Frequency-Domain Analysis

Fourier Series

Consider a continuous complex signal

$$x(t) \in [-T/2, T/2].$$

Represent $x(t)$ using an arbitrary orthonormal basis $\varphi_n(t)$:

$$x(t) = \sum_{n=-\infty}^{\infty} \alpha_n \varphi_n(t)$$

Orthonormality condition:

$$\frac{1}{T} \int_{-T/2}^{T/2} \varphi_n(t) \varphi_k^*(t) dt = \delta(n-k).$$

Multiplying the above expansion with $\varphi_k^*(t)$ and integrating over the interval, we obtain

$$\begin{aligned} \frac{1}{T} \int_{-T/2}^{T/2} x(t) \varphi_k^*(t) dt &= \frac{1}{T} \int_{-T/2}^{T/2} \sum_{n=-\infty}^{\infty} \alpha_n \varphi_n(t) \varphi_k^*(t) dt \\ &= \sum_{n=-\infty}^{\infty} \alpha_n \left(\frac{1}{T} \int_{-T/2}^{T/2} \varphi_n(t) \varphi_k^*(t) dt \right) \\ &= \sum_{n=-\infty}^{\infty} \alpha_n \delta(n-k) = \alpha_k. \end{aligned}$$

Thus, the coefficients of expansion are given by

$$\alpha_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\varphi_k^*(t)dt.$$

Proposition. *The functions* $\varphi_n(t) = \exp(j2\pi nt/T)$ *are orthonormal at the interval* $[-T/2, T/2]$.

Proof.

$$\begin{aligned}\frac{1}{T}\int_{-T/2}^{T/2} \varphi_n(t)\varphi_k^*(t)dt &= \frac{1}{T}\int_{-T/2}^{T/2} e^{j\frac{2\pi(n-k)}{T}t}dt \\ &= \frac{\sin[\pi(n-k)]}{\pi(n-k)} = \delta(n-k).\end{aligned}$$

□

Thus, we can take exponential functions $\varphi_n(t) = \exp(j2\pi nt/T)$ as orthonormal basis $\Longrightarrow$ we obtain Fourier series.

Fourier series for a periodic signal $x(t) = x(t+T)$:

$$\begin{aligned} x(t) &= \sum_{n=-\infty}^{\infty} X_n e^{j\frac{2\pi n}{T}t} \\ X_n &= \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-j\frac{2\pi n}{T}t}dt.\end{aligned}$$

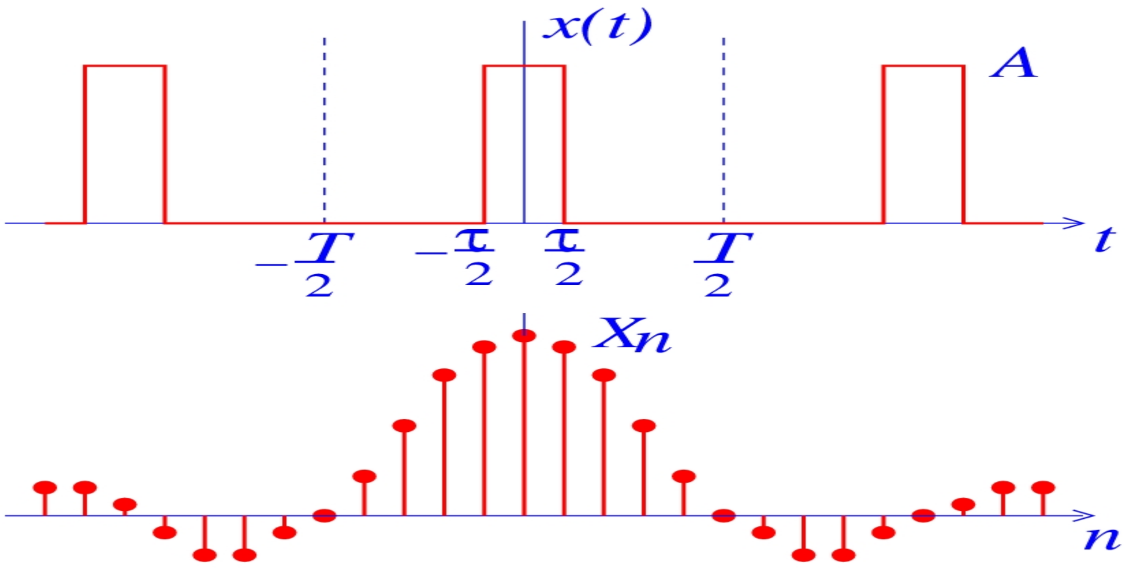
Fourier coefficients can be viewed as a signal spectrum:

$$X_n \sim X(\Omega_n), \quad \text{where} \quad \Omega_n = \frac{2\pi n}{T} \quad \Rightarrow$$

Fourier series can be applied to analyze signal spectrum! Also, this interpretation implies that periodic signals have discrete spectrum.

Example: Periodic sequence of rectangles:

$$\begin{aligned} X_n &= \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-j\frac{2\pi n}{T}t} dt = \frac{1}{T}\int_{-\tau/2}^{\tau/2} A e^{-j\frac{2\pi n}{T}t} dt \\ &= \frac{A\tau}{T} \cdot \frac{\sin(\pi n \frac{\tau}{T})}{\pi n \frac{\tau}{T}} \quad \text{real coefficients.} \end{aligned}$$



Remarks:

- In general, Fourier coefficients are complex-valued,
- For real signals, $X_{-n} = X_n^*$.
- Alternative expressions exist for trigonometric Fourier series, exploiting summation of sine and cosine functions.

Convergence of Fourier Series

Dirichlet conditions:

Condition 1. $x(t)$ is *absolutely integrable* over one period, i. e.

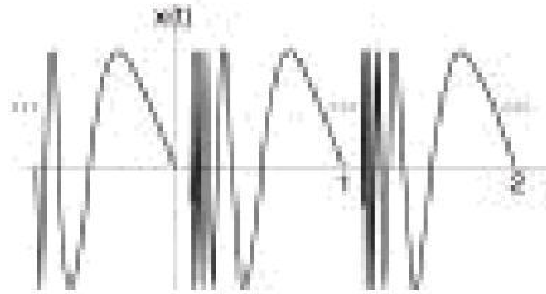
$$\int_T |x(t)|dt < \infty$$

And

Condition 2. In a finite time interval, $x(t)$ has a *finite* number of maxima and minima.

Ex. An example that violates Condition 2.

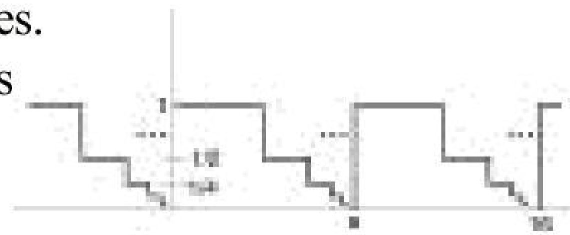
$$x(t) = \sin\left(\frac{2\pi}{t}\right) \quad 0 < t \leq 1$$



And

Condition 3. In a finite time interval, $x(t)$ has only a *finite* number of discontinuities.

Ex. An example that violates Condition 3.



Dirichlet conditions are met for most of the signals encountered in the real world.

Still, convergence has some interesting characteristics:

$$x_N(t) = \sum_{n=-N}^{N} X_n e^{j\frac{2\pi n}{T}t}.$$

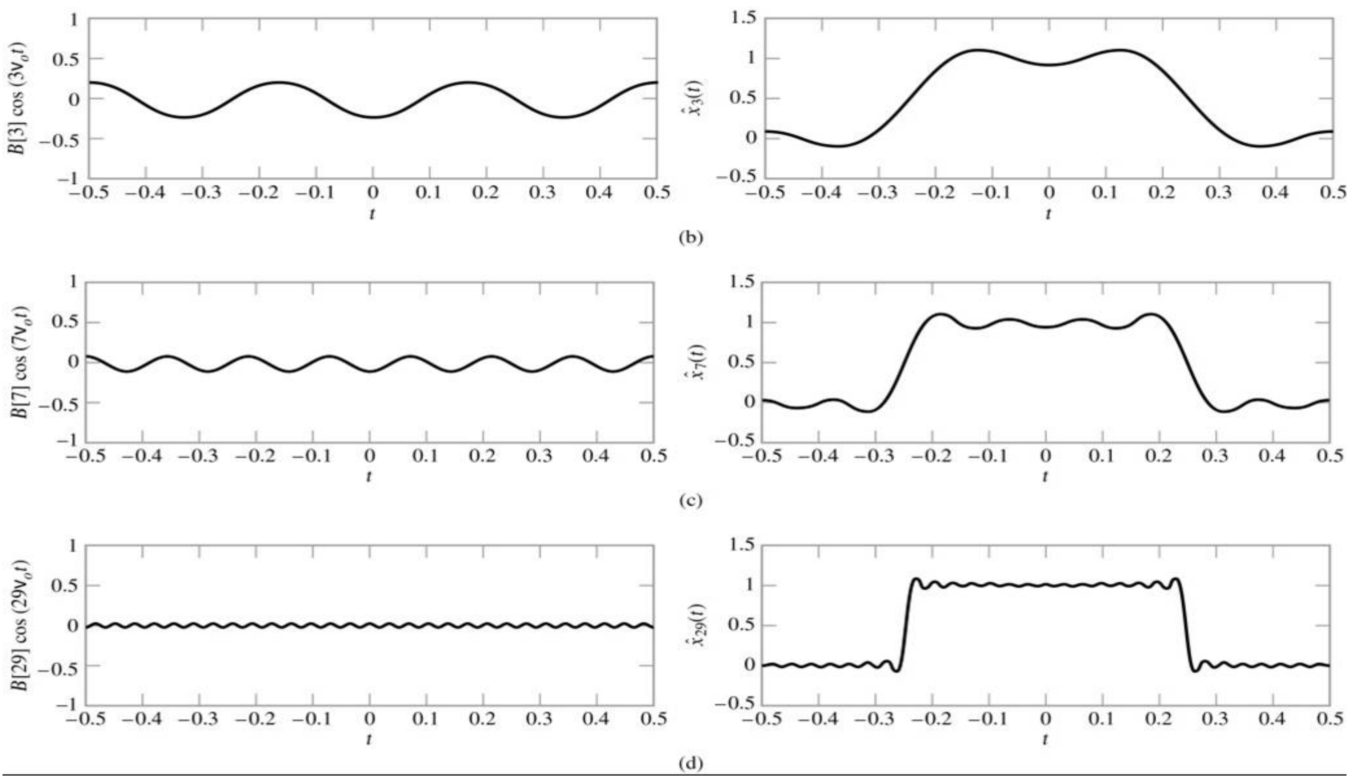
As $N \to \infty$, $x_N(t)$ exhibits Gibbs' phenomenon at points of discontinuity. Under the Dirichlet conditions:

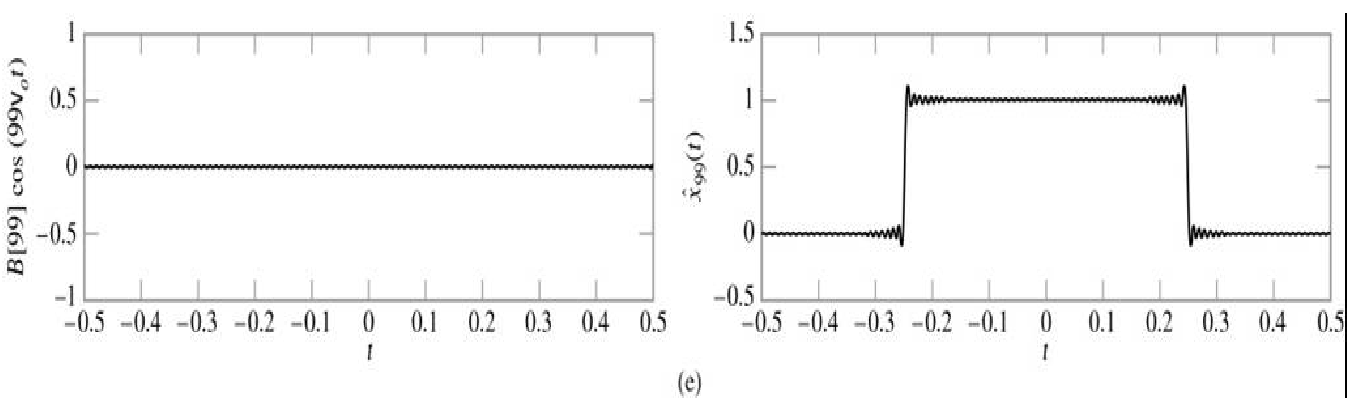
- The Fourier series $= x(t)$ at points where $x(t)$ is continuous,

- The Fourier series $=$ "midpoint" at points of discontinuity.

Demo: Fourier series for continuous-time square wave (Gibbs' phenomenon).

http://www.jhu.edu/~signals/fourier2/index.html.

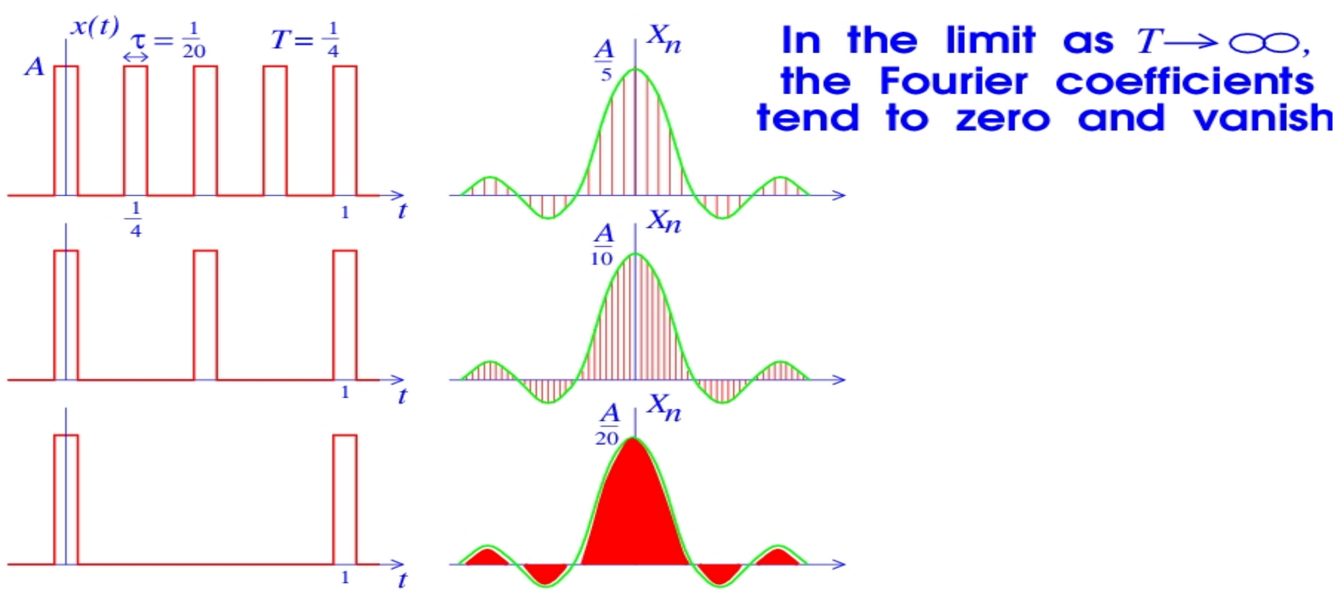




Review of Continuous-time Fourier Transform

What about Fourier representations of nonperiodic continuous-time signals?

Assuming a finite-energy signal and $T \to \infty$ in the Fouries series, we get $\lim_{T\to\infty} X_n = 0$.



Trick: To preserve the Fourier coefficients from disappearing as $T \to \infty$, introduce

$$\widetilde{X}_n = T X_n = \int_{-T/2}^{T/2} x(t) e^{-j\frac{2\pi n}{T}t} dt.$$

Transition to Fourier transform:

$$
\begin{aligned}
X(\Omega) &= \lim_{T\to\infty} \widetilde{X}_n \\
&= \lim_{T\to\infty} \int_{-T/2}^{T/2} x(t) e^{-j\frac{2\pi n}{T}t} dt \\
&= \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt,
\end{aligned}
$$

where the "discrete" frequency $2\pi n/T$ becomes the continuous frequency Ω.

Transition to inverse Fourier transform:

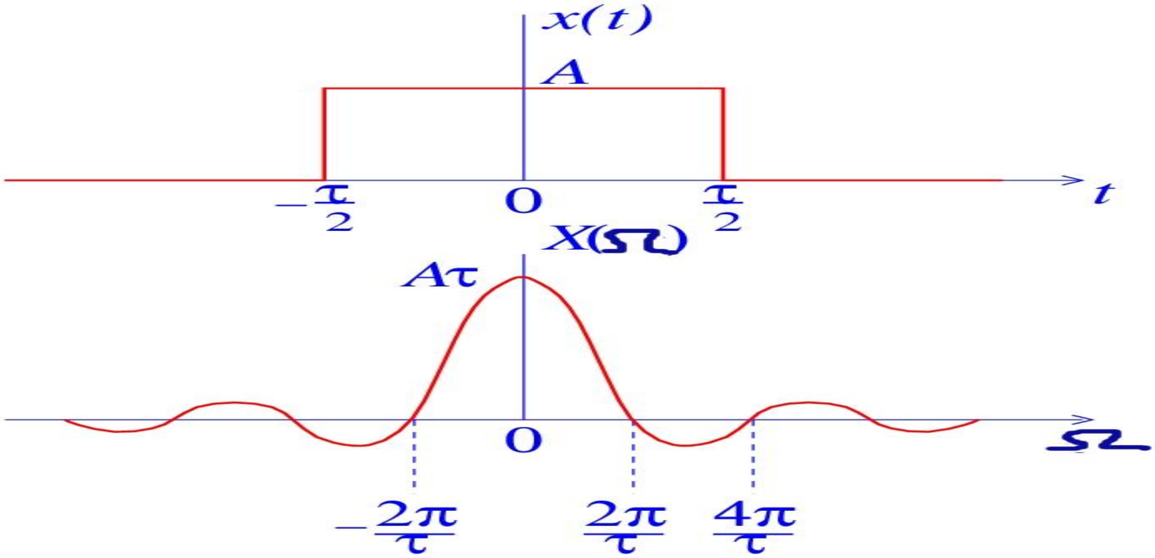
$$
\begin{aligned}
x(t) &= \lim_{T\to\infty} \sum_{n=-\infty}^{\infty} X_n e^{j\frac{2\pi n}{T}t} = \lim_{T\to\infty} \sum_{n=-\infty}^{\infty} \frac{\widetilde{X}_n}{T} e^{j\frac{2\pi n}{T}t} \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega) e^{j\Omega t} d\Omega \ \Leftarrow \ d\Omega = \frac{2\pi}{T}, \ \Omega = \frac{2\pi n}{T}.
\end{aligned}
$$

Continuous-time Fourier transform (CTFT):

$$
\begin{aligned}
X(\Omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt, \\
x(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega) e^{j\Omega t} d\Omega.
\end{aligned}
$$

Example: Finite-energy rectangular signal:

$$
\begin{aligned}
X(\Omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt \\
&= \int_{-\tau/2}^{\tau/2} A e^{-j\Omega t} dt \\
&= A\tau \frac{\sin(\Omega\tau/2)}{\Omega\tau/2} \quad \text{real spectrum.}
\end{aligned}
$$



Remarks:

- In general, Fourier spectrum is complex-valued,
- For real signals, $X(-\Omega) = X^*(\Omega)$.

Dirac Delta Function

Definition:

$$\delta(t) = \begin{cases} \infty, & t = 0, \\ 0, & t \neq 0 \end{cases}, \quad \int_{-\infty}^{\infty} \delta(t) dt = 1.$$

Do not confuse continuous-time $\delta(t)$ with discrete-time $\delta(n)$!

Sifting property:

$$\int_{-\infty}^{\infty} f(t)\delta(t-\tau) dt = f(\tau).$$

The spectrum of $\delta(t - t_0)$:

$$\begin{aligned} X(\Omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt \\ &= \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j\Omega t} dt \\ &= e^{-j\Omega t_0}. \end{aligned}$$

Delta function in time domain:

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\Omega t} d\Omega.$$

Delta function in frequency domain:

$$\delta(\Omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-j\Omega t} dt = \begin{cases} \infty, & \Omega = 0, \\ 0, & \Omega \neq 0 \end{cases} .$$

For signal

$$x(t) = A e^{j\Omega_0 t}$$

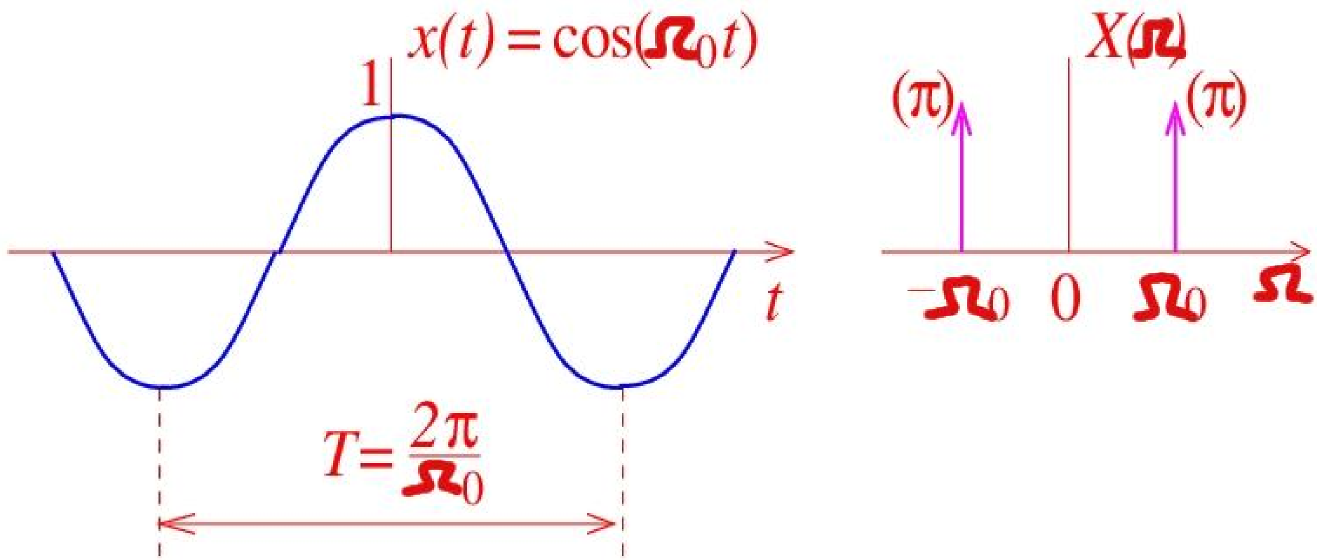
we have

$$\begin{aligned} X(\Omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt \\ &= A \int_{-\infty}^{\infty} e^{-j(\Omega - \Omega_0)t} dt \\ &= A\, 2\pi\, \delta(\Omega - \Omega_0). \end{aligned}$$

Harmonic Fourier Pairs

Delta function in frequency domain:

$$
\begin{aligned}
e^{j\Omega_0 t} &\leftrightarrow 2\pi\,\delta(\Omega - \Omega_0), \\
\cos(\Omega_0 t) &\leftrightarrow \pi\,[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)], \\
\sin(\Omega_0 t) &\leftrightarrow \frac{\pi}{j}\,[\delta(\Omega - \Omega_0) - \delta(\Omega + \Omega_0)].
\end{aligned}
$$



Parseval's Theorem for CTFT

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\Omega)|^2 d\Omega$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\Omega)|^2 d\Omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t)x^*(\tau)e^{-j\Omega(t-\tau)} dt d\tau \right\} d\Omega$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t)x^*(\tau) \underbrace{\left\{ \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\Omega(\tau-t)} d\Omega \right\}}_{\delta(\tau-t)} dt d\tau$$

$$= \int_{-\infty}^{\infty} |x(t)|^2 dt.$$

Discrete-time Fourier Transform (DTFT)

Represent continuous signal $x(t)$ via discrete sequence $x(n)$:

$$x(t) = \sum_{n=-\infty}^{\infty} x(n)\delta(t - nT).$$

Substituting this equation into the CTFT formula, we obtain:

$$\begin{aligned} X(\Omega) &= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x(n)\delta(t - nT)e^{-j\Omega t}dt \\ &= \sum_{n=-\infty}^{\infty} x(n) \int_{-\infty}^{\infty} \delta(t - nT)e^{-j\Omega t}dt \\ &= \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega nT}. \end{aligned}$$

Switch to the discrete-time frequency, i.e. use $\omega = \Omega T$:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}.$$

$X(\omega)$ is periodic with period 2π:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \underbrace{e^{-j2\pi n}}_{1}$$

$$= \sum_{n=-\infty}^{\infty} x(n) e^{-j(\omega+2\pi)n} = X(\omega + 2\pi).$$

Trick: in computing inverse DTFT, use only one period of $X(\omega)$:

$$\begin{aligned} X(\omega) &= \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \quad \text{DTFT}, \\ x(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega \quad \text{Inverse DTFT}. \end{aligned}$$

Inverse DTFT:
Proof.

$$\begin{aligned} x(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{m=-\infty}^{\infty} x(m) e^{j\omega(n-m)} d\omega \\ &= \sum_{m=-\infty}^{\infty} x(m) \underbrace{\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\omega(n-m)} d\omega}_{\delta(n-m)} = x(n). \end{aligned}$$

□

Fourier Series vs. DTFT

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{j\frac{2\pi n}{T}t}, \quad X_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-j\frac{2\pi n}{T}t} dt \text{ FS}$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}, \quad x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega \text{ DTFT}$$

Observation: Replacing, in Fourier Series

$$\begin{aligned} x(t) &\rightarrow X(\omega), \\ X_n &\rightarrow x(n), \\ t &\rightarrow -\omega, \\ T &\rightarrow 2\pi, \end{aligned}$$

we obtain DTFT!

An important conclusion: DTFT is equivalent to Fourier series but applied to the "opposite" domain. In Fourier series, a periodic continuous signal is represented as a sum of exponentials weighted by discrete Fourier (spectral) coefficients. In DTFT, a periodic continuous spectrum is represented as a sum of exponentials, weighted by discrete signal values.

Remarks:

- DTFT can be derived directly from the Fourier series,
- All Fourier series results can be applied to DTFT
- Duality between time and frequency domains.

Parseval's Theorem for DTFT

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x(n)x^*(m)e^{-j\omega(n-m)}d\omega$$

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x(n)x^*(m) \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\omega(n-m)}d\omega}_{\delta(n-m)}$$

$$= \sum_{n=-\infty}^{\infty} |x(n)|^2.$$

When Does DTFT Exist (i.e. $|X(\omega)| < \infty$)?

Sufficient condition:

$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty.$$

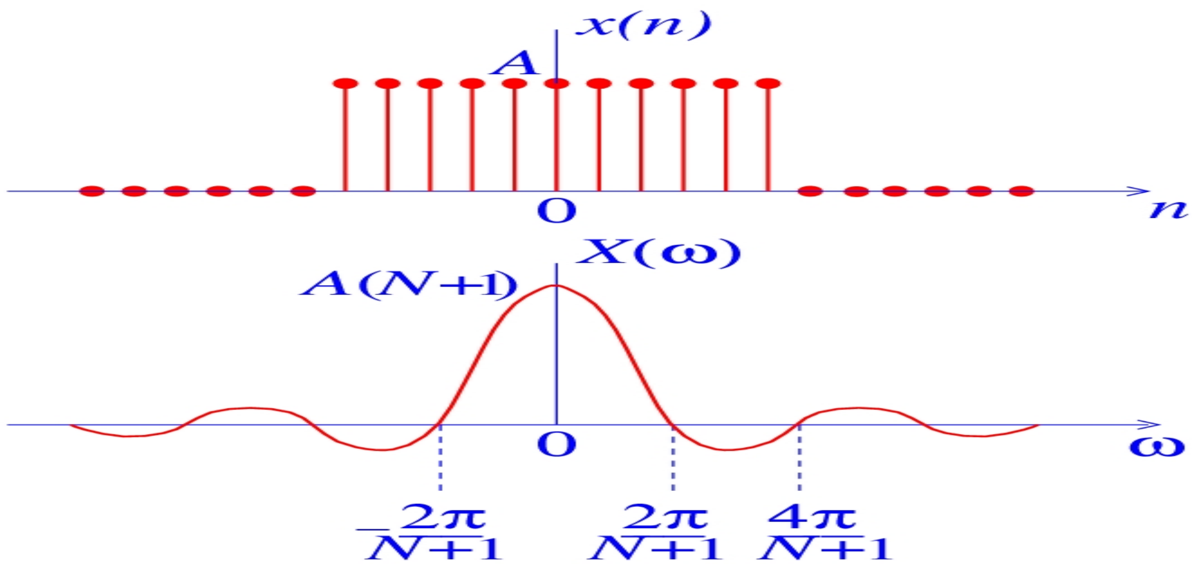
$$\begin{aligned} |X(\omega)| &= \left| \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \right| \\ &\leq \sum_{n=-\infty}^{\infty} |x(n)| \underbrace{|e^{-j\omega n}|}_{1} \\ &= \sum_{n=-\infty}^{\infty} |x(n)| < \infty. \end{aligned}$$

Example: Finite-energy rectangular signal:

$$\begin{aligned} |X(\omega)| &= \sum_{n=-N/2}^{N/2} A e^{-j\omega n} = A \sum_{n=-N/2}^{N/2} e^{-j\omega n} \\ &= A(N+1) \frac{\sin(\frac{N+1}{2}\omega)}{(N+1)\sin(\frac{\omega}{2})} \end{aligned}$$

$$\approx \quad A(N+1) \qquad \underbrace{\frac{\sin(\frac{N+1}{2}\omega)}{(N+1)\frac{\omega}{2}}}_{\text{well-known function}} \qquad \text{for } \omega \ll \pi$$

Both functions look very similar in their "mainlobe" domain.



DTFT — Convolution Theorem

If $X(\omega) = \mathcal{F}\{x(n)\}$ and $H(\omega) = \mathcal{F}\{h(n)\}$ and

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) = \{x(n)\} \star \{h(n)\}$$

then $Y(\omega) = \mathcal{F}\{y(n)\} = X(\omega)H(\omega)$.

$$\begin{aligned}
Y(\omega) &= \mathcal{F}\{y(n)\} = \sum_{n=-\infty}^{\infty} \left\{ \sum_{k=-\infty}^{\infty} x(k)h(\underbrace{n-k}_{m}) \right\} e^{-j\omega n} \\
&= \sum_{m=-\infty}^{\infty} \left\{ \sum_{k=-\infty}^{\infty} x(k)h(m)e^{-j\omega(m+k)} \right\} \\
&= \left\{ \sum_{k=-\infty}^{\infty} x(k)e^{-j\omega k} \right\} \left\{ \sum_{m=-\infty}^{\infty} h(m)e^{-j\omega m} \right\} \\
&= X(\omega)H(\omega).
\end{aligned}$$

Windowing theorem: If $X(\omega) = \mathcal{F}\{x(n)\}$, $W(\omega) = \mathcal{F}\{w(n)\}$, and $y(n) = x(n)w(n)$, then

$$Y(\omega) = \mathcal{F}\{y(n)\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\lambda)X_2(\omega - \lambda)d\lambda.$$

Frequency-Domain Characteristics of LTI Systems

Recall impulse response $h(n)$ of an LTI system:

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k).$$

Consider input sequence $x(n) = e^{j\omega n}$, $-\infty < n < \infty$.

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)e^{j\omega(n-k)} = e^{j\omega n} \underbrace{\sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}}_{H(\omega)} = e^{j\omega n} H(\omega).$$

The complex function

$$\sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}$$

is called the *frequency response* or the *transfer function* of the system.

- Impulse response and transfer function represent a DTFT pair $\Longrightarrow H(\omega)$ is a *periodic* function.

- Transfer function shows how different input frequency components are changed (e.g. attenuated) at system output.

- $Y(\omega) = X(\omega)H(\omega)$ implies that an LTI system cannot generate any new frequencies, i.e. it can only amplify or reduce/remove frequency components of the input. Conversely, if a system generates new frequencies, then it is not LTI!

- Systems that are not LTI do not have a meaningful frequency response.

Elements of Sampling Theory
Preliminaries

How are CTFT and Fourier series related for periodic signals? Consider a continuous-time signal $x_c(t)$ with CTFT

$$X(\Omega) = 2\pi\delta(\Omega - \Omega_0).$$

Then

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\Omega - \Omega_0)e^{j\Omega t}dt = e^{j\Omega_0 t}.$$

We know: periodic signal has line equispaced spectrum. Let $X(\Omega)$ be a linear combination of impulses equally spaced in frequency:

$$X(\Omega) = \sum_{n=-\infty}^{\infty} 2\pi X_n \delta(\Omega - n\Omega_0). \tag{1}$$

Using inverse CTFT, i.e. applying it to each term in the sum, we obtain:

$$\begin{aligned} x(t) &= \sum_{n=-\infty}^{\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi X_n \delta(\Omega - n\Omega_0)e^{j\Omega t}d\Omega \\ &= \sum_{n=-\infty}^{\infty} X_n e^{jn\Omega_0 t} \qquad \text{Fourier series!} \end{aligned}$$

CTFT of a periodic signal with Fourier-series coefficients $\{X_n\}$ can be interpreted as a train of impulses occurring at the harmonically-related frequencies with the weights $\{2\pi X_n\}$.

How about the following signal (periodic impulse train), defined as

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)?$$

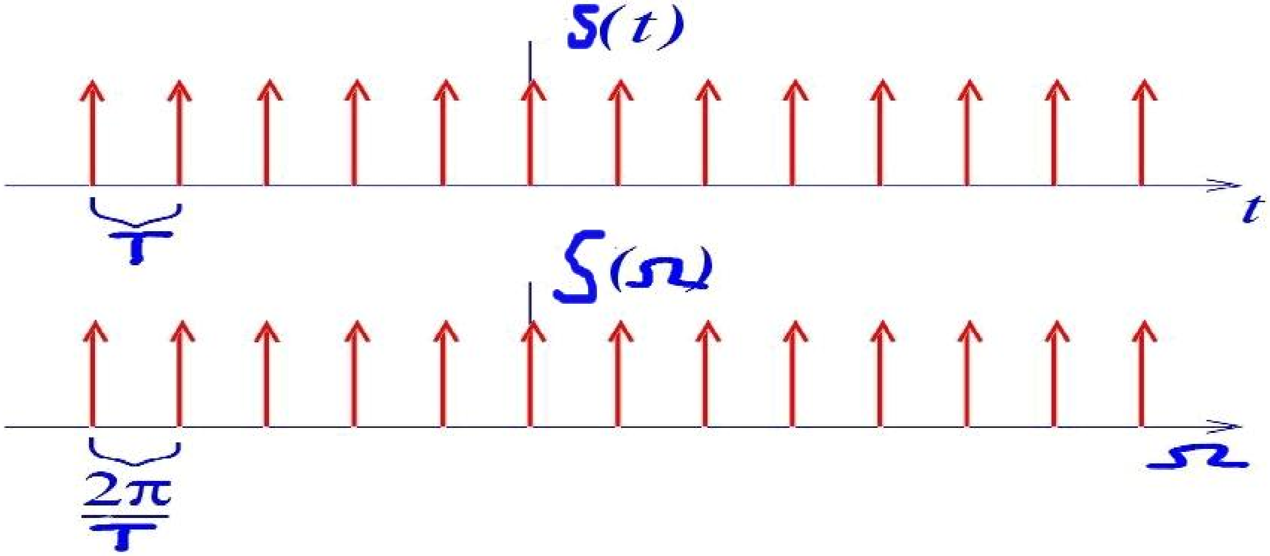
Note: The above periodic impulse train does not satisfy the Dirichlet conditions. Hence, its CTFT is introduced and understood in a limiting sense.

Here are couple of useful expressions for the periodic impulse train:

$$\begin{aligned} \sum_{k=-\infty}^{\infty} \delta(t - kT) &= \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn(2\pi/T)t}, \\ \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta\Big(\Omega - \frac{2\pi n}{T}\Big) &= \sum_{k=-\infty}^{\infty} e^{-j\Omega kT}. \end{aligned}$$

Also, the Fourier transform of a periodic impulse train is a periodic impulse train:

$$\sum_{k=-\infty}^{\infty} \delta(t - kT) \quad \leftrightarrow \quad \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta\Big(\Omega - \frac{2\pi n}{T}\Big).$$



Proof.

The impulse train $s(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$ is a periodic signal with period $T \Longrightarrow$ we can apply Fourier series and find the Fourier coefficients:

$$S_n = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-j\frac{2\pi n}{T}t} dt = \frac{1}{T}.$$

Hence

$$s(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn(2\pi/T)t}.$$

Also

$$S(\Omega) = \int_{-\infty}^{\infty} s(t) e^{-j\Omega t} dt = \sum_{k=-\infty}^{\infty} e^{-j\Omega kT}.$$

From (1), we obtain

$$S(\Omega) = \sum_{n=-\infty}^{\infty} 2\pi \underbrace{S_n}_{1/T} \delta(\omega - n \underbrace{\Omega_0}_{\frac{2\pi}{T}}) = \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta\Big(\Omega - \frac{2\pi n}{T}\Big).$$

□

The Sampling Theorem

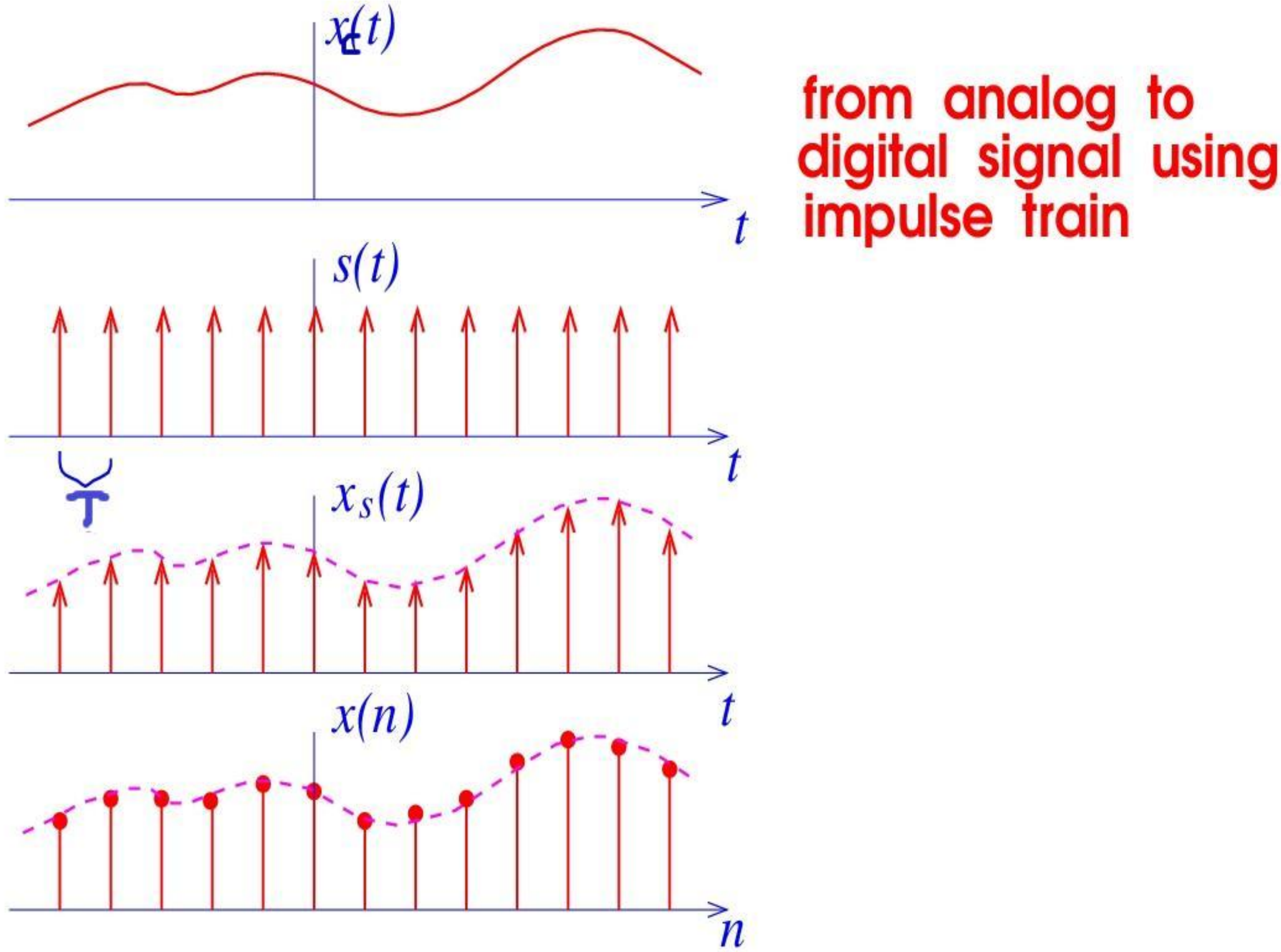
Introduce the "modulated" signal

$$x_{\mathrm{s}}(t) = x_{\mathrm{c}}(t)s(t) = x_{\mathrm{c}}(t) \sum_{n=-\infty}^{\infty} \delta(t - nT).$$

Since $x_{\mathrm{c}}(t)\,\delta(t - t_0) = x_{\mathrm{c}}(t_0)\,\delta(t - t_0)$, we obtain

$$x_{\mathrm{s}}(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\delta(t - nT) = \sum_{n=-\infty}^{\infty} x(n)\,\delta(t - nT).$$

Using this "modulated" signal, we describe the sampling operation.



$$
\begin{aligned}
x_{\rm s}(t) &= x_{\rm c}(t)\, s(t) = x_{\rm c}(t) \cdot \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn(2\pi/T)t} \\
&= \frac{1}{T} \sum_{n=-\infty}^{\infty} x_{\rm c}(t)\, e^{jn(2\pi/T)t}.
\end{aligned}
$$

It turns out that the problem is much easier to understand in the frequency domain. Hence, we compute the Fourier transform of $x_{\rm s}(t)$. Looking at each term of the summation, we have from the frequency-shift theorem:

$$x_{\rm c}(t)\, e^{jn(2\pi/T)t} \leftrightarrow X_{\rm c}\Big(\Omega - n\frac{2\pi}{T}\Big).$$

Hence, the Fourier transform of the sum is

$$\boxed{X_{\rm s}(\Omega) = \tfrac{1}{T}\textstyle\sum_{n=-\infty}^{\infty} X_{\rm c}\Big(\Omega - \tfrac{2\pi n}{T}\Big).}$$

Recall that:

$$x_{\rm s}(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\delta(t-nT) = \sum_{n=-\infty}^{\infty} x(n)\,\delta(t-nT).$$

Taking the FT of the above expression, we obtain another expression for $X_{\rm s}(\Omega)$:

$$\begin{aligned} X_{\rm s}(\Omega) &= \sum_{n=-\infty}^{\infty} x(n) \int_{-\infty}^{\infty} \delta(t-nT) e^{-j\Omega t} dt \\ &= \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega NT} = \underbrace{X(\omega)}_{\mathrm{DTFT}\{x(n)\}} \Big|_{\omega=\Omega T}. \end{aligned}$$

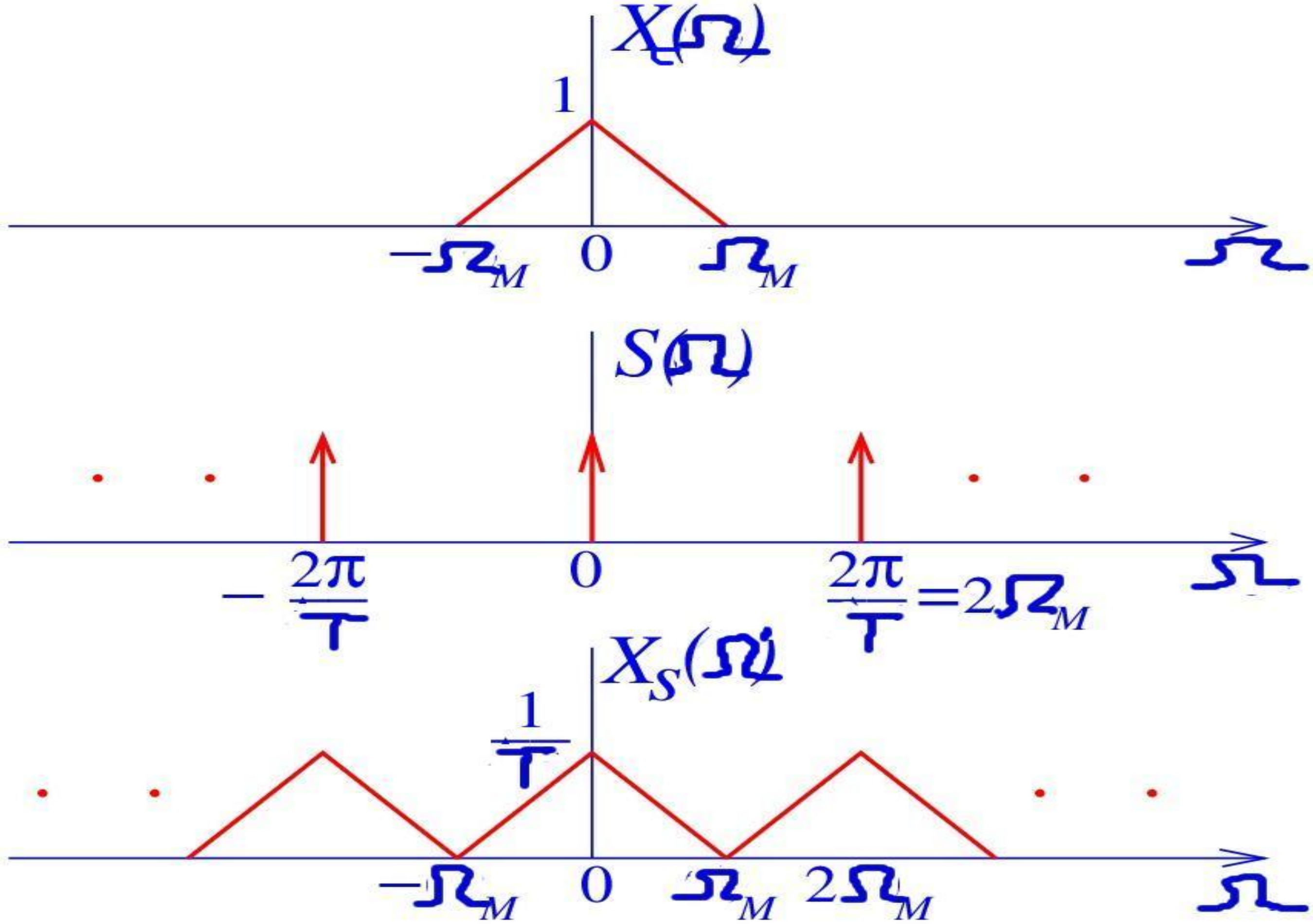
By sampling, we throw out a lot of information: all values of $x(t)$ between the sampling points are lost.

Question: Under which conditions can we reconstruct the original continuous-time signal $x(t)$ from the sampled signal $x_{\mathrm{s}}(t)$?

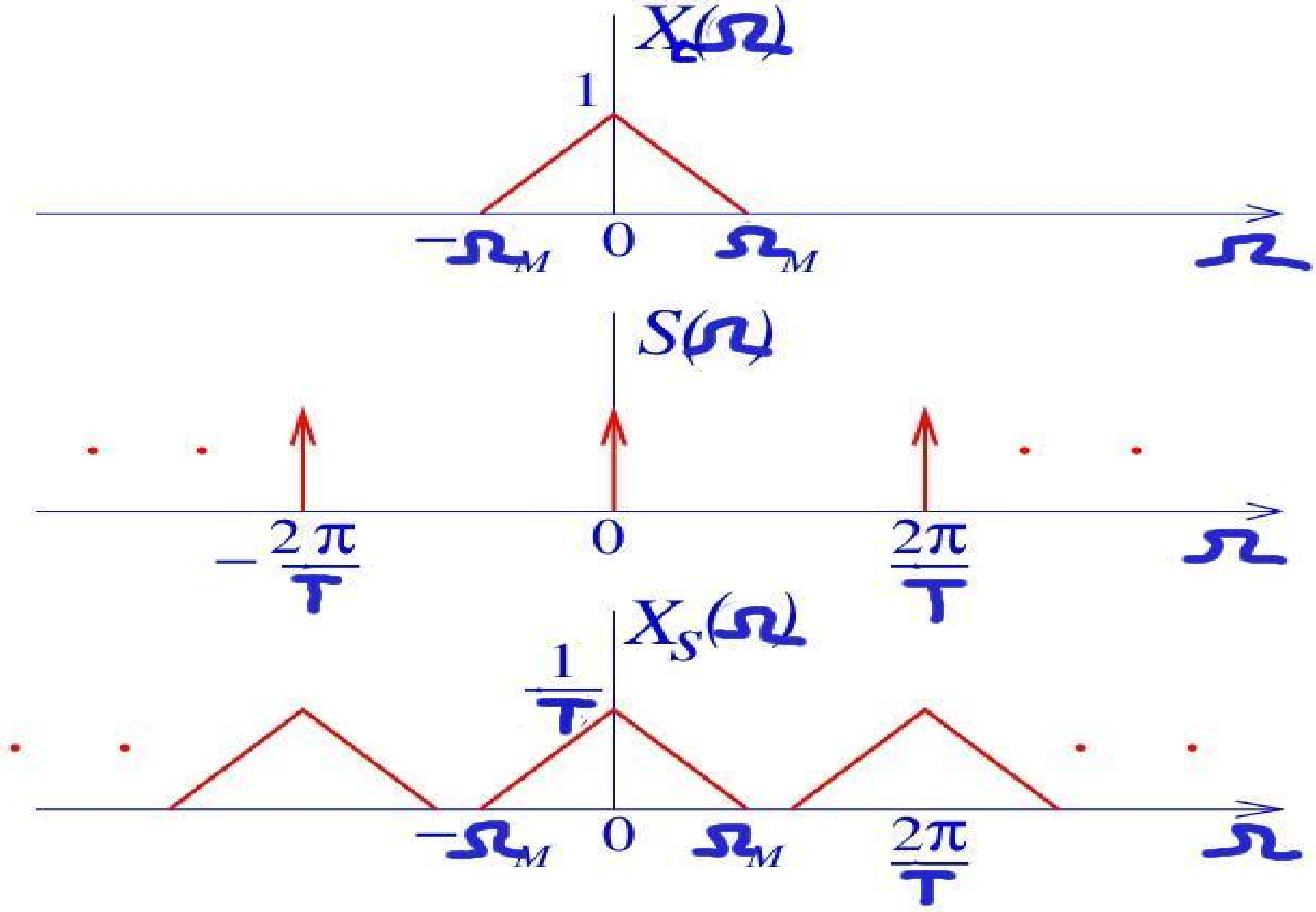
Theorem. *Suppose $x(t)$ is bandlimited, so that $X(\varOmega) = 0$ for $|\varOmega| > \varOmega_{\mathrm{M}}$. Then $x(t)$ is uniquely determined by its samples $\{x(nT)\} = \{x(n)\}$ if*

$$\boxed{\Omega_{\mathrm{s}} = \frac{2\pi}{T} > 2\Omega_{\mathrm{M}} \equiv \textit{the Nyquist rate.}}$$

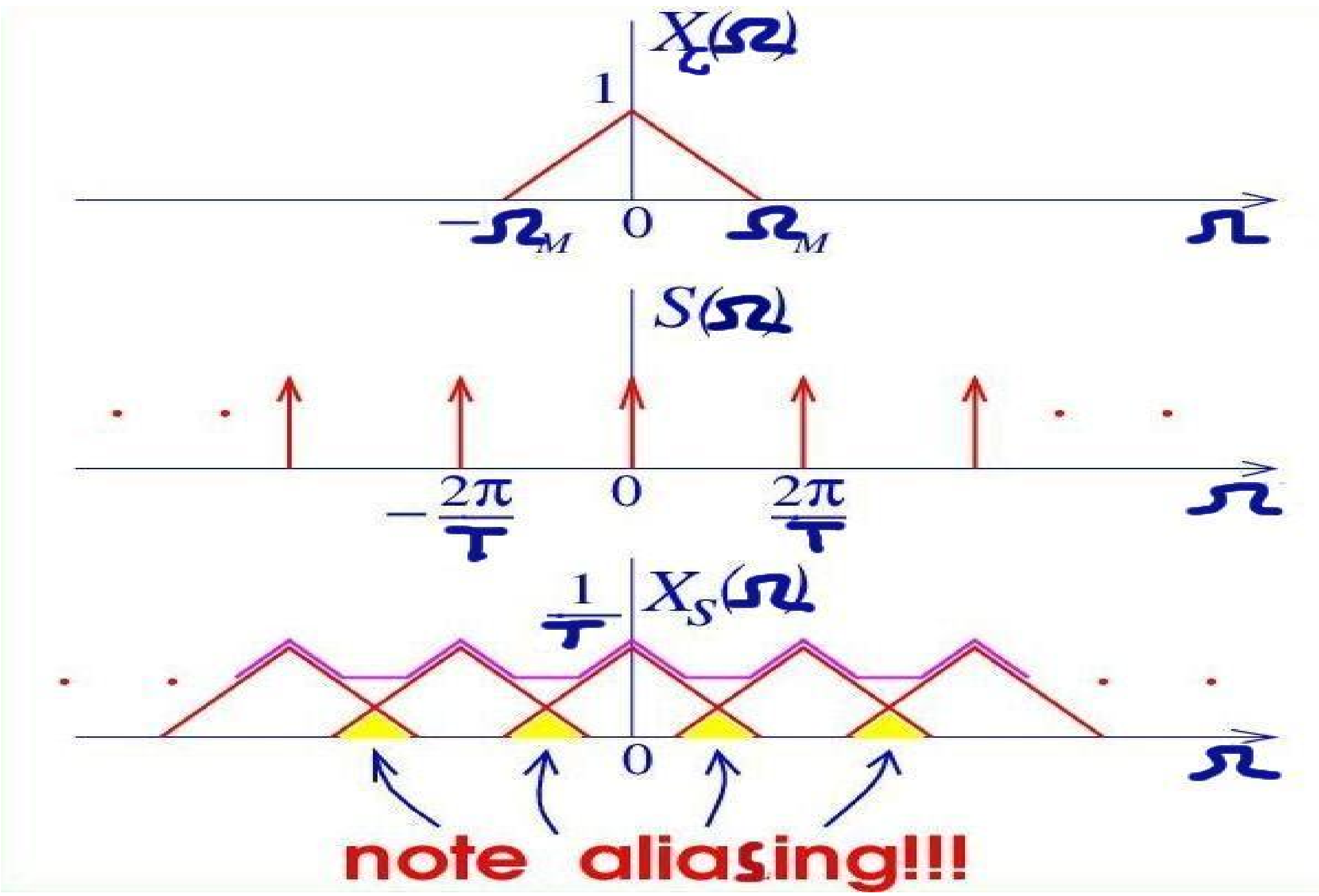
Frequency-Domain Effect for Nyquist Sampling $(2\pi/T = 2\Omega_{\mathrm{M}})$



Frequency-domain Effect for Sampling Faster than Nyquist ($2\pi/T > 2\Omega_\mathrm{M}$)



Frequency-domain Effect for Sampling Slower than Nyquist ($2\pi/T < 2\Omega_{\mathrm{M}}$)



Elements of Sampling Theory (cont.)

Introduce a lowpass filtering operation. The spectrum of the filtered signal:

$$X_{\rm f}(\Omega) = H_{\rm LP}(\Omega) X_{\rm s}(\Omega)$$

where $H_{\rm LP}(\Omega) \equiv$ ideal lowpass filter:

$$H_{\rm LP}(\Omega) = \begin{cases} T, & -\Omega_{\rm c} \le \Omega \le \Omega_{\rm c}, \\ 0, & \text{otherwise} \end{cases},$$

with the cut-off frequency $\Omega_{\rm c}$. How to reconstruct a bandlimited signal from its samples in the time domain?

Having a signal sampled at a rate higher than the Nyquist rate and infinite number of its discrete values, the signal can be exactly recovered as

$$x_{\rm f}(t) = \sum_{n=-\infty}^{\infty} x(n) \frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T} \text{ ideal interpolation formula.}$$

Proof. Start from: $X_{\rm f}(\Omega) = X_{\rm s}(\Omega) H_{\rm LP}(\Omega)$. In time domain

$$\begin{aligned} x_{\rm f}(t) &= \{x_{\rm s}(t)\} \star \{h_{\rm LP}(t)\} \\ &= \left\{ \sum_{n=-\infty}^{\infty} x_{\rm c}(nT)\delta(t-nT) \right\} \star \{h_{\rm LP}(t)\} \end{aligned}$$

$$= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x_{\mathrm{c}}(nT)\delta(\tau - nT)h_{\mathrm{LP}}(t-\tau)d\tau$$

$$= \sum_{n=-\infty}^{\infty} x(n)h_{\mathrm{LP}}(t-nT).$$

Ideal transfer function:

$$H_{\mathrm{LP}}(\Omega) = \begin{cases} T, & -\frac{\pi}{T} \leq \Omega \leq \frac{\pi}{T}, \\ 0, & \text{otherwise} \end{cases},$$

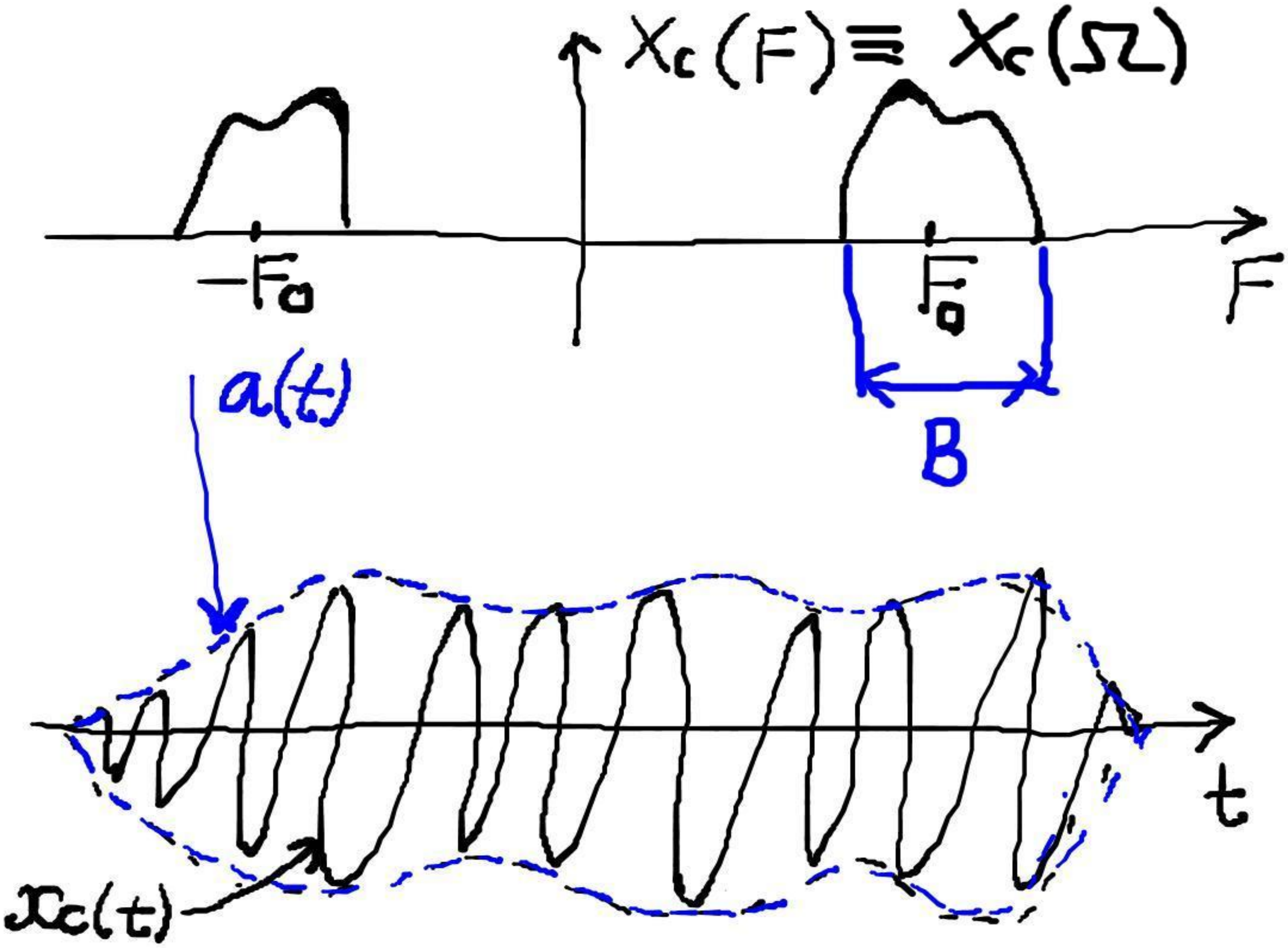
Ideal impulse response:

$$h_{\mathrm{LP}}(t) = \frac{\sin(\pi t/T)}{\pi t/T}.$$

Now, insert $h_{\mathrm{LP}}(t)$ into the equation for $x_{\mathrm{f}}(t)$. $\square$

Representations of Narrowband Signals

Narrowband signals have small bandwidth compared to the band center (carrier) frequency.



$$B \ll F_0.$$

$$x_c(t) = \underbrace{a(t)}_{\text{amplitude modulation}} \cos[\underbrace{2\pi F_0}_{\Omega_0} t + \underbrace{\theta(t)}_{\text{phase modulation}}].$$

The above representation can be used to describe any signal, but it makes sense only if $a(t)$ and $\theta(t)$ vary slowly compared with $\cos(2\pi F_0 t)$, or, equivalently, $B \ll F_0$.

- Complex-envelope and
- Quadrature-component

representations of narrowband signals.

Complex-envelope representation:

$$\begin{aligned} x_c(t) &= \mathrm{Re}\{a(t)\exp(j[\Omega_0 t + \theta(t)])\} \\ &= \mathrm{Re}\{\underbrace{a(t)\exp[j\theta(t)]}_{\widetilde{x}_c(t)}\exp(j\Omega_0 t)\}. \end{aligned}$$

The complex-valued signal $\widetilde{x}_c(t)$ contains both the amplitude and phase variations of $x_c(t)$, and is hence referred to as the *complex envelope* of the narrowband signal $x_c(t)$.

Quadrature-component representation:

$$x_{\rm c}(t) = \underbrace{a(t)\cos\theta(t)}_{x_{\rm cI}(t)}\cos(\Omega_0 t) - \underbrace{a(t)\sin\theta(t)}_{x_{\rm cQ}(t)}\sin(\Omega_0 t).$$

$x_{\rm cI}(t)$ and $x_{\rm cQ}(t)$ are termed the *in-phase* and *quadrature* components of narrowband signal $x_{\rm c}(t)$, respectively.

Note that

$$\widetilde{x}_{\rm c}(t) = x_{\rm cI}(t) + j x_{\rm cQ}(t).$$

If we "blindly" apply the Nyquist theorem, we would choose

$$F_{\rm N} = 2(F_0 + \frac{B}{2}) \approx 2F_0 \quad \text{for } B \ll F_0.$$

However, since the effective bandwidth of $x_{\rm c}(t)$ [and $\widetilde{x}_{\rm c}(t)$] is $B/2$, the optimal rate should be B!

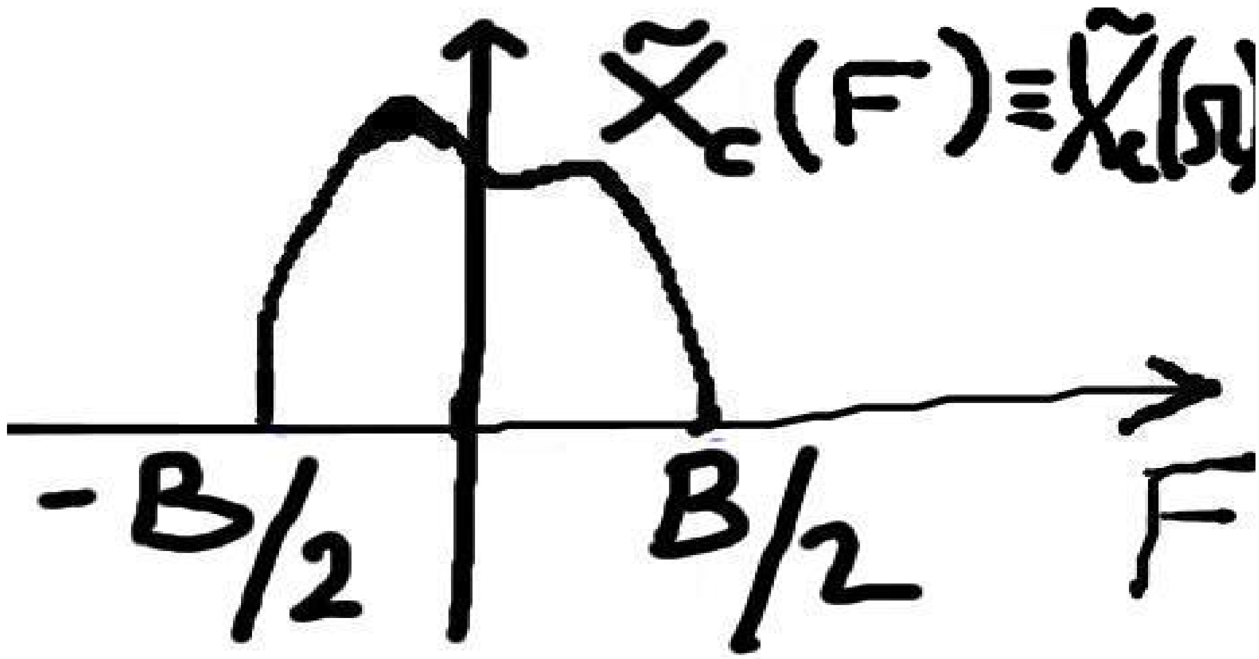
Recall

$$\begin{aligned} x_{\rm c}(t) &= a(t)\cos[2\pi F_0 t + \theta(t)] \\ &= a(t)\cdot\frac{\exp\{j[\Omega_0 t + \theta(t)]\} + \exp\{-j[\Omega_0 t + \theta(t)]\}}{2} \\ &= \underbrace{\frac{a(t)\exp[j\theta(t)]}{2}}_{\frac{1}{2}\widetilde{x}_{\rm c}(t)}\exp(j\Omega_0 t) + \underbrace{\frac{a(t)\exp[-j\theta(t)]}{2}}_{\frac{1}{2}\widetilde{x}_{\rm c}^{*}(t)}\exp(-j\Omega_0 t) \end{aligned}$$

and hence

$$X_c(\Omega) = \frac{1}{2}[\widetilde{X}_c(\Omega - \Omega_0) + \widetilde{X}_c^*(-\Omega - \Omega_0)],$$

implying that $\widetilde{x}_c$ is a *baseband* complex-valued signal (occupying the band $[-B/2, B/2]$):



$$\widetilde{x}_c(t) = \sum_{n=-\infty}^{\infty} \widetilde{x}_c\Big(\frac{n}{B}\Big) \frac{\sin[\pi B(t - n/B)]}{\pi B(t - n/B)}$$

Now,

$$
\begin{aligned}
x_{\mathrm{c}}(t) &= \mathrm{Re}\{\widetilde{x}_{\mathrm{c}}(t)\exp(j2\pi F_0 t)\} \\
&= \mathrm{Re}\left\{\sum_{n=-\infty}^{\infty} \widetilde{x}_{\mathrm{c}}\Big(\frac{n}{B}\Big)\frac{\sin[\pi B(t-n/B)]}{\pi B(t-n/B)}\exp(j2\pi F_0 t)\right\}
\end{aligned}
$$

$$
= \mathrm{Re}\left\{\sum_{n=-\infty}^{\infty} a\Big(\frac{n}{B}\Big)\exp\{j[\theta(n/B)+2\pi F_0 t]\}\frac{\sin[\pi B(t-n/B)]}{\pi B(t-n/B)}\right\}
$$

$$
= \sum_{n=-\infty}^{\infty} a\Big(\frac{n}{B}\Big)\cos[2\pi F_0 t+\theta(n/B)]\frac{\sin[\pi B(t-n/B)]}{\pi B(t-n/B)}
$$

$$
= \sum_{n=-\infty}^{\infty} \Big[x_{\mathrm{cI}}\Big(\frac{n}{B}\Big)\cos(2\pi F_0 t) - x_{\mathrm{cQ}}\Big(\frac{n}{B}\Big)\sin(2\pi F_0 t)\Big]
\cdot\frac{\sin[\pi B(t-n/B)]}{\pi B(t-n/B)}.
$$

Z Transform

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) z^{-n}.$$

Relationship between the z transform and DTFT: substitute $z = re^{j\omega}$,

$$\begin{aligned} X(z)|_{z=re^{j\omega}} &= \sum_{n=-\infty}^{\infty} x(n)(re^{j\omega})^{-n} \\ &= \sum_{n=-\infty}^{\infty} \{x(n)r^{-n}\} e^{-j\omega n} \\ &= \mathcal{F}\{x(n)r^{-n}\} \quad \Longrightarrow \end{aligned}$$

The z transform of an arbitrary sequence $x(n)$ is equivalent to DTFT of the exponentially weighted sequence $x(n)r^{-n}$.

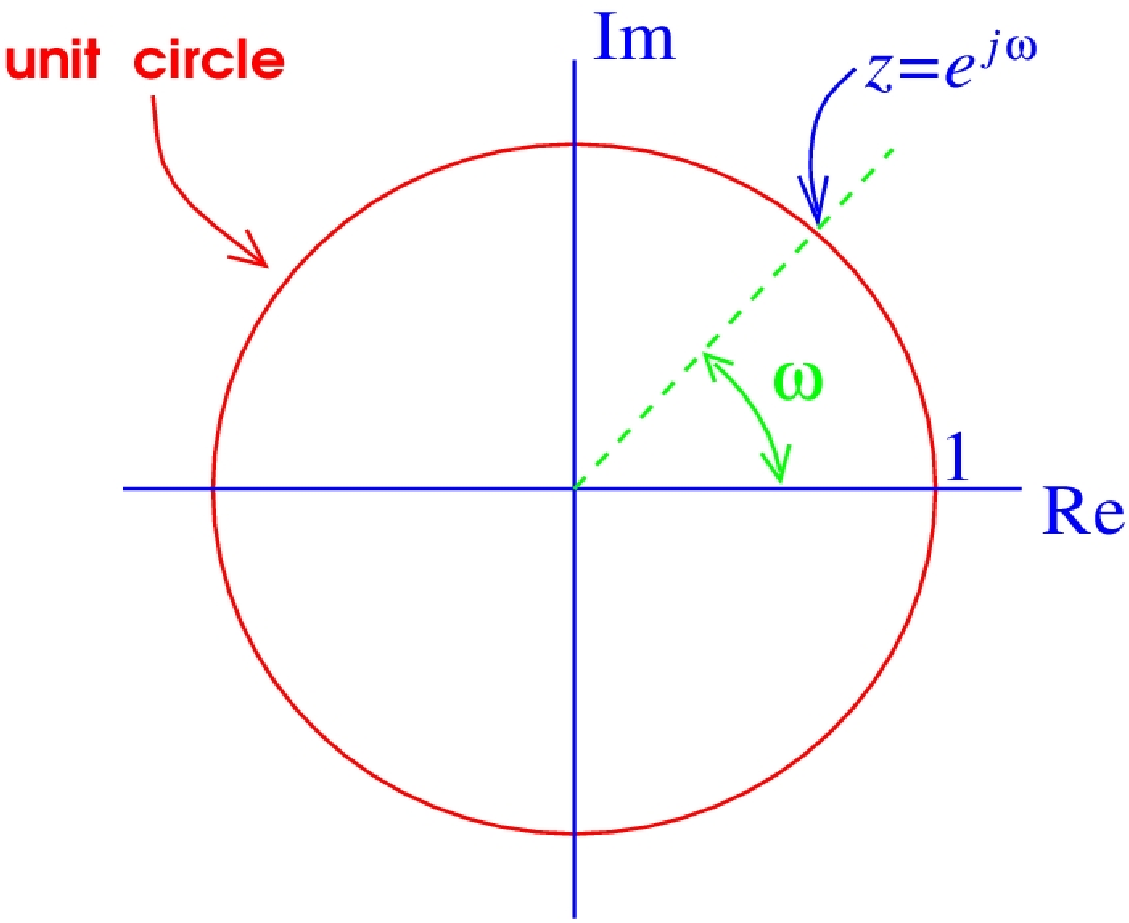
If $r = 1$ then

$$X(z)|_{z=e^{j\omega}} = X(\omega) = \mathcal{F}\{x(n)\} \quad \Longrightarrow$$

DTFT corresponds to z transform with $|z| = 1$.

Notation: Observe that $X(e^{j\omega}) \equiv X(\omega)$.

The z transform reduces to the DTFT for values of z on the unit circle:



Question: When does the z transform converge?

Region of convergence (ROC) $\equiv$ range of values of z for which $|X(z)| < \infty$.

Example: The z transform of the signal $x(n) = a^n u(n)$ is

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n.$$

For convergence, we require that

$$\sum_{n=0}^{\infty} |az^{-1}|^n < \infty,$$

which holds if $|az^{-1}| < 1$ or, equivalently, $|z| > |a|$. Note:

$$X(z) = \frac{1}{1 - az^{-1}}.$$

Example: The z transform of the signal

$$x(n) = -a^n u(-n-1) = \begin{cases} 0, & n \geq 0, \\ -a^n, & n \leq -1 \end{cases}$$

is

$$\begin{aligned} X(z) &= -\sum_{n=-\infty}^{-1} a^n z^{-n} = -\sum_{n=1}^{\infty} a^{-n} z^n = -\sum_{n=1}^{\infty} (a^{-1}z)^n \\ &= -\frac{a^{-1}z}{1 - a^{-1}z} = \frac{1}{1 - az^{-1}} \equiv \text{same as previous ex.} \end{aligned}$$

But, ROC is now $|z| < |a|$.

Remark: a discrete-time signal $x(n)$ is uniquely determined by its z transform $X(z)$ *and* its ROC.

ROC Properties

- The ROC of $X(z)$ consists of a ring in the z plane centered about the origin,
- The ROC does not contain any poles,
- If $x(n)$ is of finite duration, then the ROC is the entire z plane, except possibly $z = 0$ and/or $z = \infty$,
- If $x(n)$ is a right-sided sequence, and if the circle $|z| = r_0$ is in the ROC, then all finite values of z for which $|z| > r_0$ will also be in the ROC (need to check $z = \infty$),
- If $x(n)$ is a left-sided sequence, and if the circle $|z| = r_0$ is in the ROC, then all finite values of z for which $0 < |z| < r_0$ will also be in the ROC (need to check $z = 0$),
- If $x(n)$ is a two-sided sequence, and if the circle $|z| = r_0$ is in the ROC, then the ROC will be a ring in the z plane that includes the circle $|z| = r_0$ (we can represent this sequence as right-sided sequence + left-sided sequence).

Z Transform (cont.)

Inverse Z transform:

Recall that

$$X(z)\Big|_{z=re^{j\omega}} = \mathcal{F}\{x[n]r^{-n}\}.$$

Applying the inverse DTFT, we get

$$\begin{aligned}
x[n] &= r^n \mathcal{F}^{-1}\{X(re^{j\omega})\} \\
&= r^n \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi} X(re^{j\omega})e^{j\omega n}\,d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\underbrace{re^{j\omega}}_{z})(\underbrace{re^{j\omega}}_{z})^n\,d\omega \\
&= \frac{1}{2\pi j}\oint X(z)z^{n-1}\,dz \qquad \longleftarrow dz = jre^{j\omega}d\omega.
\end{aligned}$$

Comments:

- $\oint \cdots dz$ denotes integration around a closed circular contour centered at the origin and having radius r,
- r must be chosen so that the contour of integration $|z| = r$ belongs to the ROC,
- contour integration in complex plane may be a complicated task; simpler alternative procedures exist for obtaining a sequence from a Z transform.

LTI system analysis:

$$y(n) = \{h(n)\} \star \{x(n)\} \quad \leftrightarrow \quad Y(z) = H(z)X(z)$$

Results:

- A discrete-time LTI system is causal if and only if the ROC of its transfer function is the exterior of a circle including infinity.

- A discrete-time LTI system is stable if and only if the ROC of its transfer function includes the unit circle $|z| = 1$.

Rational Z transforms

Recall LCCD equations of ARMA processes

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k).$$

Taking z-transforms of both sides, we get

$$\sum_{k=0}^{N} \mathcal{Z}\{a_k y(n-k)\} = \sum_{k=0}^{M} b_k \mathcal{Z}\{x(n-k)\},$$

yielding

$$Y(z) \sum_{k=0}^{N} a_k z^{-k} = X(z) \sum_{k=0}^{M} b_k z^{-k}.$$

Hence, the transfer function of an ARMA process is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}.$$