RANDOM SIGNALS

Random Variables

A random variable $x(\xi)$ is a mapping that assigns a real number x to every outcome ξ from an abstract probability space. The mapping should satisfy the following two conditions:

- the interval $\{x(\xi) \leq x\}$ is an event in the abstract probabilty space for every x;
- $\Pr[x(\xi) < \infty] = 1$ and $\Pr[x(\xi) = -\infty] = 0$.

Cumulative distribution function (cdf) of a random variable $x(\xi)$:

$$F_x(x) = \Pr\{x(\xi) \leq x\}.$$

Probability density function (pdf):

$$f_x(x) = \frac{dF_x(x)}{dx}.$$

Then

$$F_x(x) = \int_{-\infty}^{x} f_x(x)\, dx.$$

Since $F_x(\infty) = 1$, we have *normalization condition:*

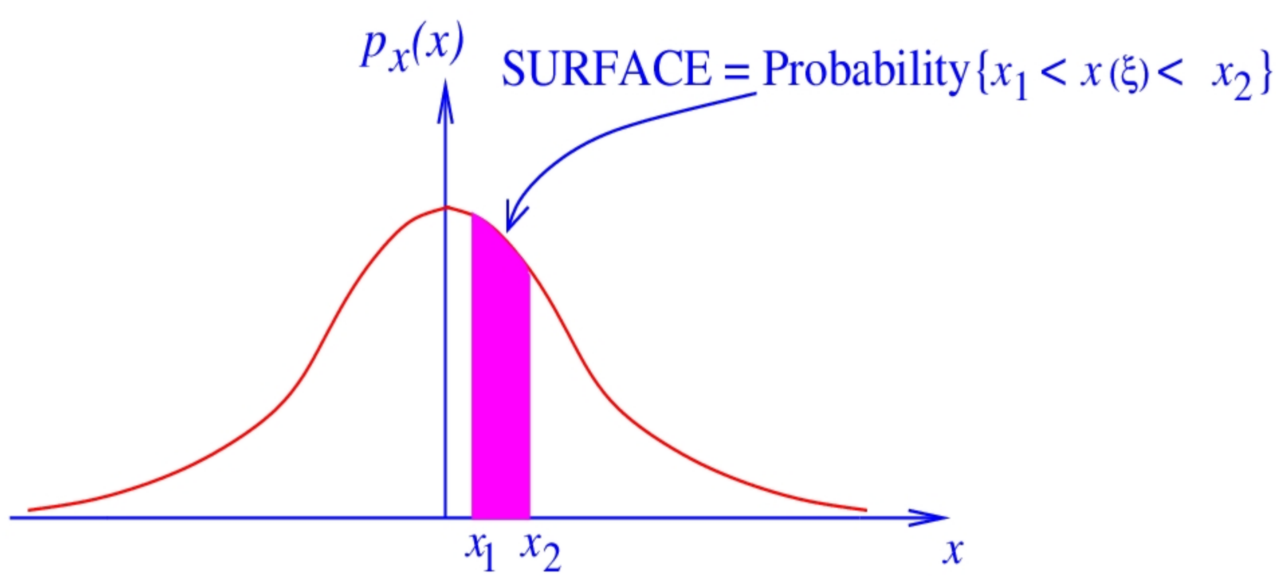
$$\int_{-\infty}^{\infty} f_x(x) dx = 1.$$

Several important properties:

$$0 \le F_x(x) \le 1, \quad F_x(-\infty) = 0, \quad F_x(\infty) = 1,$$
$$f_x(x) \ge 0, \quad \int_{-\infty}^{\infty} f_x(x)\, dx = 1.$$

Simple interpretation:

$$f_x(x) = \lim_{\Delta \to 0} \frac{\Pr\{x - \Delta/2 \le x(\xi) \le x + \Delta/2\}}{\Delta}.$$



Expectation of an arbitrary function $g(x(\xi))$:

$$\mathrm{E}\{g(x(\xi))\} = \int_{-\infty}^{\infty} g(x) f_x(x)\, dx.$$

Mean:

$$\mu_x = \mathrm{E}\{x(\xi)\} = \int_{-\infty}^{\infty} x f_x(x)\, dx.$$

Variance of a real random variable $x(\xi)$:

$$\begin{aligned} \mathrm{var}\{x\} &= \sigma_x^2 = \mathrm{E}\{(x - \mathrm{E}\{x\})^2\} \\ &= \mathrm{E}\{x^2 - 2x\mathrm{E}\{x\} + \mathrm{E}\{x\}^2\} \\ &= \mathrm{E}\{x^2\} - (\mathrm{E}\{x\})^2 \\ &= \mathrm{E}\{x^2\} - \mu_x^2. \end{aligned}$$

Complex random variables: A complex random variable

$$x(\xi) = x_{\mathrm{R}}(\xi) + j x_{\mathrm{I}}(\xi).$$

Although the definition of the mean remains unchanged, the definition of variance changes for complex $x(\xi)$:

$$\begin{aligned} \mathrm{var}\{x\} &= \sigma_x^2 = \mathrm{E}\{|x - \mathrm{E}\{x\}|^2\} \\ &= \mathrm{E}\{|x|^2 - x\mathrm{E}\{x\}^* - x^*\mathrm{E}\{x\} + |\mathrm{E}\{x\}|^2\} \\ &= \mathrm{E}\{|x|^2\} - |\mathrm{E}\{x\}|^2 \\ &= \mathrm{E}\{|x|^2\} - |\mu_x|^2. \end{aligned}$$

Random Vectors

A real-valued vector containing N random variables

$$\boldsymbol{x}(\xi) = \begin{bmatrix} x_1(\xi) \\ x_2(\xi) \\ \vdots \\ x_N(\xi) \end{bmatrix}$$

is called a *random N vector* or a random vector when dimensionality is unimportant. A real-valued random vector $\equiv$ mapping from an abstract probability space to a vector-valued real space $\mathcal{R}^N$.

A random vector is completely characterized by its *joint* cumulative distribution function, which is defined by

$$F_{\boldsymbol{x}}(x_1, x_2, \ldots, x_N) \stackrel{\triangle}{=} P[\{x_1(\xi) \leq x_1\} \cap \ldots \cap \{x_N(\xi) \leq x_N\}]$$

and is often written as

$$F_{\boldsymbol{x}}(\boldsymbol{x}) = P[\boldsymbol{x}(\xi) \leq \boldsymbol{x}].$$

A random vector can also be characterized by its *joint* probability density function (pdf), defined as follows:

$$f_{\boldsymbol{x}}(\boldsymbol{x}) = \lim_{\substack{\Delta x_1 \to 0 \\ \Delta x_2 \to 0 \\ \vdots \\ \Delta x_N \to 0}} \frac{P[\{x_1 < x_1(\xi) \le x_1 + \Delta x_1\} \cap \ldots \cap \{x_N < x_N(\xi) \le x_N + \Delta x_N\}]}{\Delta x_1 \cdots \Delta x_N}$$

$$= \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_N} F_{\boldsymbol{x}}(\boldsymbol{x}).$$

The function

$$f_{x_i}(x_i) = \underbrace{\int \cdots \int}_{(N-1)} f_{\boldsymbol{x}}(\boldsymbol{x})\, dx_1 \cdots dx_{i-1}\, dx_{i+1} \cdots dx_N$$

is known as *marginal* pdf and describes individual random variables.

The cdf of $\boldsymbol{x}$ can be computed from the joint pdf as:

$$F_{\boldsymbol{x}}(\boldsymbol{x}) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_N} f_{\boldsymbol{x}}(\boldsymbol{v})\, dv_1 dv_2 \cdots dv_N \triangleq \int_{-\infty}^{\boldsymbol{x}} f_{\boldsymbol{x}}(\boldsymbol{v})\, d\boldsymbol{v}.$$

Complex random vectors:

$$\boldsymbol{x}(\xi) = \boldsymbol{x}_{\rm R}(\xi) + j\boldsymbol{x}_{\rm I}(\xi) = \begin{bmatrix} x_{\rm R,1}(\xi) \\ x_{\rm R,2}(\xi) \\ \vdots \\ x_{{\rm R},N}(\xi) \end{bmatrix} + j \begin{bmatrix} x_{\rm I,1}(\xi) \\ x_{\rm I,2}(\xi) \\ \vdots \\ x_{{\rm I},N}(\xi) \end{bmatrix}.$$

Complex random vector $\equiv$ mapping from an abstract probability space to a vector-valued complex space $\mathcal{C}^N$. The cdf of a complex-valued random vector $\boldsymbol{x}(\xi)$ is defined as:

$$\begin{aligned} F_x(\boldsymbol{x}) &\stackrel{\triangle}{=} P[\boldsymbol{x}(\xi) \le \boldsymbol{x}] \\ &\stackrel{\triangle}{=} P[\{\boldsymbol{x}_{\rm R}(\xi) \le \boldsymbol{x}_{\rm R}\} \cap \{\boldsymbol{x}_{\rm I}(\xi) \le \boldsymbol{x}_{\rm I}\}] \end{aligned}$$

and its *joint* pdf is defined as

$$\begin{aligned} f_{\boldsymbol{x}}(\boldsymbol{x}) &= \lim_{\substack{\Delta x_{\rm R,1} \to 0 \\ \Delta x_{\rm I,1} \to 0 \\ \vdots \\ \Delta x_{{\rm R},N} \to 0 \\ \Delta x_{{\rm I},N} \to 0}} \\ &\frac{P[\{\boldsymbol{x}_{\rm R} < \boldsymbol{x}_{\rm R}(\xi) \le \boldsymbol{x}_{\rm R} + \Delta\boldsymbol{x}_{\rm R}\} \cap \{\boldsymbol{x}_{\rm I} < \boldsymbol{x}_{\rm I}(\xi) \le \boldsymbol{x}_{\rm I} + \Delta\boldsymbol{x}_{\rm I}\}]}{\Delta x_1 \cdots \Delta x_N} \\ &= \frac{\partial}{\partial x_{\rm R,1}} \frac{\partial}{\partial x_{\rm I,1}} \cdots \frac{\partial}{\partial x_{{\rm R},N}} \frac{\partial}{\partial x_{{\rm I},N}} F_{\boldsymbol{x}}(\boldsymbol{x}). \end{aligned}$$

The cdf of $\boldsymbol{x}$ can be computed from the joint pdf as:

$$\begin{aligned} F_{\boldsymbol{x}}(\boldsymbol{x}) &= \int_{-\infty}^{x_{\mathrm{R},1}} \cdots \int_{-\infty}^{x_{\mathrm{I},N}} f_{\boldsymbol{x}}(\boldsymbol{v})\, dv_{\mathrm{R},1} dv_{\mathrm{I},1} \cdots dv_{\mathrm{R},N} dv_{\mathrm{I},N} \\ &\stackrel{\triangle}{=} \int_{-\infty}^{x_N} f_{\boldsymbol{x}}(\boldsymbol{v})\, d\boldsymbol{v}, \end{aligned}$$

where the single integral in the last expression is used as a compact notation for a multidimensional integral and should not be confused with a complex contour integral.

Note that

$$F_x(\boldsymbol{x}) = \int_{-\infty}^{\widetilde{\boldsymbol{x}}} f_x(\widetilde{\boldsymbol{x}})\, d\widetilde{\boldsymbol{x}} = \int_{-\infty}^{\boldsymbol{x}} f_x(\boldsymbol{x})\, d\boldsymbol{x}.$$

where $\widetilde{\boldsymbol{x}} = [\boldsymbol{x}_{\mathrm{R}}^T, \boldsymbol{x}_{\mathrm{I}}^T]^T$.

For two random variables, $\boldsymbol{x} = [x, y]^T$: $f_{\boldsymbol{x}}(\boldsymbol{x}) = f_{x,y}(x, y)$.

x and y are independent if

$$f_{x,y}(x, y) = f_x(x) \cdot f_y(y) \quad \Longrightarrow \quad \mathrm{E}\{xy\} = \mathrm{E}\{x\}\mathrm{E}\{y\}.$$

Expectation of a function $g(\boldsymbol{x}(\xi))$:

$$\mathrm{E}\{g(\boldsymbol{x})\} = \int_{-\infty}^{\infty} g(\boldsymbol{x}) f_x(\boldsymbol{x}) d\boldsymbol{x}.$$

For two random variables, $\boldsymbol{x} = [x, y]^T$:

$$\mathrm{E}\{g(x, y)\} = \int_{-\infty}^{\infty} g(x, y) f_{x,y}(x, y)\, dxdy.$$

Correlation:

Real correlation:

$$r_{x,y} = \mathrm{E}\{xy\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{x,y}(x, y)\, dxdy.$$

Real covariance:

$$\begin{aligned} r_{x,y} &= \mathrm{E}\{(x - \mu_x)(y - \mu_y)\} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x - \mu_x)(y - \mu_y)\, f_{x,y}(x, y)\, dxdy. \end{aligned}$$

Complex correlation:

$$r_{x,y} = \mathrm{E}\{xy^*\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy^* f_{x,y}(x, y)\, dxdy.$$

Complex covariance:

$$\begin{aligned} r_{x,y} &= \mathrm{E}\{(x - \mu_x)(y - \mu_y)^*\} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x - \mu_x)(y - \mu_y)^*\, f_{x,y}(x, y)\, dxdy. \end{aligned}$$

Covariance Matrix:

Mean vector:

$$\boldsymbol{\mu}_x = \mathrm{E}\{\boldsymbol{x}\}.$$

Real covariance matrix:

$$\begin{aligned} R_x &= \mathrm{E}\{(\boldsymbol{x} - \mathrm{E}\{x\})(\boldsymbol{x} - \mathrm{E}\{x\})^T\} \\ &= \mathrm{E}\{\boldsymbol{x}\boldsymbol{x}^T\} - \mathrm{E}\{\boldsymbol{x}\}\mathrm{E}\{\boldsymbol{x}\}^T, \\ R_x &= \mathrm{E}\{\boldsymbol{x}\boldsymbol{x}^T\} \quad \text{if} \quad \mathrm{E}\{\boldsymbol{x}\} = \mathbf{0}. \end{aligned}$$

Complex covariance matrix:

$$\begin{aligned} R_x &= \mathrm{E}\{(\boldsymbol{x} - \mathrm{E}\{x\})(\boldsymbol{x} - \mathrm{E}\{x\})^H\} \\ &= \mathrm{E}\{\boldsymbol{x}\boldsymbol{x}^H\} - \mathrm{E}\{\boldsymbol{x}\}\mathrm{E}\{\boldsymbol{x}\}^H, \\ R_x &= \mathrm{E}\{\boldsymbol{x}\boldsymbol{x}^H\} \quad \text{if} \quad \mathrm{E}\{\boldsymbol{x}\} = \mathbf{0}. \end{aligned}$$

Observe the following property of complex correlation:

$$r_{i,k} = \mathrm{E}\{x_i x_k^*\} = \mathrm{E}\{x_k x_i^*\}^* = r_{k,i}^*.$$

Then, for $\mathrm{E}\{\boldsymbol{x}\} = \boldsymbol{0}$:

$$R_x = \mathrm{E}\{\boldsymbol{x}\boldsymbol{x}^H\} = \begin{bmatrix} r_{1,1} & r_{1,2} & \cdots & \cdots & r_{1,N} \\ r_{2,1} & r_{2,2} & \cdots & \cdots & r_{2,N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ r_{N,1} & r_{N,2} & \cdots & \cdots & r_{N,N} \end{bmatrix}$$

$$= \begin{bmatrix} r_{1,1} & r_{1,2} & \cdots & \cdots & r_{1,N} \\ r_{1,2}^* & r_{2,2} & \cdots & \cdots & r_{2,N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ r_{1,N}^* & r_{2,N}^* & \cdots & \cdots & r_{N,N} \end{bmatrix}.$$

The covariance matrix is *Hermitian*. It is *positive semidefinite* because

$$\begin{aligned} \boldsymbol{b}^H R_x \boldsymbol{b} &= \boldsymbol{b}^H \mathrm{E}\{\underbrace{(\boldsymbol{x} - \mathrm{E}\{\boldsymbol{x}\})}_{\boldsymbol{z}}(\boldsymbol{x} - \mathrm{E}\{\boldsymbol{x}\})^H\}\boldsymbol{b} \\ &= \boldsymbol{b}^H \mathrm{E}\{\boldsymbol{z}\boldsymbol{z}^H\}\boldsymbol{b} = \mathrm{E}\{|\boldsymbol{b}^H\boldsymbol{z}|^2\} \geq 0. \end{aligned}$$

Linear Transformation of Random Vectors

Linear Transformation:

$$\boldsymbol{y} = g(\boldsymbol{x}) = A\boldsymbol{x}.$$

Mean Vector:

$$\boldsymbol{\mu}_y = \mathrm{E}\{A\boldsymbol{x}\} = A\boldsymbol{\mu}_x.$$

Covariance Matrix:

$$
\begin{aligned}
R_y &= \mathrm{E}\left\{\boldsymbol{y}\boldsymbol{y}^H\right\} - \boldsymbol{\mu}_y\boldsymbol{\mu}_y^H \\
&= \mathrm{E}\left\{A\boldsymbol{x}\boldsymbol{x}^H A^H\right\} - A\boldsymbol{\mu}_x\boldsymbol{\mu}_x^H A^H \\
&= A\Big(\mathrm{E}\left\{\boldsymbol{x}\boldsymbol{x}^H\right\} - \boldsymbol{\mu}_x\boldsymbol{\mu}_x^H\Big)A^H \\
&= AR_{xx}A^H.
\end{aligned}
$$

Gaussian Random Vectors

Gaussian random variables:

$$\begin{aligned} f_x(x) &= \frac{1}{\sigma_x\sqrt{2\pi}} \exp\left\{ -\frac{(x-\mu_x)^2}{2\sigma_x^2} \right\} \quad \text{for real } x, \\ f_x(x) &= \frac{1}{\sigma_x^2 \pi} \exp\left\{ -\frac{|x-\mu_x|^2}{\sigma_x^2} \right\} \quad \text{for complex } x. \end{aligned}$$

Real Gaussian random vectors:

$$f_x(\boldsymbol{x}) = \frac{1}{(2\pi)^{N/2}|R_x|^{1/2}} \exp\left\{ -\tfrac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_x)^T R_x^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_x) \right\}.$$

Complex Gaussian random vectors:

$$f_x(\boldsymbol{x}) = \frac{1}{\pi^N |R_x|} \exp\left\{ -(\boldsymbol{x}-\boldsymbol{\mu}_x)^H R_x^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_x) \right\}.$$

Symbolic notation for *real* and *complex* Gaussian random vectors:

$$\begin{aligned} \boldsymbol{x} &\sim \mathcal{N}_{\mathrm{r}}(\boldsymbol{\mu}_x, R_x), \qquad \text{real}, \\ \boldsymbol{x} &\sim \mathcal{N}_{\mathrm{c}}(\boldsymbol{\mu}_x, R_x), \qquad \text{complex}. \end{aligned}$$

A *linear transformation* of Gaussian vector is also Gaussian, i.e. if

$$\boldsymbol{y} = A\boldsymbol{x}$$

then

$$\boldsymbol{y} \sim \mathcal{N}_{\rm r}(A\boldsymbol{\mu}_x, AR_xA^T) \qquad \text{real,}$$
$$\boldsymbol{y} \sim \mathcal{N}_{\rm c}(A\boldsymbol{\mu}_x, AR_xA^H) \qquad \text{complex.}$$

Complex Gaussian Distribution

Consider joint pdf of real and imaginary part of a complex vector $\boldsymbol{x}$

$$\boldsymbol{x} = \boldsymbol{u} + j\boldsymbol{v}.$$

Assume $\boldsymbol{z} = [\boldsymbol{u}^T, \boldsymbol{v}^T]^T$. The $2n$-variate Gaussian pdf of the (real!) vector $\boldsymbol{z}$ is

$$f_z(\boldsymbol{z}) = \frac{1}{\sqrt{(2\pi)^{2n}|R_z|}} \exp\left[-\tfrac{1}{2}(\boldsymbol{z} - \boldsymbol{\mu}_z)^T R_z^{-1}(\boldsymbol{z} - \boldsymbol{\mu}_z)\right],$$

where

$$\boldsymbol{\mu}_z = \begin{bmatrix} \boldsymbol{\mu}_u \\ \boldsymbol{\mu}_v \end{bmatrix}, \quad R_z = \begin{bmatrix} R_{uu} & R_{uv} \\ R_{vu} & R_{vv} \end{bmatrix}.$$

That is

$$P[\boldsymbol{z} \in \Omega] = \int_{\boldsymbol{z} \in \Omega} p_Z(\boldsymbol{z})\, d\boldsymbol{z}.$$

Complex Gaussian Distribution (cont.)

Suppose R_z happens to have a special structure:

$$R_{uu} = R_{vv} \quad \text{and} \quad R_{uv} = -R_{vu}.$$

(Note that $R_{uv} = R_{vu}^T$ by construction.) Then, we can define a complex Gaussian pdf

$$f_x(\boldsymbol{x}) = \frac{1}{\pi^n |R_x|} \exp\left[-(\boldsymbol{x} - \boldsymbol{\mu}_x)^H R_x^{-1} (\boldsymbol{x} - \boldsymbol{\mu}_x)\right],$$

where

$$\begin{aligned}
\boldsymbol{\mu}_x &= \boldsymbol{\mu}_u + j\boldsymbol{\mu}_v \\
R_x &= \mathrm{E}\left\{(\boldsymbol{x} - \boldsymbol{\mu}_x)(\boldsymbol{x} - \boldsymbol{\mu}_x)^H\right\} = 2(R_{uu} + jR_{vu}) \\
\mathbf{0} &= \mathrm{E}\left\{(\boldsymbol{x} - \boldsymbol{\mu}_x)(\boldsymbol{x} - \boldsymbol{\mu}_x)^T\right\}.
\end{aligned}$$

Covariance of White Noise and Prewhitening Operation

Covariance matrix for white uniform zero-mean noise:

$$R_x = \sigma^2 I,$$

where σ^2 is the noise variance.

Very important operation is *prewhitening* of a nonwhite process. Assume that the process has the covariance matrix $R_x \neq \sigma^2 I$. Then, prewhitening operation can be written as

$$\boldsymbol{y} = W R_x^{-1/2} \boldsymbol{x}$$

where W is any *unitary* matrix ($W^H = W^{-1}$).

$$\begin{aligned}
R_y &= \mathrm{E}\{\boldsymbol{y}\boldsymbol{y}^H\} \\
&= \mathrm{E}\{W R_x^{-1/2} \boldsymbol{x}\boldsymbol{x}^H R_x^{-H/2} W^H\} \\
&= W R_x^{-1/2} \mathrm{E}\{\boldsymbol{x}\boldsymbol{x}^H\} \mathrm{E}\, R_x^{-H/2} W^H \\
&= W \left(R_x^{-1/2} R_x R_x^{-1/2} \right) W^H \\
&= W W^H = I.
\end{aligned}$$

We can define an arbitrary (noninteger) power of R_x as

$$R_x^q = \sum_{i=1}^{N} \lambda_i^q \boldsymbol{u}_i \boldsymbol{u}_i^H.$$

Prewhitening matrix is *not unique*!

Discrete-time Stochastic Processes

Definition. [Wide-Sense Stationarity (WSS)] *A random process $x(n)$ is WSS if*

its mean is a constant:

$$\mathrm{E}\left[x(n)\right] = \mu_x,$$

its autocorrelation $r_x(n_1, n_2)$ depends only on the difference $n_1 - n_2$:

$$r_x(n, n-l) = \mathrm{E}\left[x(n)x^*(n-l)\right] = r_x(l),$$

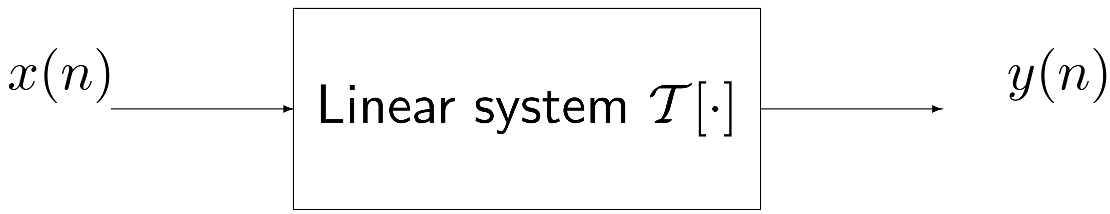
and its variance is finite:

$$c_x(0) = \mathrm{E}\left\{|x(n) - \mu_x|^2\right\} < \infty.$$

Power Spectral Density (PSD): The PSD (or auto-PSD) of a stationary stochastic process is a Fourier transform of its autocorrelation sequence:

$$P_x(e^{j\omega}) = \sum_{l=-\infty}^{\infty} r_x(l) e^{-j\omega l}.$$

Linear Systems with Stationary Random Inputs:



Output mean value:

$$\mu_y = \mathrm{E}\left[y(n)\right] = \mathrm{E}\Big[\sum_{k=-\infty}^{\infty} h(k)x(n-k)\Big]$$

$$= \sum_{k=-\infty}^{\infty} h(k)\mathrm{E}\big[x(n-k)\big] = \mu_x \sum_{k=-\infty}^{\infty} h(k) = \mu_x H(e^{j0}).$$

Output autocorrelation:

$$r_{y,x}(n+k,n) = \mathrm{E}\left[y(n+k)x^*(n)\right]$$

$$= \mathrm{E}\,[\sum_{l=-\infty}^{\infty} h(l)x(n+k-l)\cdot x^*(n)]$$

$$= \sum_{l=-\infty}^{\infty} h(l)\underbrace{\mathrm{E}\left[x(n+k-l)\cdot x^*(n)\right]}_{r_x(k-l)}$$

$$= \sum_{l=-\infty}^{\infty} h(l)r_x(k-l).$$

$$r_{y,x}(n+k,n) = r_{y,x}(k) = r_x(k) \star h(k).$$

$$\begin{aligned}
r_y(n+k,n) &= \mathrm{E}\left[y(n+k)y^*(n)\right] \\
&= \mathrm{E}\left[y(n+k) \sum_{l=-\infty}^{\infty} x^*(l)h^*(n-l)\right] \\
&= \sum_{l=-\infty}^{\infty} h^*(n-l) \underbrace{\mathrm{E}\left[y(n+k)x^*(l)\right]}_{r_{y,x}(n+k-l)} \\
&\overset{m=n-l}{=} \sum_{m=-\infty}^{\infty} h^*(m) r_{y,x}(m+k) \\
&\overset{p=-m}{=} \sum_{p=-\infty}^{\infty} h^*(-p) r_{y,x}(k-p) \\
&= r_{y,x}(k) \star h^*(-k) = r_y(k).
\end{aligned}$$

So

$$r_y(k) = r_{y,x}(k) \star h^*(-k) = \{r_x(k)\} \star \{h(k)\} \star \{h^*(-k)\}.$$

The corresponding PSD:

$$P_y(z) = P_x(z)H(z)H^*\Big(\frac{1}{z^*}\Big).$$

Wide-Sense Stationary Process

For wide-sense stationary zero-mean sequence $\{x_i\}$:

$$\mathrm{E}\{x_i\} = 0, \quad r_{i,k} = r_{i-k},$$

the covariance matrix is Toeplitz:

$$R_x = \mathrm{E}\{\boldsymbol{x}\boldsymbol{x}^H\} = \begin{bmatrix} r_0 & r_1^* & r_2^* & \cdots & r_{N-2}^* & r_{N-1}^* \\ r_1 & r_0 & r_1^* & \cdots & r_{N-3}^* & r_{N-2}^* \\ r_2 & r_1 & r_0 & \cdots & r_{N-4}^* & r_{N-3}^* \\ r_3 & r_2 & r_1 & \cdots & r_{N-5}^* & r_{N-4}^* \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ r_{N-1} & r_{N-2} & r_{N-3} & \cdots & r_1 & r_0 \end{bmatrix}.$$

Example: Consider

$$\boldsymbol{x} = \begin{bmatrix} x(n) \\ x(n-1) \\ \vdots \\ x(n-N+1) \end{bmatrix}.$$

In this case, the covariance matrix is

$$R_x = \mathrm{E}\left\{\boldsymbol{x}\boldsymbol{x}^H\right\} =$$

$$\begin{bmatrix} \mathrm{E}\left\{|x(n)|^2\right\} & \mathrm{E}\left\{x(n)x^*(n-1)\right. & \cdots & \mathrm{E}\left\{x(n)x^*(n-N+1)\right. \\ \mathrm{E}\left\{x(n-1)x^*(n)\right\} & \mathrm{E}\left\{|x(n-1)|^2\right\} & \cdots & \mathrm{E}\left\{x(n-1)x^*(n-N+1)\right\} \\ \vdots & \vdots & \vdots & \vdots \\ \mathrm{E}\left\{x(n-N+1)x^*(n)\right\} & \cdots & \cdots & \mathrm{E}\left\{|x(n-N+1)|^2\right\} \end{bmatrix}$$

and, therefore, the stationarity assumption means

$$\mathrm{E}\left\{x(k)x^*(k-m)\right\} = r_x(m).$$