The Continuous Wavelet Transform

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The space L^2

$$\|f\| = \left\{ \int_{-\infty}^{\infty} |f(t)|^2 \, dt \right\}^{1/2}$$

$$L^2(\mathbb{R}) = \{ f : \|f\| < \infty \}$$

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(t) \overline{g(t)} \, dt$$

translation: $T_a f(t) = f(t-a)$

modulation: $E_a f(t) = e^{iat} f(t)$

dilation: $D_s f(t) = |s|^{-1/2} f(t/s) \qquad s \neq 0$

note: $\|T_a f\| = \|E_a f\| = \|D_s f\| = \|f\|$

The Fourier Transform

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi t} f(t)\,dt$$

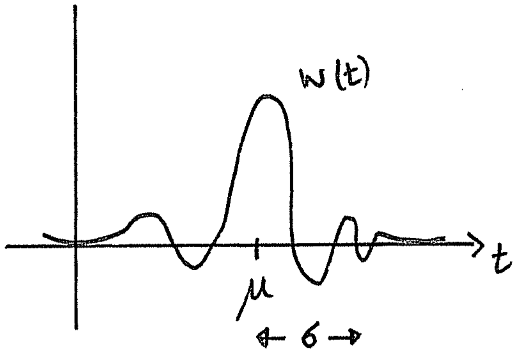
inversion: $$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi t} \hat{f}(\xi)\,d\xi$$

Parseval - Plancherel: $$\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle$$

$$(T_a f)^\wedge = E_{-a} \hat{f} \qquad [f(t-a)]^\wedge = e^{-ia\xi} \hat{f}(\xi)$$

$$(E_a f)^\wedge = T_a \hat{f} \qquad [e^{iat} f(t)]^\wedge = \hat{f}(\xi - a)$$

$$(D_s f)^\wedge = D_{1/s} \hat{f} \qquad \left[\frac{1}{\sqrt{|s|}} f\left(\frac{t}{s}\right)\right]^\wedge = \sqrt{|s|}\, \hat{f}(s\xi)$$



μ = center
σ = spread

If $w(t) \in L^2$, then $|w(t)|^2 / \|w\|^2$ is ≥ 0, has integral 1

$\Rightarrow$ probability density

$$\mu = \frac{1}{\|w\|^2} \int_{-\infty}^{\infty} t \, |w(t)|^2 \, dt$$

$$\sigma = \frac{1}{\|w\|^2} \left\{ \int_{-\infty}^{\infty} (t-\mu)^2 \, |w(t)|^2 \, dt \right\}^{1/2}$$

(mean, standard deviation of $|w(t)|^2 / \|w\|^2$)

Likewise, $\hat{\mu}$, $\hat{\sigma}$ = center, spread of $\hat{w}(\xi)$.

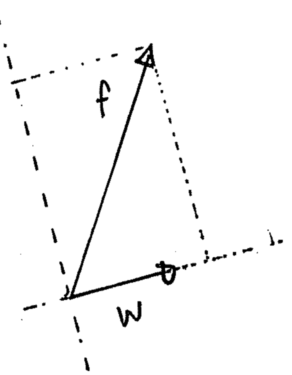
<u>Heisenberg Uncertainty Principle</u>

$$\sigma \cdot \hat{\sigma} \geq \frac{1}{2}$$

(equality $\Leftrightarrow$ $w(t)$ is some form of Gaussian)

Interpretation of Inner Product

vectors



$\langle f, w \rangle \approx$ component of f in direction w

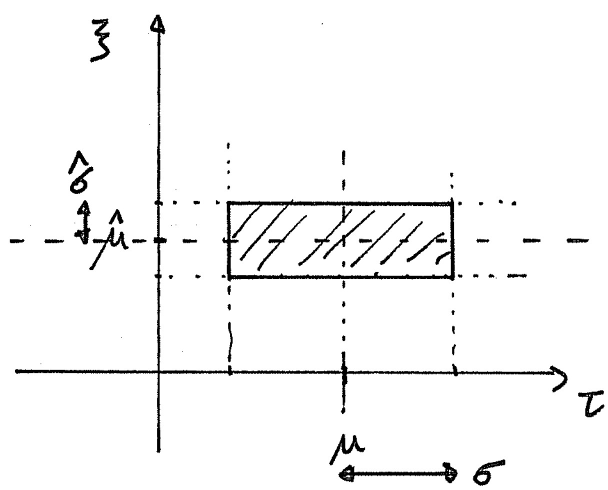
functions if w has center μ, spread σ,

$\langle f, w \rangle \approx$ part of f localized in interval $\mu \pm \sigma = [\mu - \sigma, \mu + \sigma]$

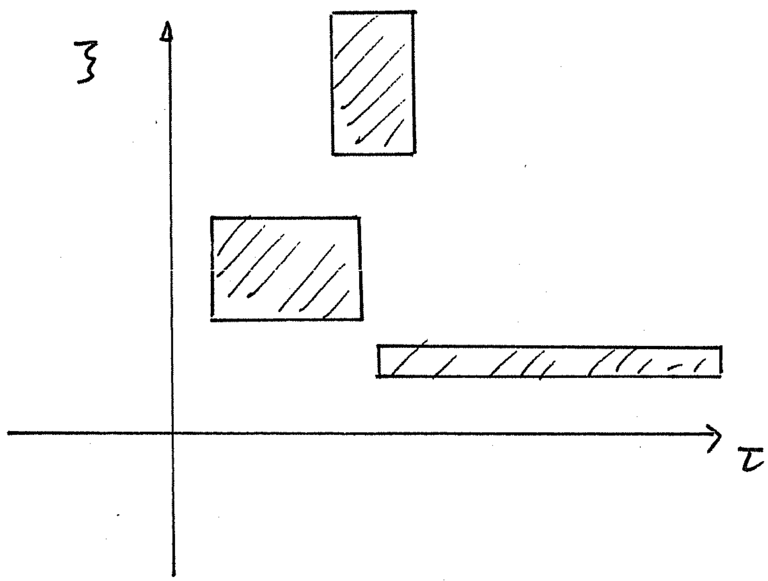
$\langle \hat{f}, \hat{w} \rangle \approx$ part of $\hat{f}$ localized in $\hat{\mu} \pm \hat{\sigma}$

but $\langle f, w \rangle = \langle \hat{f}, \hat{w} \rangle$, so

$\langle f, w \rangle \approx$ part of f localized in time $\mu \pm \sigma$, frequency $\hat{\mu} \pm \hat{\sigma}$



w	$\mu \pm \sigma$	$\hat{\mu} \pm \hat{\sigma}$
$T_a w$	$\mu + a \pm \sigma$	$\hat{\mu} \pm \hat{\sigma}$
$E_a w$	$\mu \pm \sigma$	$\hat{\mu} + a \pm \hat{\sigma}$
$D_s w$	$s\mu \pm s\sigma$	$\frac{1}{s}\hat{\mu} \pm \frac{1}{s}\hat{\sigma}$



By choice of $w(t)$, and by applying shifts, modulations, dilations, we can change the shape of rectangles and move them around, but always area ≥ 2.

Short-Term Fourier Transform

$w(t)$ = window function

$$\Psi_w f(\tau, \xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\xi t}\, \overline{w(t-\tau)}\, dt$$

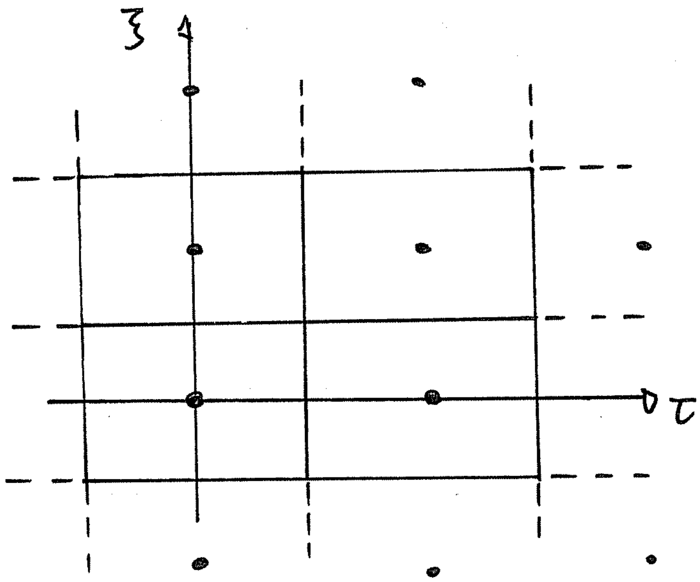
$$= \langle f, \frac{1}{\sqrt{2\pi}} e^{i\xi t} w(t-\tau) \rangle$$

$$= \langle f, \frac{1}{\sqrt{2\pi}} E_\xi T_\tau w \rangle$$

$$= \langle \hat{f}, \frac{1}{\sqrt{2\pi}} T_\xi E_{-\tau} \hat{w} \rangle$$

contains information on f in $\mu + \tau \pm \sigma$, $\hat{\mu} + \xi \pm \hat{\sigma}$

= original time-frequency rectangle for w, shifted in time and frequency.



we can cover entire time-frequency plane with samples equally spaced in time, frequency.

Same resolution at all times/frequencies

<u>Continuous Wavelet Transform</u>

$$\emptyset_w f(\tau, s) = \int_{-\infty}^{\infty} f(t)\ |s|^{-\frac{1}{2}}\ \overline{w\left(\frac{t-\tau}{s}\right)}\ dt$$

$$= \langle f, T_\tau D_s w \rangle$$

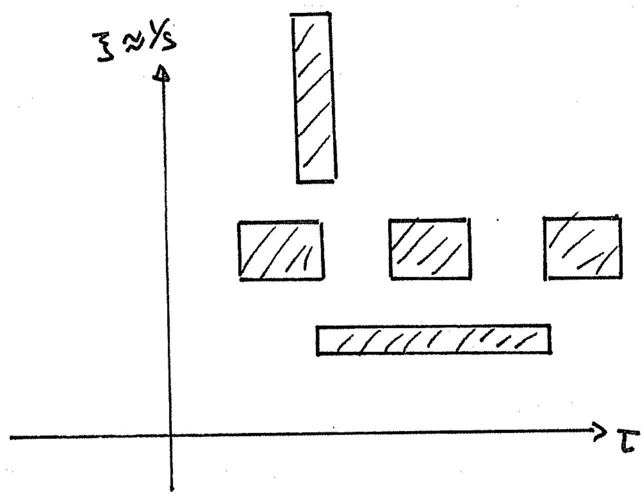
$$= \langle \hat{f}, E_{-\tau} D_{1/s} \hat{w} \rangle$$

$w(t)$ = mother wavelet

τ = time

s = scale $\approx$ 1/frequency

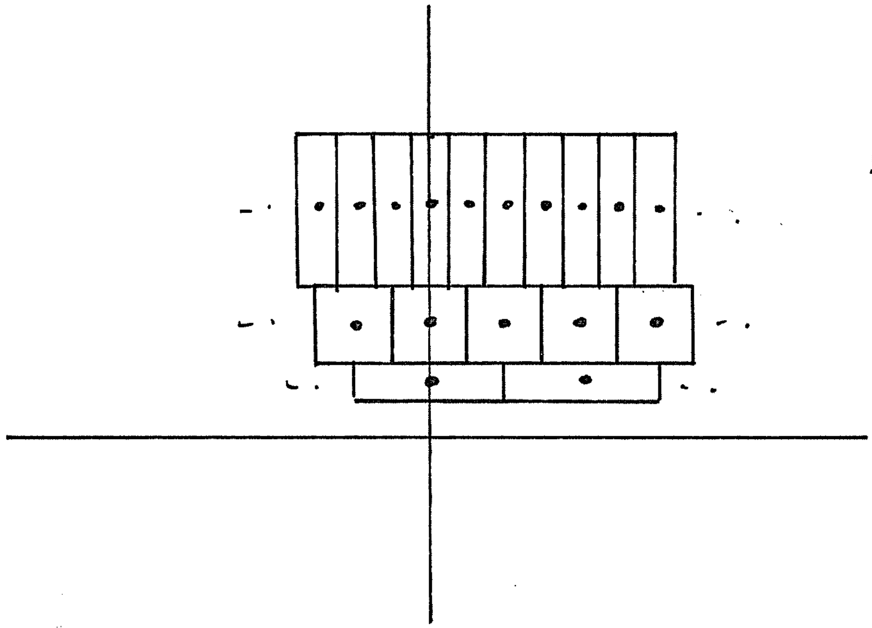
contains information on f in $s\mu + \tau \pm s\sigma$, $\frac{1}{s}\hat{\mu} \pm \frac{1}{s}\hat{\sigma}$



same shape for fixed s

low frequency: bad time resolution
good frequency resolution

high frequency: good time resolution
bad frequency resolution



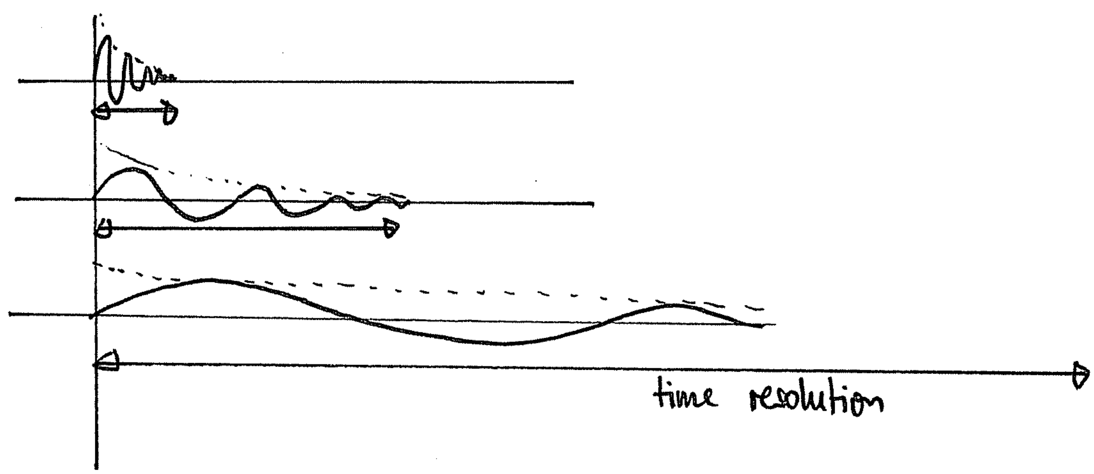
frequency resolution in octaves only

To cover the entire time-frequency plane, we need dyadic grid : $\left\{\begin{matrix}\text{double}\\ \text{halve}\end{matrix}\right\}$ scale, $\left\{\begin{matrix}\text{double}\\ \text{halve}\end{matrix}\right\}$ time step

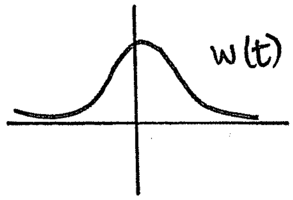
<u>another way to look at it:</u>

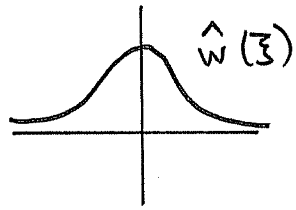
scale $\approx$ wave length

The time resolution at a given frequency is proportional to wave length.



Good Window for STFT



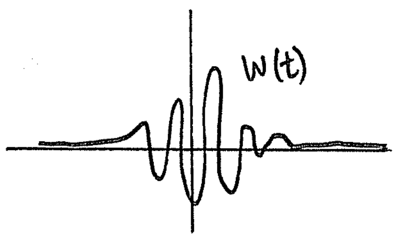


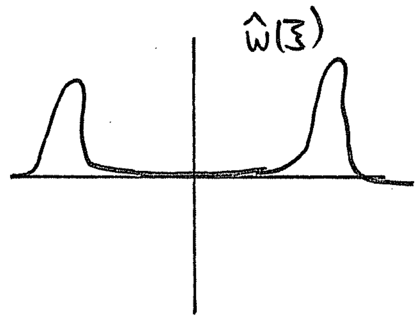
$\mu = \hat{\mu} = 0$

$\sigma \cdot \hat{\sigma} \approx \frac{1}{2}$

low-pass filter

Good Window for CWT





$\mu = 0$

$\hat{\mu} \neq 0$

$\sigma \cdot \hat{\sigma} \approx \frac{1}{2}$

$\hat{\mu} \approx \hat{\sigma}$

bandpass filter

Inversion and Redundancy

Q: Can we invert Fourier transform?

A: yes

Q: Can we invert it based on a subset of data?

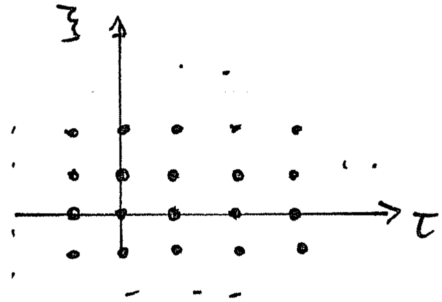
A: not in general. $f, \hat{f}$ are both functions of one variable: no redundancy.

Q: Can we invert STFT?

A: Yes,

$$f(t) = \frac{1}{\|w\|^2} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Psi_w f(\tau, \xi) e^{i\xi t} w(t-\tau)\, d\tau\, d\xi$$

Q: Can we invert it based on a subset of data, for example a regular grid?



A: No, in general, but yes for suitable $w(t)$

STFT takes $f(t)$ into $\Psi_w f(\tau, \xi)$: redundancy.

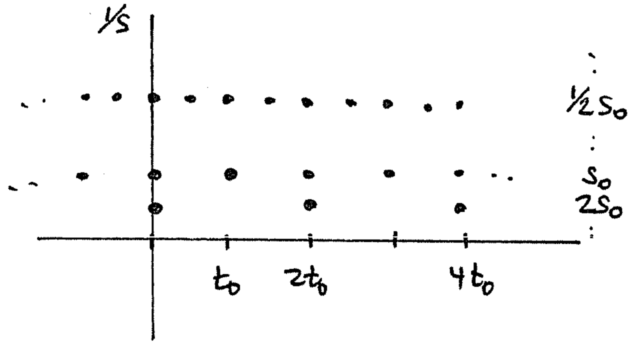
Q: Can we invert CWT?

A: Yes, if $C_w = \sqrt{2\pi \int |\hat{w}(\xi)|^2 / |\xi| \, d\xi} < \infty$

$$f(t) = \frac{1}{C_w^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \emptyset_w f(\tau, s)\, |s|^{-\frac{1}{2}} w\left(\frac{t-\tau}{s}\right) \cdot |s|^{-2} \, ds \, d\tau$$

(Note: $w \in L^1 \Rightarrow \hat{w}$ continuous $\Rightarrow \hat{w}(0) = 0 \Leftrightarrow \int w(t)\, dt = 0$)

Q: Can we invert it based on a subset of data, for example a dyadic grid?



A: In general no, but yes for suitable w

This is related to the <u>discrete wavelet transform</u>

References

I found it surprisingly hard to find readable introductions to the continuous wavelet transform. Most books and survey articles on wavelets have a section on the CWT at the beginning, but they are often very short and/or hard to read.

I would recommend.

[1] C.K. Chui, Waveletts: A Mathematical Tool for Signal Analysis, SIAM 1997, (chapters 1, 2.)

[2]. F. Hlawatsch & G.F. Boudreaux-Bartels, Linear and Quadratic Time-Frequency Signal Representations, IEEE Signal Processing Magazine, April 1992, p.21 (beginning part)