, does it necessarily follow that* $P_1 \, \sigma \, P_2$? An affirmative solution is known only in the case of simple polytopes.[82] Another result of this nature is Schneider's ingenious characterisation[83] of the polytopes P with the property that, for all vectors t, if $P \cap (P+t) \neq \emptyset$, then $(P \cap (P+t)) \, \sigma \, P$.

The above results on angle sums have been used in connection with a combinatorial problem, which we shall now describe.[84] Let P be a d-polytope. Then P is called *facet-forming* if there exists a $(d+1)$-polytope Q whose facets (d-faces) are all combinatorially equivalent to P. Otherwise P is called a *nonfacet*. In the case $d = 2$ it is obvious that triangles, quadrilaterals and pentagons are facet-forming polygons (for we may take Q to be a tetrahedron, cube, or regular dodecahedron, respectively) but, from inequality (1) of §2.4, it is impossible for all the facets of a 3-polytope to be n-gons for $n \geqslant 6$. Thus hexagons, heptagons, octagons, ..., are nonfacets. For $d > 3$ many facet-forming polytopes are known (for example, simplexes, cubes, d-polytopes with $d+2$ vertices, duals of the odd-dimensional cyclic polytopes, etc.), but it is only recently that nonfacets have been discovered. So far, the only known method of proving that a given d-polytope is a non-facet depends upon the properties of its angle-sums. The basic idea is as follows.

Let $P_1, \ldots, P_r$ be the facets of a $(d+1)$-polytope Q (so that $r = f_d(Q)$). Then $\phi_j(P_i)/f_j(P_i)$ may be called the *average angle* of P_i at its j-faces. Since every j-face of a $(d+1)$-polytope is incident with at least $d+1-j$ j-faces, using the fact that the angle deficiencies are strictly positive for $j \leqslant d-2$, we can easily deduce that

$$\sum_{i=1}^{r} \phi_j(P_i) < \frac{1}{d+1-j} \sum_{i=1}^{r} f_j(P_i)$$

for all j such that $0 \leqslant j \leqslant d-2$. Using the inequalities for angle sums implied by Theorem 24, and remembering that we are interested in the case where all the P_i are combinatorially equivalent to a given d-polytope P, we arrive at the following result:

THEOREM 26. *If P is a facet-forming d-polytope, then for each $j = 0, \ldots, d-3$,*

$$f_j(P) < \frac{d+1-j}{d-1-j} m_j(P)$$

where $m_j(P)$ is the maximum number of j-faces in any regular projection of any polytope combinatorially equivalent to P.

To discover nonfacets we therefore find polytopes which violate the above condition. For example one can show that the d-crosspolytope is a nonfacet for $d \geqslant 6$ (the cases $d = 4, 5$ are undecided), and that the 600-cell (one of the regular polytopes, see §3.4) are nonfacets. We can also deduce that for each d, the cyclic polytope $C(v, d)$ is a nonfacet if v is sufficiently large, and it has been conjectured that it is only necessary to take $v \geqslant d+3$.

The case $d = 3$ is one of the most interesting, and, perhaps the most difficult. The "smallest" 3-dimensional nonfacet so far discovered has 14 vertices, and is illustrated in Figure 1 of §2.4; it is possible that 3-dimensional nonfacets with as few as 6 vertices may exist.

To verify that the 3-polytope P with 14 vertices just mentioned is a nonfacet, we need only note that as the longest edge-path on P contains 12 edges, it necessarily follows that $m_1(P) \leqslant 12$. Substituting $d = 3$, $j = 1$, $f_1(P) = 36$, we see that the inequality of Theorem 26 does not hold, and therefore P is a nonfacet.

This example illustrates the (unexpected) connection between 3-polytopes that are nonfacets, and those whose longest edge-circuits contain comparatively few vertices. Bearing in mind the remarks of §2.4 it is not surprising that it is comparatively difficult to find simple 3-dimensional nonfacets. Using Theorem 14, along with the estimates of α and c given, one can show that there exists a simple nonfacet with about 10^{977} vertices. The fact that this number is so large illustrates the weakness of the angle-sum method just described; it is unlikely that any substantially better estimate of the size of the smallest 3-dimensional simple nonfacet will be found until more powerful techniques have been developed.

Barnette has recently discovered a simple 4-polytope which is a non-facet, using an exact analogue of the above methods, that is, by the study of the largest simple manifold which is a union of 2-faces of the polytope.[(85)]

We conclude by remarking that many of the results on angle-sums have analogues for spherical polytopes (that is polytopes on the surface of a sphere). For these we refer the reader to the original papers.[(86)]

3.3. *Analogues of Euler's Theorem*

The reader will have noticed the remarkable resemblance between the form of the equations in Theorem 1 (Euler's Theorem for the f-vectors) and Theorem 22 (for the angle-sum vectors). It is natural to enquire whether other functions that arise in connection with the study of convex sets have similar properties. Let us say that a function ψ, defined on the set $\mathscr{P}$ of all polytopes of dimension $\leqslant d$ in E^d satisfies an *Euler-type relation* if, for all d-polytopes P, and for some choice of signs on the right,

$$\sum_{j=0}^{d-1} (-1)^j \, \Sigma\psi(F^j) = \pm\psi(\pm P),$$

where the second summation on the left is over all the j-faces F^j of P. A number of such functions were discovered "by accident" but they are now known to be consequences of a fundamental identity,[(87)] which also leads to an alternative proof of Theorem 22:

THEOREM 27. *For any d-polytope P, and any vector u,*

$$\sum_{j=0}^{d} (-1)^j \, \Sigma h(F^j, u) = -h(-P, u)$$

where $h(P, u)$ is the supporting function of P at u, and the second summation on the left is over all the j-faces F^j of P.

We mention two examples. Firstly we have a relation between the mixed volumes (see §3.1) of a polytope and its faces:

THEOREM 28. *For any d-polytope P, and any convex bodies $K_{r+1}, \ldots, K_d$,*

$$\sum_{j=0}^{d} (-1)^j \, \Sigma v(F^j, \ldots, F^j, K_{r+1}, \ldots, K_d) = (-1)^r v(-P, \ldots, -P, K_{r+1}, \ldots, K_d)$$

where, in each mixed volume, the first argument is repeated r times.

In particular, taking $r = 1$ and $K_2 = \ldots = K_d = B$ (the unit ball) leads to the fact that the mean width of a polytope is a function that satisfies an Euler-type relation.

The second example[88] concerns the point now known as the *Steiner point* $s(K)$ of a convex set K. Originally introduced by Steiner in connection with an extremal problem,[89] it has, more recently,[90] been shown to have the important additivity property

$$s(K_1+K_2) = s(K_1)+s(K_2) \tag{1}$$

where, on each side + denotes vector addition.

In the case of a polytope $s(P)$ may be defined in at least two ways. Firstly,

$$s(P) = \frac{1}{\sigma_d} \int_{S^{d-1}} uh(K, u)\, d\omega \tag{2}$$

where σ_d is the volume of the unit ball B^d in E^d, and $d\omega$ is an element of surface area of the unit sphere S^{d-1} centred at the origin. Secondly, $s(K)$ may be defined as the centroid of the vertices of P, each vertex being weighted with the exterior angle of P at the vertex (see §3.2). From either definition, (1) follows easily. Multiplying the equation of Theorem 27 by u and integrating over S^{d-1} yields immeidately:

THEOREM 29. *For any d-polytope P,*

$$\sum_{j=0}^{d} (-1)^j \,\Sigma s(F^j) = s(P). \tag{3}$$

This has the interesting feature that it is a vector relation; all the other known Euler-type relations are scalar equations.

Because of (1), the Steiner point has proved useful in several investigations.[91] It is worth noting that $s(K)$ is a uniformly continuous function of K with respect to the Hausdorff metric (in contrast to, for example, the centroid of K which is a continuous, but not uniformly continuous function).[92] It is an unsolved and interesting problem whether $s(K)$ is uniquely determined by property (1) and the fact that

$$Ts(K) = s(TK)$$

for all congruence transformations T. The answer to this question is known to be in the affirmative if we also require continuity.[93]

Following Sallee,[94] a function ψ defined on $\mathscr{P}$ will be called a *valuation* if

$$\psi(P_1)+\psi(P_2) = \psi(P_1 \cap P_2)+\psi(P_1 \cup P_2)$$

whenever P_1, P_2 and $P_1 \cup P_2$ belong to $\mathscr{P}$. Comparing Theorem 27 with the easily proved relation

$$h(P_1, u)+h(P_2, u) = h(P_1 \cap P_2, u)+h(P_1 \cup P_2, u)$$

it is not surprising that there is a close connection between valuations and functions that satisfy Euler-type relations. We have the following:

THEOREM 30. (i) *A continuous function ψ defined on $\mathscr{P}$ which satisfies*

$$\sum_{j=0}^{d-1} (-1)^j \Sigma\psi(F^j) = \varepsilon\psi(P) \tag{4}$$

for all d-polytopes P, with $\varepsilon = \pm 1$, is a valuation on $\mathscr{P}$.

(ii) *Every valuation on $\mathscr{P}$ can be expressed as the sum of two valuations, one of which satisfies* (4) *with $\varepsilon = +1$, and the other satisfies* (4) *with $\varepsilon = -1$.*

In particular, Steiner points are valuations.(95) The central problem in this area is, of course, the characterisation of functions satisfying these conditions. Such questions are closely related to certain classical problems discussed by Hadwiger.(96)

3.4. *Symmetry and regularity*

Let P be a given d-polytope and T be any isometry (congruence transformation) such that $TP = P$. Then T is called a *symmetry* of P, and the group of all symmetries of P is denoted by $\mathscr{S}(P)$. Conversely, given any finite group $\mathscr{G}$ of isometries of E^d, a family of polytopes may be constructed by taking the convex hull of the orbit of any point under $\mathscr{G}$. For any such polytope P, clearly $\mathscr{G}$ is a subgroup of $\mathscr{S}(P)$. In view of the enormous amount of classical literature on the symmetry groups of polytopes, it is surprising that a systematic investigation of the polytopes associated with each of the finite isometry groups of E^3 has only recently been carried out. Robertson and Carter have shown that, after factoring out by a suitable equivalence relation and introducing a topology, the family of all 3-polytopes may be regarded as the points of a finite connected CW-complex in which the cells correspond to different combinatorial types. These facts emerge from a general topological theory of the orbits of finite subgroups of the orthogonal group in n dimensions due to Robertson, Carter and Morton.(97) In this general theory, open manifolds replace the cells of the 3-dimensional version.

If the symmetry group of P is non-trivial and all the symmetries of P leave an affine subspace A point-wise invariant, then A is called an axis of symmetry of P. If $\mathscr{S}(P)$ is transitive on the set vert P, then P possesses an axis of symmetry of at least 0 dimensions, for the centroid of the set vert P is left invariant by each symmetry of P. If P has a centre (as defined in §2.2) then $\mathscr{S}(P)$ necessarily contains a subgroup of two elements, namely that generated by central reflection in its centre.

Many problems on polytopes may be modified (and sometimes simplified) by restricting attention to polytopes with some property related to $\mathscr{S}(P)$ such as possessing an axis of symmetry of some specified dimension a, or a centre of symmetry. We have already discussed a number of instances of the latter restriction in earlier parts of this paper. Recently it has been shown that restrictions of this type enable a number of enumerative results to be established.(98) We cannot quote all of these here, but remark on some curious coincidences for which there is no geometrical explanation. For example, the number of combinatorial types of $(2a+2)$-polytopes with $2(a+2)$ vertices and an a-dimensional axis of symmetry, is equal to the number of combinatorial types of $(a+2)$-polytopes with $a+4$ vertices.

Many of these results have been achieved through the use of Gale diagrams and c.s. diagrams. Firstly we remark that it is possible to construct a Gale diagram of a given polytope P in such a way that $\mathscr{S}(P)$ is faithfully represented in the diagram. We illustrate this by reference to a particular example, namely that of the octahedron (Figure 2 of §2.5). By a *symmetry* of a Gale diagram $\bar{V} \subset E^{n-d-1}$ we mean any permutation of the points of $\bar{V}$ which induces a congruence transformation of E^{n-d-1}. In the example,

$$(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, \bar{x}_5, \bar{x}_6) \to (\bar{x}_5, \bar{x}_6, \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4)$$

and

$$(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, \bar{x}_5, \bar{x}_6) \to (\bar{x}_2, \bar{x}_1, \bar{x}_3, \bar{x}_4, \bar{x}_5, \bar{x}_6)$$

are symmetries of $\bar{V}$. The latter is called an *implicit symmetry* since it corresponds to the identity mapping on E^2. We define $\mathscr{S}(\bar{V})$ to be the group of all symmetries (including implicit symmetries) of $\bar{V}$. Then,

THEOREM 31. *For every polytope P there exists a Gale diagram $\bar{V}$ such that $\mathscr{S}(P) \cong \mathscr{S}(\bar{V})$.*

When P has a 2-fold axis of symmetry A, then vert P possesses a Gale transform with special properties. Let each point of E^d be represented in the form (x, y) with $x \in A$ and $y \in A^\perp$ (the orthogonal complement of A through o). Then the vertices of P may be written in the form

$$u_i = (z_i, o) \qquad i = 1, \ldots, m$$

$$\left.\begin{aligned} w_j &= (x_j, y_j) \\ w_j' &= (x_j, -y_j) \end{aligned}\right\} \quad y_j \neq o, \qquad j = 1, \ldots, n.$$

P has $m+2n$ vertices, m of which lie on A and the remaining $2n$ are *paired* with respect to the axis A. The set of points $V_A = \{z_1, \ldots, z_m, x_1, \ldots, x_n\} \subset A$ is called the *axis figure* of P, and the set $V_C = \{\pm y_1, \ldots, \pm y_n\} \subset A^\perp$ is called the *coaxis figure* of P. In general neither the axis figure nor the coaxis figure is the set of vertices of a convex polytope, but we can still define their transforms algebraically as in §2.5. The main result is the following:

THEOREM 32. *If P is the d-polytope with $m+2n$ vertices defined above, then there exists a Gale transform $\bar{V} \subset E^{m+2n-d-1}$ of* vert P *which has an $(m+n-a-1)$-dimensional axis of symmetry $\bar{A}$ containing the points $\bar{u}_1, \ldots, \bar{u}_m$. The set*

$$\{\bar{u}_1, \ldots, \bar{u}_m, \bar{w}_1+\bar{w}_1', \ldots, \bar{w}_n+\bar{w}_n'\} \subset \bar{A}$$

is a Gale transform of the axis figure V_A, and the centrally symmetric set of points

$$\{\pm(\bar{w}_1-\bar{w}_1'), \ldots, \pm(\bar{w}_n-\bar{w}_n')\} \subset \bar{A}^1$$

is a c.s. transform of the coaxis figure V_C.

We have quoted these results in some detail for they initiate a new method of investigating the symmetries and axes of symmetry of polytopes. This method may turn out to be of wide application.

The idea of a symmetry may be extended. By an affine (projective) symmetry of a d-polytope P we mean any affine (permissible projective) transformation $T : E^d \to E^d$ such that $TP = P$. In this way the affine symmetry group $\mathscr{S}_A(P)$ and projective symmetry group $\mathscr{S}_P(P)$ of P may be defined. Each of these symmetries corresponds to an automorphism T^* of the face-lattice $\mathscr{F}(P)$, and $\mathscr{S}(P)$, $\mathscr{S}_A(P)$, $\mathscr{S}_P(P)$ each correspond to subgroups of $\mathscr{A}(P)$, the group of all automorphisms of $\mathscr{F}(P)$. This brings us to a problem of obvious importance, which is unsolved[98a] even for $d = 3$.

Given any d-polytope P, does there always exist a polytope P′ combinatorially equivalent to P such that $\mathscr{S}(P') \cong \mathscr{A}(P')$ $\left(\cong \mathscr{A}(P)\right)$?

In other words, we are asking if the automorphism group of P can always be realised as the symmetry group of some $P' \approx P$?

The investigation of symmetry groups leads naturally to the idea of regularity of polytopes.[99] Several equivalent definitions of regularity have been used. An example of an inductive definition [100] is the following: A d-polytope P is *regular* if its facets are regular and all its vertices are regular. (By a regular vertex F^0 of P we mean one with the property that if $F_1{}^0, \ldots, F_r{}^0$ are the end-points of the edges of P that meet at F^0, then $\operatorname{conv}\{F_1{}^0, \ldots, F_r{}^0\}$ is a regular $(d-1)$-polytope.) For $d = 2$, the regular polytopes are, of course, the familiar regular polygons. There are five regular 3-polytopes (tetrahedron, cube, octahedron, isosahedron, dodecahedron), six regular 4-polytopes (simplex, cube, crosspolytope, 24-cell, 120-cell, 600-cell) and three regular d-polytopes for $d \geqslant 5$ (simplex, cube, crosspolytope). An alternative definition of regularity has recently been given by McMullen:[101] P is said to be *regular* if $\mathscr{S}(P)$ is transitive on the maximal towers of faces of P (see §2.1). Not only has this the advantage of simplicity, but it is readily generalised. We say that P is *affinely, projectively,* or *combinatorially regular,* if $\mathscr{S}_A(P)$, $\mathscr{S}_P(P)$ or $\mathscr{A}(P)$, respectively, is transitive on the set of maximal towers of faces of P. McMullen recently proved the following powerful result:

THEOREM 33. *An affinely (projectively, combinatorially) regular polytope is affinely (projectively, combinatorially) equivalent to a regular polytope.*

Relaxing some of the requirements in the definition of regularity we obtain larger classes of polytopes, many of which are investigated in the classical literature. For example, if all the facets of P are regular and $\mathscr{S}(P)$ is transitive on the set vert P, then P is called *semi-regular*. A further generalisation which has been investigated recently is that of *regular-faced* polytopes, that is, polytopes whose facets (and so all proper faces) are regular. Each of these concepts (and others which have not been mentioned) lead to corresponding enumeration problems: How many d-polytopes are there of the given types? The answers are known completely[102] for $d = 3$, but our knowledge is very fragmentary for $d \geqslant 4$.

NOTES

(1) In particular, see Grünbaum [3] for a comprehensive account of the theory up to 1967, and Grünbaum [5] for a survey of the most recent results. An alternative treatment of many of the combinatorial theorems can be found in McMullen-Shephard [2]. Other relevant texts are Alexandrov [1], Coxeter [1], Fejes Tóth [1], and Hadwiger [2, 3].

(2) Coxeter [1] p. 13.

(3) Klee [2].

(4) Coxeter [1] p. 141.

(5) According to Coxeter [1] p. 144, the regular polytopes were independently discovered at least nine times between 1881 and 1900.

(6) Minkowski [1].

(7) Grünbaum [3].

(8) For a detailed account, with proofs, of the material in this section, see Grünbaum [3] Chapters 1–4, or McMullen–Shephard [2] Chapters 1 and 2.

(9) See Grünbaum [3] §4.8. The corresponding problem for centrally symmetric polytopes has been completely solved by McMullen [4].

(10) No $(r+s)$-polytopes of type (r, s) are known for $r \geqslant 4$, $s \geqslant 4$, and $r+s \geqslant 9$. For d-polytopes of types $(2, d-2)$ and $(3, d-3)$ see Grünbaum [3] pp. 65–66.

(11) $c(6, 3) = 7$, $c(7, 4) = 31$, $c(8, 5) = 116$, $c(9, 6) = 379$, $c(7, 3) = 34$, $c(8, 3) = 257$ and the unchecked result is $c(9, 3) = 2607$. See Grünbaum [3] p. 424, for a survey of earlier results, and Federico [1] for a recent enumeration.

(12) $c_s(7, 3) = 5$, $c_s(8, 3) = 14$, $c_s(9, 3) = 50$, $c_s(10, 3) = 233$, $c_s(11, 3) = 1249$, $c_s(12, 3) = 7595$. The last value has not been independently checked. See Grünbaum [3] p. 424, for earlier work, and Bowen–Fisk [1] for a recent enumeration.

(13) Grünbaum–Sreedharan [1].

(13a) Altshuler [1].

(14) McMullen–Shephard [2] Chapter 2, Theorem 16.

(15) Grünbaum [3] p. 207.

(16) McMullen [1]. See also §3.4.

(17) For proofs of Euler's Theorem, with especial reference to convex polytopes, see Grünbaum [3] Chapter 8, and McMullen–Shepherd [2] §2.4. Several " elementary proofs " of Euler's Theorem that have been published are incomplete in that they implicitly assume certain topological properties of polytopes. This remark applies to proofs based on the idea of " building up " a polytope facet-by-facet, an operation of questionable validity in $d \geqslant 4$ dimensions.

(18) Proofs of Theorem 2 are given in Grünbaum [3] Chapter 9, Hadwiger [5], and McMullen–Shephard [2] §2.4. In the latter, the simplified method of solving the Dehn–Sommerville equations recently devised by MacDonald [1] and modified by McMullen [3], is described. See also Riordan [1] for other solutions. Analogues of Theorem 2 for cubical polytopes are given in Grünbaum [3] §9.4. For similar equations in more general systems see Klee [1].

(19) Steinitz [1], Grünbaum [3] §10.3.

(20) Grünbaum [3] §10.4.

(20a) For an analogous investigation of pairs (f_0, f_2) see Reay [1].

(21) Cyclic polytopes were originally discovered by Carathéodory [1, 2] and rediscovered more recently by Gale [1] and Motzkin [1]. For further information see Grünbaum [3] §4.7 and Chapter 7, and McMullen–Shephard [2] §2.3 (vi). In these two references different methods of determining the numbers $f_j(C(v, d))$ are described, the latter following Shephard [11].

(22) See Grünbaum [3] §10.1 and, for the parts that have been proved since 1967, McMullen–Shephard [2] Chapter 4, Grünbaum [7], and McMullen [3].

(23) Grünbaum [3] §10.2.

(24) Grünbaum [3] §10.2; of the more recent results, the cases $d = 4, 5$ are due to Walkup [1], while the cases with $v \geqslant d+4$ have been established in unpublished work of M. A. Perles.

(25) McMullen–Shephard [1] prove this for $n = 1, 2$. However, the analogous conjecture for centrally symmetric triangulations of the $(d-1)$-sphere ($d \geqslant 4$) is known to be false (Grünbaum [9, 10]).

(26) Grünbaum [5].

(27) Grünbaum [3, 5].

(28) Grünbaum [3] Chapter 11.

(28a) For an affirmative answer in special cases see Zaks [1], Grünbaum [11].

(29) Balinski [1].

(30) Grünbaum [2].

(31) Larman–Mani [1].

(32) This result is a consequence of Menger's Theorem, see Whitney [1], Dirac [1].

(33) Sallee [1], Grünbaum [3] §11.3. For a different generalisation see Barnette [5].

(33a) The first nontrivial case of the conjecture, $d = 4$, has been settled in the affirmative by Jung [1] and Mani [1].

(34) Grünbaum [3] §11.1.

(35) Grünbaum [3] §12.2.

(36) Grünbaum [3] §12.3.

(37) See Grünbaum [3] §3.3 for illustrations of Schlegel diagrams. Brückner [1] attempted to enumerate the simplicial 4-polytopes with 8 vertices by considering possible Schlegel diagrams. For the reason mentioned he arrived at an incorrect value of $c_s(8, 4)$, later corrected by Grünbaum–Sreedharan [1]. See also the discussion in Barnette [3]. For a related topic see Peterson [1].

(38) For an elementary exposition of some of these, see Shephard [7].

(39) Grünbaum [5].

(40) Steinitz [2], Steinitz–Rademacher [1], Grünbaum [3] §13.1, Barnette–Grünbaum [2].

(41) Grünbaum [3] p. 244.

(42) Barnette–Grünbaum [1]. Very interesting is also the following recent result of Barnette [4]: For every simple circuit in the graph of a 3-polytope P there exists a 3-polytope P' of the same combinatorial type as P, such that the corresponding circuit in P' is the inverse image of the boundary of some regular projection (see §3.2 for definition) of P'.

(43) Named after Sir William Hamilton who, in 1857, posed the problem of finding simple edge-circuits on a regular dodecahedron which visited every vertex. His solution of this problem was an early application of group theory.

(44) The argument given here is due to Brown [1]. For further results see Moon–Moser [1] The " smallest " 3-polytope with no Hamiltonian circuit has 11 vertices, and the smallest known 3-polytope with no Hamiltonian path has 14 vertices. See Grünbaum [5], Shephard [7], and Barnette–Jucovič [1].

(45) Tait [1, 2, 3].

(46) Grünbaum–Motzkin [1]. The smallest known simple 3-polytopes with no Hamiltonian circuit, and with no Hamiltonian path, have 38 and 88 vertices respectively. The former were discovered independently by Lederberg [1], Bosak [1] and Barnette, and the latter by T. Zamfirescu (private communication). For extensions of these results and further references, see Grünbaum [5] §1.4.

(47) These values are due to R. Forcade (unpublished).

(48) Tutte [1].

(49) Grünbaum [3] Chapter 17 (written by Victor Klee).

(50) Barnette [1].

(50a) See Hawkins–Hill–Reeve–Tyrrell [1]. The fact that the permissible values of p_5 are exactly double those of p_4 has been explained geometrically by Meek [1] and Crowe–Molnár [1].

(51) Grünbaum [3] §13.4.

(51a) See Crowe [1], Malkevitch [1, 2], Gallai [1].

(52) Eberhard's original proof is at the end of his book [1]. Simpler proofs are given in Grünbaum [3] §13.3.

(53) Grünbaum [6].

(54) Grünbaum [3] §13.4.

(55) Barnette [2]. A different lower bound for p_6 was found by Jucovič [3].

(56) For a solution corresponding to case (5) see Grünbaum [8]. For the case of self-dual 3-polytopes see Jucovič [4].

(57) For a survey of the most recent results, see Grünbaum [5] §1.3.

(58) This section follows closely McMullen–Shephard [1].

(59) Gale [1].

(60) Notice that this differs from the usage in Grünbaum [3] §5.4.

(61) Polya [1], DeBruijn [1].

(62) In fact Perles enumerated the distended Gale diagrams. No details of his method have been published except for the brief account in Grünbaum [3] §6.3.

(62a) Since this was written, E. K. Lloyd (private communication) has used a modified form of Polya's Theorem to find an expression for $c(d+3, d)$.

(63) McMullen–Shephard [2] §3.4.

(64) For full details, see Grünbaum [3] §5.5.

(65) The construction described is due to M. A. Perles. Before this example was discovered in 1966 many authors regarded it as " almost obvious " that every polytope could be rationally embedded. It was Klee who originally raised doubts as to whether this is, in fact, the case. Gale diagrams enable us to establish easily that every d-polytope with at most $d+3$ vertices can be rationally embedded.

(66) Grünbaum [3] §5.5.

(67) See, for example, Blaschke [1], Bonnesen–Fenchel [1], Hadwiger [1, 2].

(68) For the classical theory of mixed volumes see Bonnesen–Fenchel [1]. For some more recent results, see Shephard [1].

(68a) For detailed and attractive accounts of some such problems and results, and for references to the very substantial literature, see Fejes Tóth [1, 2].

(69) Melzak [1] has shown that if we restrict ourselves to tetrahedra, then the regular tetrahedron is the solution to the first problem, but nothing seems to be known when arbitrary combinatorial types are considered. A 2-dimensional problem similar to the second question (the isoperimetric problem for " honeycombs ") has been solved by Bleicher–Fejes–Tóth [1]. See also Levy [1] for the problem of maximising the area of a polygon whose sides have prescribed lengths.

(70) Rogers [1].

(71) A related, still unsolved problem is: Does every combinatorial type of 3-polytope have a representative P such that, for a suitable point o, each perpendicular from o to the plane of a 2-face of P has its foot in the 2-face?

(72) For a survey, see Grünbaum [3] §13.5. Also Jucovič [1, 2].

(73) Besicovitch–Eggleston [1], Shephard [2].

(74) Shephard–Webster [1].

(75) Grünbaum [3] Chapter 15, Shephard [3], Perles–Shephard [3], Berg [1].

(76) It is known that $f_j(Q) \geq f_j(P)$ for $j = 0, \ldots, d-1$ is a necessary condition, see Eggleston–Grünbaum–Klee [1] and Grünbaum [3] §5.3.

(77) Early proofs of Theorem 22 (see Grünbaum [3] Chapter 14) were long and depended upon the idea of proving the result for simplexes, and then showing that it remained true for any polytope " built-up " from simplexes. Recent proofs are much simpler, making use of the

methods of integral geometry, the main idea being indicated later in this section. See Perles–Shephard [2], Shephard [5].

(78) This definition and the following discussion of Grassmann angles closely follows Grünbaum [4].

(79) Shephard [8].

(80) The following treatment follows Perles–Shephard [2]. For the corresponding properties of Grassmann angles see Grünbaum [4].

(81) For zonotopes, see Shephard [6]. Polytopes with $q = 1$ have been called equi-projective, but no characterisation of equi-projective polytopes is known, even for $d = 3$.

(81a) For newer results on Grassmann angles see Larman–Mani [2].

(82) Stoker [1], Karcher [1]. This question is closely related to the classical Cauchy rigidity theorem, Lyusternik [1]. See also Barnette [6] for a related result.

(83) Schneider [1]. A more general problem recently discussed by Gruber [1, 2] and Schneider [2] is that of determining all polytopes P such that, for all vectors t, either $P \cap (P+t) = \emptyset$ or $P \cap (P+t)\, \sigma P$ for equivalence relations σ other than that mentioned. If σ is homothety then an old result of Rogers and Shephard [1] states that P must be a simplex. If $P_1\, \sigma P_2$ means that P_1 is a non-singular affine transform of P_2, or that P_1 has the same number of vertices as P_2, or σ is the relation mentioned in the text, then the corresponding class of polytopes consists of all direct sums of simplexes.

(84) The following treatment follows Perles–Shephard [1].

(85) Barnette [1].

(86) Perles–Shephard [2], Grünbaum [4].

(87) For a proof of this identity, and a survey of all the known Euler-type relations, see Shephard [9].

(88) Grünbaum [3] §14.3.

(89) Steiner [1].

(90) Grünbaum [1].

(91) For example, Firey–Grünbaum [1], Flanders [1], Shephard [4].

(92) Shephard–Webster [1].

(93) For the case $d = 2$ see Shephard [10], and for $d > 2$ see Schmitt [1]. C. Berg has recently shown that Schmitt's proof is incomplete. For another characterisation of Steiner points see Hadwiger [4].

(94) Sallee [3].

(95) Sallee [2].

(96) Hadwiger [1] and [3] pp. 236–243.

(97) Robertson–Carter [1], Robertson–Carter–Morton [1].

(98) McMullen–Shephard [3].

(98a) Barnette [7] established an affirmative answer for 3-polytopes P such that $\mathscr{A}(P)$ is the 2-element group.

(99) See Coxeter [1], which contains an extensive bibliography of the classical literature on this subject.

(100) Fejes Tóth [1].

(101) McMullen [1, 2].

(102) For $d = 3$ there are 13 semiregular polytopes (apart from the five regular ones, the polygonal prisms and antiprisms). There are 92 regular-faced 3-polytopes apart from the semiregular ones. See Johnson [1], Zalgaller [1], and Grünbaum [3] §19.1.

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