1.0 Normalization of voltage equations

We desire to normalize the voltage equations, that is, we desire to express them in per-unit. The advantages of doing so are:

- The per-unit system offers computational simplicity by eliminating units and expressing system quantities as dimensionless ratios.
- The numerical values of currents and voltages are related to their rated values irrespective of machine size.
- It eliminates the need for using arbitrary constants and simplifies some of the mathematical expressions so that they may be expressed in terms of equivalent circuits.
- Impedances, when given on the machine base, lie on a relatively narrow range so that errors can be easily detected.

There are several different possible normalization schemes. What Anderson and Fouad (A&F) have done (see Appendix C) is to carefully compare the merits of all of these schemes. In doing so, they developed some criteria, guidelines, the most important of which is that the equations must be independent of whether they are in pu or MKS for both the voltage equations as well as the power expression (power invariance).

Note that machine manufacturers, when expressing their machine data in per unit, may use a different system that does not satisfy the power invariance property - they use Park’s original transformation (called “Q” in A&F, eq. (4.22), instead of our “P”).

The choice made by A&F satisfies the above criteria; in addition, the A&F choice ensures that the numerical values of the per-unit impedances are the same as those provided by manufacturers using their system of normalization.
In most undergraduate power system analyses courses, we learn that per-unitization requires selection of two base quantities out of the following four: V, I, Z, and S, and then the base quantities for the other two are computed. The situation is the same here, except that we also must deal with speed (or frequency). This necessitates that we must also select a base for either frequency (ω or f) or time, t.

In addition, we will also have need to compute base quantities associated with flux linkage (λ) and inductance (L or M).

Our approach will be to obtain the bases for the stator side and then the bases for the rotor side.

One may note two excellent references on the subject of per-unitizing synchronous machine models:

Other references that address this subject, besides A&F, include those by
- Sauer and Pai
- Concordia
- Padiyar
- Kundur
and also course notes from de Mello.

1.1 Stator side per-unitization:

We select our stator-side bases as:
- \( V_B \): the stator rated line-neutral voltage, rms.
- \( S_B \): the stator rated per-phase power, volt-amps
- \( \omega_B \): the generator rated speed, in electrical rad/sec (\( \omega_{Re}=377 \))
Then we may compute bases for the following 5 quantities:

- **current:** \( I_B = \frac{S_B}{V_B} \)

- **impedance:** \( X_B = R_B = Z_B = \frac{V_B}{I_B} = \frac{V_B^2}{S_B} \)

- **time:** \( t_B = \frac{1}{\omega_B} \) (Note we could have used \( t_B = \frac{2\pi}{\omega_B} \) but this would simply provide a different scaling and is therefore arbitrary). Note that our choice of \( t_B \) is the time required for the rotor to move one electrical radian.

- **Flux linkage:** \( \lambda_B = V_B t_B \) (This comes from the fact that \( v = \frac{d\lambda}{dt} \approx \frac{\Delta\lambda}{\Delta t} \Rightarrow \Delta\lambda = V\Delta t \))

- **Inductance:** \( L_B = \frac{\lambda_B}{I_B} = \frac{X_B}{\omega_B} = \frac{V_B}{\omega_B I_B} = \frac{V_B t_B}{I_B} \)

**Question:** How does our choice of stator base quantities affect the per-unit values of the d- and q-axis quantities?

To answer this question, let \( V \) and \( I \) be the rms magnitudes of the a-phase line-neutral voltage \( V \angle \alpha \) and a-phase line current \( I \angle \gamma \), respectively. Then the per-unit phasors are

\[
\overline{V}_u = \frac{V}{V_B} \angle \alpha = V_u \angle \alpha \quad \quad \quad \overline{I}_u = \frac{I}{I_B} \angle \gamma = I_u \angle \gamma
\]

Now let’s investigate the 0dq quantities.
To begin, note that the expressions for instantaneous voltages and currents for each phase are:

\[ v_a = \sqrt{2}V \sin(\theta + \alpha) \quad i_a = \sqrt{2}I \sin(\theta + \gamma) \]
\[ v_b = \sqrt{2}V \sin(\theta + \alpha - 120^\circ) \quad i_b = \sqrt{2}I \sin(\theta + \gamma - 120^\circ) \]
\[ v_c = \sqrt{2}V \sin(\theta + \alpha + 120^\circ) \quad i_c = \sqrt{2}I \sin(\theta + \gamma + 120^\circ) \]

Use the Park’s transformation on the above to obtain:

\[
\begin{bmatrix}
0 \\
\sqrt{3}V \sin \alpha \\
\sqrt{3}V \cos \alpha
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 \\
\sqrt{3}I \sin \gamma \\
\sqrt{3}I \cos \gamma
\end{bmatrix}
\]

(This confirms our conclusion at the end of the last set of notes that, for balanced conditions, the 0dq quantities are constants, i.e., DC.)

Now, per-unitize by dividing by \(V_B\) and \(I_B\):

\[
\begin{bmatrix}
0 \\
\frac{\sqrt{3}V}{V_B} \sin \alpha \\
\frac{\sqrt{3}V}{V_B} \cos \alpha
\end{bmatrix}
\begin{bmatrix}
0 \\
\frac{\sqrt{3}I}{I_B} \sin \gamma \\
\frac{\sqrt{3}I}{I_B} \cos \gamma
\end{bmatrix}
\]

Observe about the above that

1. The per-unit d and q voltages are equal to the per-unit a-phase voltage scaled by \(\sqrt{3} \sin \alpha\) and \(\sqrt{3} \cos \alpha\), respectively.
2. The per-unit d and q currents are equal to the per-unit line current scaled by \(\sqrt{3} \sin \gamma\) and \(\sqrt{3} \cos \gamma\), respectively.
1.2 Rotor-side per-unitization:

Recall that in system per-unitization, we must select a single power base for the entire system, independent of the fact that some sections of the system are magnetically coupled through transformers, i.e., we do NOT choose different power bases for different sides of a transformer.

The same restriction applies here, where the rotor circuit is magnetically coupled to the stator circuit, i.e., the power base selected for the stator side must also be the power base used on the rotor side. This is $S_B$.

In addition, we are required to select the same time (or frequency) base for both the stator side and the rotor side. This is $t_B$ (or $\omega_B$).

On the rotor side, we have 1 base left to choose. For transformers, we typically choose the 1 remaining base as the voltage base (or current base) according to the turns ratio. Here, however, we do not know a “turns ratio,” and therefore we are left with problem of what, and how, to choose. (One text treats the problem under the assumption that a “turns ratio” is known between stator and rotor circuits - see the text by Padiyar, “Power System Dynamics,” pp. 73-77.)

In making this choice, a problem results from the fact that stator power levels are typically several times the rotor power levels. A&F give an interesting comparison (see pg 95) of a typical stator-side per-phase power rating of 100 MVA and field winding ratings of 250v, 1000A (250kw). What are our choices of the one remaining rotor-side base quantity in this case?

- Choose voltage base=rated voltage=250v, but then the current base is $I_B=100E6/250=400000$ amps, and per-unit values of field currents will be very small.
- Choose current base=rated current=1000A, but then voltage base is $V_B=100E6/1000=100000$ volts, and per-unit values of field voltages will be very small.
Analogy to transformers:
With transformers, we choose the base voltage (or current) on side 1, and then we choose the base voltage (or current) on side 2 as that voltage that is produced by the transformer on side 2 when the base voltage on side 1 is applied. We did this because we wanted a per-unit circuit of the transformer where the ideal transformer was eliminated.

Another way to think about what we do with transformers is that we select current bases that would produce the same mutual flux between the two windings, i.e., we choose $I_{B1}$ and $I_{B2}$ such that:
- $I_{B1}$ produces $\lambda_{21}$ (the flux linkage from current $I_{B1}$ in coil 1 that links coil 2),
- $I_{B2}$ produces $\lambda_{12}$ (the flux linkage from current $I_{B2}$ in coil 2 that links coil 1)

and $\lambda_{21}=\lambda_{12}$.

Note that $\lambda_{21}=M_{21}I_{B1}$ and $\lambda_{12}=M_{12}I_{B2}$, where $M_{21}=M_{21}=M$ is the mutual inductance between the two transformer coils. If coil 1 creates $\lambda_1$ and of this, only the mutual flux $\lambda_{21}$ links with coil 2, then the difference is the leakage flux given by

$$\lambda_{L1}=\lambda_1-\lambda_{21}$$

(eq. 1)

and illustrated in Fig. 1.

![Fig. 1: Mutual and Leakage flux](image)

Here, each of these three fluxes are given by

$$\lambda_{L1}=l_1I_{B1}, \lambda_1=L_1I_{B1}, \text{ and } \lambda_{21}=MI_{B1}$$

(eq. 2)

where $l_1$, $L_1$, and $M$, are the leakage, self, and mutual inductances, respectively. Substitution of (2) into (1) results in:
and canceling $I_{B1}$ gives:

$$l_1 = L_1 - M \implies L_1 = l_1 + M$$

which implies that the self inductance is comprised of the leakage inductance plus the mutual inductance.

Similar analysis results in $L_2 = l_2 + M$.

**Back to synchronous machines:**

We can apply the same concept to the synchronous machine as we applied to the transformer above. That is,

*We select the base currents for the four rotor-side windings $F, D (Q, G)$ to produce the same mutual flux in the air gap as produced by the stator-side base current $I_B$ flowing in the corresponding fictitious $d$-axis ($q$-axis) coil.*

We will begin by applying this idea to obtain the base current for the main field winding.

**Base-current for main field winding, approach 1:**

One can visualize the above concept for the case of the relationship between the F-winding and the d-winding, in Fig. 2.

![Fig. 2: Base currents in d and F windings](image-url)
We see from Fig. 2 that we select \( I_{FB} \), the field winding base current, as that current when flowing in the F-winding will produce a mutual flux \( \lambda_{md} \) equal to the same mutual flux that is produced by a current \( I_B \) flowing in the d-winding.

But how do we compute \( I_{FB} \)?

From our previous set of notes (see page 20), and also eq. 4.20 in text, we derived

\[
\begin{bmatrix}
L_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & L_d & 0 & \sqrt{3} M_F & \sqrt{3} M_D & 0 & 0 \\
0 & 0 & L_q & 0 & 0 & \sqrt{3} M_Q & \sqrt{3} M_G \\
0 & \sqrt{3} M_F & 0 & L_F & M_R & 0 & 0 \\
0 & \sqrt{3} M_D & 0 & M_R & L_D & 0 & 0 \\
0 & 0 & \sqrt{3} M_Q & 0 & 0 & L_Q & M_Y \\
0 & 0 & \sqrt{3} M_G & 0 & 0 & M_Y & L_G \\
\end{bmatrix}
\begin{bmatrix}
i_0 \\
i_d \\
i_q \\
i_F \\
i_D \\
i_Q \\
i_G \\
\end{bmatrix}
= eq. (4.20')
\]

From this, we can see that

\[
\lambda_d = L_d i_d + kM_F i_F + kM_D i_D
\]

(eq. 3)

where \( k = \sqrt{(3/2)} \).

\( L_d i_d \) is all of the flux produced by the d-winding, but only a part of this flux, the mutual flux, links with the F-winding. Call this flux from the d-winding that links with the F-winding \( \lambda_{md} \) given by \( \lambda_{md} = L_{md} i_d \), where \( L_{md} \) is the mutual inductance associated with this flux (A&F call it a magnetizing inductance).
As with the case of the transformer, the difference between the total flux from the d-winding and the mutual flux is attributed to the leakage flux $\lambda_L$, so that,
\[ l_d i_d = L_d i_d - L_{md} i_d \] (eq. 4)

Canceling the current $i_d$, we see that
\[ l_d = L_d - L_{md} \Rightarrow L_d = l_d + L_{md} \] (eq. 5)

When $I_B$ flows in the d-winding, so that $i_d = I_B$, the mutual flux is given by
\[ \lambda_{md} = L_{md} I_B \] (eq. 6)

Looking back at eq (3), we see that the flux from the F-winding that links the d-axis winding is just $kM_F i_F$.

Our criteria for selecting $I_{FB}$ says that when $I_{FB}$ flows in the F-winding, the mutual flux linking the d-winding should equal the mutual flux from the d-winding linking the F-winding when it carries $I_B$. Thus, we write that
\[ \lambda_{md} = L_{md} I_B = kM_F I_{FB} \] (eq. 7)

And we see that
\[ I_{FB} = \frac{L_{md}}{kM_F} I_B \] (eq. 8)

$M_F$ and $L_{md}$ are generally provided in (or can be obtained from) manufacturer’s data for a given machine.
- $M_F$ can be computed as illustrated in Section 2.3 below, using the magnetization curve,
- $L_{md} = L_d - l_d$, where manufacturer’s data sheets contain $L_d$ and $l_d$.

Therefore, once $I_B$ is selected, $I_{FB}$ may be computed.

Note an important point: in MKS units (i.e., henries), $L_{md}$ is not the same as $kM_F$, i.e., the reciprocal mutuals are not equal!

**Base current for main field winding, approach 2:**
One may also develop a relation for $I_{FB}$ from the perspective of the flux linking the field winding, i.e., instead of using eq. (3) from (4.22’), use:
\[ \lambda_F = kM_F i_d + L_F i_F + M_R i_D \] (eq. 9)
Similar to eq. (5), the self inductance $L_F$ is comprised of the leakage and the mutual, i.e.,

$$L_F = l_F + L_{mF} \quad (\text{eq. } 10)$$

Inspecting eq. (9), we see that the flux from the d-winding linking with the F-winding is $kM_Fi_d$, so that when $i_d = I_B$ and $i_F = I_{FB}$, we have that

$$L_{mF}I_{FB} = kM_F I_B \quad (\text{eq. } 11)$$

and we see that

$$I_{FB} = \frac{kM_F}{L_{mF}} I_B \quad (\text{eq. } 12)$$

where, as before, $M_F$ is obtained per Section 2.3 below, $L_{mF} = L_F - l_F$, and $L_F$, $l_F$ are obtained from manufacturer’s data sheet.

**Base current for D-winding:**

We select the D-winding base current, $I_{DB}$ according to the following criteria:

We select $I_{DB}$, the D-winding base current, as that current when flowing in the D-winding will produce a mutual flux $\lambda_{md}$ equal to the same mutual flux that is produced by a current $I_B$ flowing in the d-winding.

Similar analysis as for the F-winding results in

$$I_{DB} = \frac{L_{md}}{kM_D} I_B, \quad I_{DB} = \frac{kM_D}{L_{mF}} I_B \quad (\text{eq. } 13)$$

We may also utilize a similar procedure between D and F windings to obtain

$$I_{DB} = \frac{L_{mF}}{M_R} I_{FB} \quad (\text{eq. } 14)$$
Base current for Q-winding:
We select Q-winding base current, \( I_{QB} \) according to the following criteria:

We select \( I_{QB} \), the Q-winding base current, as that current when flowing in the Q-winding will produce a mutual flux \( \lambda_{mq} \) equal to the same mutual flux that is produced by a current \( I_B \) flowing in the q-winding.

Similar analysis as for the F-winding results in

\[
I_{QB} = \frac{L_{mq}}{kM_Q} I_B, \quad I_{QB} = \frac{kM_Q}{L_{mQ}} I_B \quad \text{(eq. 15)}
\]

Base current for G-winding:
We select the G-winding base current, \( I_{GB} \) according to the following criteria:

We select \( I_{GB} \), G-winding base current, as that current when flowing in the G-winding will produce a mutual flux \( \lambda_{mq} \) equal to the same mutual flux that is produced by a current \( I_B \) flowing in the q-winding.

Similar analysis as for the F-winding results in

\[
I_{GB} = \frac{L_{mq}}{kM_G} I_B, \quad I_{GB} = \frac{kM_G}{L_{mG}} I_B \quad \text{(eq. 16)}
\]

We may also utilize a similar procedure between Q and G windings to obtain

\[
I_{GB} = \frac{L_{mQ}}{M_Y} I_{QB} \quad \text{(eq. 17)}
\]
Summary:

Eqt. (8) together with eqts. (12-17) provide the ability to develop any of the equations given as (4.54) in the text. These equations are:

\[
\begin{align*}
L_{md}I_B^2 &= L_{mF}I_{FB}^2 = L_{mD}I_{DB}^2 = kM_F I_B I_{FB} = kM_D I_B I_{DB} = M_R I_{FB} I_{DB} \\
L_{mq}I_B^2 &= kM_Q I_B I_{QB} = L_{mQ}I_{QB}^2 = M_Y I_{QB} I_{GB}
\end{align*}
\]

For example, recalling (8) is \( I_{FB} = \frac{L_{md}}{kM_F} I_B \) and (12) is \( I_{FB} = \frac{kM_F}{L_{mF}} I_B \), we can multiply the left-hand-sides together and the right-hand-sides together to obtain:

\[
I_{FB}^2 = \frac{L_{md}kM_F}{kM_FL_{mF}} I_B^2 \Rightarrow L_{mF}I_{FB}^2 = L_{md}I_B^2.
\]

Now define the following factors:

\[
k_F = \frac{I_B}{I_{FB}}, \quad k_D = \frac{I_B}{I_{DB}}, \quad k_Q = \frac{I_B}{I_{QB}}, \quad k_G = \frac{I_B}{I_{GB}}
\]

Because we have the same power base on all stator and rotor circuits, we obtain:

\[
S_B = V_B I_B = V_{FB} I_{FB} = V_{DB} I_{DB} = V_{QB} I_{QB} = V_{GB} I_{GB}
\]

Then

\[
k_F = \frac{V_{FB}}{V_B}, \quad k_D = \frac{V_{DB}}{V_B}, \quad k_Q = \frac{V_{QB}}{V_B}, \quad k_G = \frac{V_{GB}}{V_B}
\]

Note that these k-factors may be considered to be effective turns ratios.

We may also derive expressions for the resistance and inductance bases. Note our desire is to be able to compute rotor-side bases as a function of stator-side bases. The k-factors given above will be very handy here.
Rotor-side resistance bases:

\[ R_{FB} \equiv \frac{V_{FB}}{I_{FB}} = \frac{V_{FB}}{V_B} \frac{I_B}{I_{FB}} \frac{V_B}{I_B} = k_F^2 R_B \]

Likewise,

\[ R_{DB} = k_D^2 R_B, \quad R_{QB} = k_Q^2 R_B, \quad R_{GB} = k_G^2 R_B \]

Rotor-side inductance bases:

\[ L_{FB} \equiv \frac{V_{FB} t_B}{I_{FB}} = \frac{V_{FB}}{V_B} \frac{I_B}{I_{FB}} \frac{V_B t_B}{I_B} = k_F^2 L_B \]

Likewise,

\[ L_{DB} = k_D^2 L_B, \quad L_{QB} = k_Q^2 L_B, \quad L_{GB} = k_G^2 L_B \]

Rotor-stator mutuals:

Your text, pg. 95 refers to HW problem 4.18 which states that base mutuals must be the geometric mean of the base self-inductances, i.e.,

\[ M_{12} = \sqrt{L_{1B} L_{2B}} \]

Thus, we have that the base for the field winding to stator winding mutual terms is given by (see eq. 4.57 in text):

\[ M_{FB} = \sqrt{L_B L_{FB}} = \sqrt{L_B^2 k_F^2 L_B} = k_F L_B \]

Note that it is not the same as the base self inductance \( L_{FB} \) given above. Likewise, we get (see eq. 4.57 in text, except here we included \( M_{GB} \)):

\[ M_{DB} = k_D L_B, \quad M_{QB} = k_Q L_B, \quad M_{GB} = k_G L_B \]
Rotor-rotor mutuals:
There are just 2 of them (see eq. 4.57 in text, except here we included $M_{YB}$):

$$M_{RB} = \sqrt{L_{FB}L_{DB}} = \sqrt{k_F^2 L_B k_D^2 L_B} = k_F k_D L_B$$

Likewise,

$$M_{YB} = k_G k_Q L_B$$

2.3 Example 4.1, pg 97 of text

This is a good example that you should review carefully. The only thing that is perhaps not too clear is the computation of $M_F$. I will just review that part of it here.

Computation of $M_F$: A&F make the statement,

“From the no-load magnetization curve, the value of field current corresponding to the rated voltage on the air-gap line is 365 A.”

The “open-circuit characteristic” or “magnetization curve” plots

- Something proportional to exciting (field) current on horizontal axis
- Something proportional to the flux on the vertical axis.

under open-circuit conditions (the a-phase winding is open). Figure 3 below illustrates.

![Figure 3](image-url)
The air-gap line is the $V_a$ vs. $i_F$ relation that results if the iron has constant permeability. The solid line that bends to the right is the actual characteristic that occurs, which shows the terminal voltage falls away from the air-gap line as the field current is raised beyond a certain point. This falling away is caused by saturation of the ferromagnetic material, resulting from the decrease in permeability under high flux conditions. Figure 4 illustrates a magnetization curve for a real 13.8 kV synchronous machine. The vertical axis is line-to-line voltage.

Fig. 4
What is done in Ex. 4.1 (and what is actually done in industry to obtain \( M_F \)), is that the field current is determined corresponding to steady-state rated open circuit terminal voltage. This voltage is \( V_B = V_{LL\text{-rated}}/\sqrt{s} \). For Ex. 4.1, this is \( V_B = 15 \text{kV}/\sqrt{3} = 8660 \text{ volts} \). This is the rms voltage, but A&F indicate that we need the corresponding peak voltage: \( V_{\text{peak}} = \sqrt{2}(8660) = 12,247.1 \text{ volts} \). But why do we need the peak voltage?

Let’s consider this question.

From first page of previous notes titled “Machine Equations,” or from eq. (4.11’) in A&F, we have

\[
\lambda_a = L_{aa} i_a + L_{ab} i_b + L_{ac} i_c + L_{aF} i_F + L_{aD} i_D + L_{aQ} i_Q + L_{aG} i_G
\]

But \( i_a = i_b = i_c = 0 \) under open circuit conditions.
And \( i_D = i_Q = 0 \) under steady-state conditions. Therefore

\[
\lambda_a = L_{aF} i_F + L_{aQ} i_G
\]

Recall that the g-winding models the Q-axis flux produced by the eddy-current effects in the rotor during the transient period. But since we are now considering only the steady-state condition, \( i_G = 0 \). Therefore

\[
\lambda_a = L_{aF} i_F \quad (*)
\]

Now recall from first page of previous notes titled “Machine Equations,” or from eq. (4.16’) in A&F, that \( L_{af} = M_F \cos \theta \), and substitution into (*) yields

\[
\lambda_a = M_F i_F \cos \theta \quad (**)\]

Differentiating (**) results in

\[
\frac{d\lambda_a}{dt} = -M_F i_F \sin \theta \frac{d\theta}{dt} = -\omega_{\text{Re}} M_F i_F \sin \theta \quad (***)
\]

Now recall the voltage equation for the a-phase:

\[
v_a = -i_a r_a - \dot{\lambda}_a + v_n \quad (#)
\]

Substituting (***) into (#), we obtain
\[ v_a = -i_a r_a + \omega_{Re} M_F i_F \sin \theta + v_n \]

But under open circuit conditions, \( i_a = 0, i_n = 0 \) (implying \( v_n = 0 \)) and we have
\[ v_a = \omega_{Re} M_F i_F \sin \theta \quad \text{(#*).} \]

From (#*), we see that
\[ V_{peak} = \omega_{Re} M_F i_F \implies M_F = \frac{V_{peak}}{i_F \omega_{Re}} \]

So we choose a point off the magnetization curve, for example, A&F choose \( i_F = 365 \) A, \( V_{peak} = 12,247.1 \) volts (365 A is the value of field current corresponding to the rated voltage on the air-gap line, and 12,247.1/\( \sqrt{2} \)=8660 volts is the rated RMS line-to-neutral voltage (corresponding to 8660\( \sqrt{3} \)=15kV). Then
\[ M_F = \frac{V_{peak}}{i_F \omega_{Re}} = \frac{12,247.1}{(365)(377)} = 89.006 \times 10^{-3} \text{ henries} \]

And from this we can compute
\[ k_F = \frac{k M_F}{L_{md}} = \frac{k M_F}{L_d - l_d} \]

where the denominator is comprised of data provided by the manufacturer. The rest of Ex. 4.1 is just an application of our per-unitization formula.

There is an interesting paragraph in Appendix C, pg. 552 of your text, to which I want to draw your attention. It says,

“Note that a key element in determining the factor \( k_F \), and hence all the rotor base quantities, is the value of \( M_F \) (in H). This is obtained from the air gap line of the magnetization curve provided by the manufacturer. Unfortunately, no such data is given for any of the amortisseur circuits. Thus, while the pu values of the various amortisseur elements can be determined, their corresponding MKS data are not known.

I provide some comments on certain sentences in this paragraph:
“Note that a key element in determining the factor $k_F$, and hence all the rotor base quantities,” refers to the fact that we obtain $L_{FB}$ and $R_{FB}$ from:

\[
L_{FB} = k_F^2 L_B \\
R_{FB} = k_F^2 R_B
\]

“This is obtained from the air gap line of the magnetization curve provided by the manufacturer,” as we have seen above by using

\[
M_F = \frac{V_{peak}}{i_F \omega_{Re}}
\]

We are able to get $M_F$ in this way because we can directly control the current $i_F$, with no other circuits energized (as a result of the open-circuit, steady-state conditions), and directly measure the induced voltage at the a-phase terminals.

“Unfortunately, no such data is given for any of the amortisseur circuits.” It is not possible to directly control the currents $i_D$, $i_Q$, and $i_G$, since their corresponding circuits do not have sources. The only way to energize these circuits is via a transient condition, but there is no way to provide a transient condition that will also not energize other circuits, which would result in the measured terminal voltage being induced from the mutual inductance between itself and the other circuits as well.

“Thus, while the pu values of the various amortisseur elements can be determined, their corresponding MKS data are not known.” In example 4.1, the text puts an asterisk by some of the parameters ($L_D$, $L_Q$, $kM_D$, $kM_Q$, $r_D$, and $r_Q$), indicating they were “estimated for academic study”). This is because manufacturer’s datasheets do not always include the parameters for the amortisseur (and g-winding) circuits, simply because they are hard to measure (based on the comments of the previous bullet). However, it is possible to obtain the per-unit value (not the MKS value) of some of the amortisseur circuit parameters (specifically, the mutual inductances), because, as we shall see in Section 3.0 below, in per-unit, all direct-axis rotor mutuals are equal and all quadrature-axis mutuals are equal! In other words:
- D-axis mutuals:
  
  - F-d winding mutual, $kM_F$
  - D-d winding mutual, $kM_D$
  - F-D winding mutual, $M_R$ (can be called $M_X$ in some texts)

  That is, we will show that in per-unit, $kM_F u = kM_D u = M_R u$

- Q-axis mutuals:
  
  - G-q winding mutual, $kM_G$
  - Q-q winding mutual, $kM_Q$
  - G-Q winding mutual, $M_Y$

  That is, we will show that in per-unit, $kM_Q u = kM_G u = M_Y u$

2.4 Applying the bases to voltage equations:

Recall our voltage equation as written in MKS units:

$$
\begin{bmatrix}
  v_o \\
v_d \\
v_q \\
- v_F \\
0 \\
0 \\
0
\end{bmatrix} = 
\begin{bmatrix}
  r_a + 3r_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & r_b & \omega L_q & 0 & 0 & \omega \sqrt{\frac{3}{2}} M_Q & \omega \sqrt{\frac{3}{2}} M_G \\
  0 & - \omega L_D & r_c & - \omega \sqrt{\frac{3}{2}} M_F & - \omega \sqrt{\frac{3}{2}} M_D & 0 & 0 \\
  0 & 0 & 0 & r_F & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & r_F & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & r_Q & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & r_G
\end{bmatrix}
\begin{bmatrix}
i_o \\
i_d \\
i_q \\
i_F \\
i_D \\
i_Q \\
i_G
\end{bmatrix}
$$

$$
\begin{bmatrix}
  L_a + 3L_n & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & L_d & 0 & \sqrt{\frac{3}{2}} M_F & \sqrt{\frac{3}{2}} M_D & 0 & 0 \\
  0 & 0 & L_q & 0 & 0 & \sqrt{\frac{3}{2}} M_Q & \sqrt{\frac{3}{2}} M_G \\
  0 & \sqrt{\frac{3}{2}} M_F & 0 & L_F & M_R & 0 & 0 \\
  0 & \sqrt{\frac{3}{2}} M_D & 0 & M_R & L_D & 0 & 0 \\
  0 & \sqrt{\frac{3}{2}} M_Q & 0 & 0 & L_Q & M_Y & 0 \\
  0 & \sqrt{\frac{3}{2}} M_G & 0 & 0 & M_Y & L_G & 0
\end{bmatrix}
\begin{bmatrix}
i_o \\
i_d \\
i_q \\
i_F \\
i_D \\
i_Q \\
i_G
\end{bmatrix}
$$
Let’s normalize them using our chosen bases to obtain the equations in per-unit. The per-unit equations should appear as above when done, except that everything must be in per-unit.

**Step 1:** Replace all MKS voltages on the left with
- the product of their per-unit value and their base value (use $V_B$ for the first 3 equations and $V_{FB}$, $V_{DB}$, $V_{QB}$, $V_{GB}$ for the last four equations),

and replace all currents on the right with
- the product of their per-unit value and their base value (use $I_B$ for the first 3 equations and $I_{FB}$, $I_{DB}$, $I_{QB}$, $I_{GB}$ for the last four equations).

This results in a relation similar to eq. 4.60 in the text, as follows:

$$
\begin{bmatrix}
\begin{array}{cccccccc}
  v_{oa}V_B & v_{da}V_B & v_{qa}V_B & -v_{ra}V_{FB} \\
  r_a + 3r_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & r_b & \omega L_q & 0 & 0 & \omega \frac{3}{2} M_Q & \omega \frac{3}{2} M_G \\
  0 & -\omega L_d & r_c & -\omega \sqrt{\frac{3}{2}} M_F & -\omega \sqrt{\frac{3}{2}} M_D & 0 & 0 \\
  0 & 0 & 0 & r_f & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & r_p & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & r_Q \\
  0 & 0 & 0 & 0 & 0 & 0 & r_G \\
\end{array}
\end{bmatrix}
\begin{bmatrix}
i_{oa}I_B \\
i_{da}I_B \\
i_{qa}I_B \\
i_{ra}I_{FB} \\
i_{fb}I_{FB} \\
i_{fb}I_{FB} \\
i_{fb}I_{FB} \\
i_{ga}I_{GB}
\end{bmatrix}
- \\
\begin{bmatrix}
L_o + 3L_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & L_d & 0 & \sqrt{\frac{3}{2}} M_F & \sqrt{\frac{3}{2}} M_D & 0 & 0 \\
0 & 0 & L_q & 0 & 0 & \sqrt{\frac{3}{2}} M_Q & \sqrt{\frac{3}{2}} M_G \\
0 & \sqrt{\frac{3}{2}} M_F & 0 & L_F & M_R & 0 & 0 \\
0 & \sqrt{\frac{3}{2}} M_D & 0 & M_R & L_D & 0 & 0 \\
0 & 0 & \sqrt{\frac{3}{2}} M_Q & 0 & 0 & L_Q & M_Y \\
0 & 0 & \sqrt{\frac{3}{2}} M_G & 0 & 0 & M_Y & L_G
\end{bmatrix}
\begin{bmatrix}
i_{oa}I_B \\
i_{da}I_B \\
i_{qa}I_B \\
i_{ra}I_{FB} \\
i_{fb}I_{FB} \\
i_{fb}I_{FB} \\
i_{fb}I_{FB} \\
i_{ga}I_{GB}
\end{bmatrix}

(eq. 4.60')
**Step 2:** For each of the equations in the above, we need to divide through by the voltage base. For those equations containing $\omega$, we replace it with $\omega = \omega_u \omega_B \ (\omega_B = \omega_{Re})$. Then we do some algebra-work on each equation to express the coefficients of each current and current derivative as per-unitized self or mutual inductances. As an example, the second equation is done for you in the text; here, I will do the last equation, corresponding to the G-winding….

$$0(V_{GB}) = -r_G i_{Gu} I_{GB} - \sqrt{\frac{3}{2}} M_G i_{qu} I_B - M_Y i_{Qu} I_{QB} - L_G i_{Gu} I_{GB}$$

**Step 2a:** Divide through by $V_{GB}$ to obtain:

$$0 = \frac{-r_G}{V_{GB}} i_{Gu} - \sqrt{\frac{3}{2}} \frac{M_G}{V_{GB}} \frac{i_{qu}}{I_B} - \frac{M_Y}{V_{GB}} \frac{i_{Qu}}{I_{QB}} - \frac{L_G}{V_{GB}} \frac{i_{Gu}}{I_{GB}}$$

The first term has a denominator of $R_{GB}$. The last three terms are not so obvious. We desire them to have denominators of $M_{GB}$, $M_{YB}$, and $L_{GB}$, respectively, where, from above, we recall $M_{GB} = k_G L_B$, $M_{YB} = k_G k_Q L_B$, $L_{GB} = k_G^2 L_B$, where $k_G = \frac{V_{GB}}{V_B} = \frac{I_B}{I_{GB}}$, and $k_Q = \frac{I_B}{I_{QB}}$.

**Step 2b:** Let’s multiply the denominator of the last three terms by $V_B / V_B$. This results in:

$$0 = \frac{-r_G}{R_{GB}} i_{Gu} - \sqrt{\frac{3}{2}} \frac{M_G}{V_{GB}} \frac{V_B}{V_B} \frac{i_{qu}}{I_B} - \frac{M_Y}{V_{GB}} \frac{V_B}{V_B} \frac{i_{Qu}}{I_{QB}} - \frac{L_G}{V_{GB}} \frac{V_B}{V_B} \frac{i_{Gu}}{I_{GB}}$$

**Step 2c:** Let’s multiply the denominator of the last two terms by $I_B / I_B$. This results in:
\[ 0 = -\frac{r_G}{R_{GB}} i_{Gu} - \sqrt{\frac{3}{2}} \frac{M_G}{V_{GB} V_B} i_{qu} - \frac{M_Y}{V_{GB} V_B I_B} i_{Qu} - \frac{L_G}{V_{GB} V_B I_B I_{GB}} i_{Gu} \]

**Step 2d:** Recall the k-factors (pg 96 of text): \( k_G = \frac{V_{GB}}{V_B} = \frac{I_B}{I_{GB}} \), and \( k_Q = \frac{I_B}{I_{QB}} \).

Substitution yields:

\[ 0 = -\frac{r_G}{R_{GB}} i_{Gu} - \sqrt{\frac{3}{2}} \frac{M_G}{k_G V_B I_B} i_{qu} - \frac{M_Y}{k_G k_Q V_B I_B} i_{Qu} - \frac{L_G}{k_G^2 V_B I_B} i_{Gu} \]

**Step 2e:** We are close now, as we need \( M_{GB} = k_G L_B \), \( M_{YB} = k_G k_Q L_B \), and \( L_{GB} = (k_G)^2 L_B \), respectively, on the denominator of the last three terms. Recall that \( L_B = \frac{V_B}{(\omega_B I_B)} \), so we need to divide top and bottom on the denominators of the last three terms by \( \omega_B \). Doing so yields:

\[ 0 = -\frac{r_G}{R_{GB}} i_{Gu} - \sqrt{\frac{3}{2}} \frac{M_G}{k_G L_B \omega_B} i_{qu} - \frac{M_Y}{k_G k_Q L_B \omega_B} i_{Qu} - \frac{L_G}{k_G^2 L_B \omega_B} i_{Gu} \]

**Step 2f:** And substituting in \( L_B \) results in:

\[ 0 = -\frac{r_G}{R_{GB}} i_{Gu} - \sqrt{\frac{3}{2}} \frac{M_G}{M_{GB} \omega_B} i_{qu} - \frac{M_Y}{M_{YB} \omega_B} i_{Qu} - \frac{L_G}{L_{GB} \omega_B} i_{Gu} \]

**Step 2g:** Recalling that \( M_{GB} = k_G L_B \), \( M_{YB} = k_G k_Q L_B \), and \( L_{GB} = (k_G)^2 L_B \), we may write:

\[ 0 = -\frac{r_G}{R_{GB}} i_{Gu} - \sqrt{\frac{3}{2}} \frac{M_G}{M_{GB} \omega_B} i_{qu} - \frac{M_Y}{M_{YB} \omega_B} i_{Qu} - \frac{L_G}{L_{GB} \omega_B} i_{Gu} \]

which results in
\[ 0 = -r_G i_{Gu} - \sqrt{\frac{3}{2} M_{Gu} i_{qu}} - \frac{M_{Yu}}{\omega_B} i_{Qu} - \frac{L_{Gu}}{\omega_B} i_{Gu} \]

**Step 2h:** However, we still have one problem. Recall that we want the equations to be identical in pu to their form in MKS units. But in the last equation, we still have \(\omega_B\), which does not appear in our MKS equation. We can take care of it, however, by recalling that \(\omega_B = 1/t_B\), so that:

\[
\frac{1}{\omega_B} i_{qu} = \frac{1}{t_B} \frac{di_{qu}}{dt} = \frac{d}{d\tau} \left( \frac{t}{t_B} \right) \frac{di_{qu}}{dt},
\]

\[
\frac{1}{\omega_B} i_{Qu} = \frac{1}{t_B} \frac{di_{Qu}}{dt} = \frac{d}{d\tau} \left( \frac{t}{t_B} \right) \frac{di_{Qu}}{dt},
\]

\[
\frac{1}{\omega_B} i_{Gu} = \frac{1}{t_B} \frac{di_{Gu}}{dt} = \frac{d}{d\tau} \left( \frac{t}{t_B} \right) \frac{di_{Gu}}{dt},
\]

where \(\tau = t/t_B\) is the normalized time.

With this last change, we can write, finally, that

\[ 0 = -r_G i_{Gu} - \sqrt{\frac{3}{2} M_{Gu} i_{qu}} - \frac{M_{Yu}}{\omega_B} i_{Qu} - \frac{L_{Gu}}{\omega_B} i_{Gu} \]

which is the per-unitized form of the last equation in eq. (4.60').

Note that it is exactly the same form as the original equation in MKS units.

Similar work can be done for the other equations, resulting in equations similar to eq. 4.74 in your text:
\[
\begin{bmatrix}
\nu_d \\
-v_F \\
0 \\
v_q \\
0
\end{bmatrix} = -
\begin{bmatrix}
r & 0 & 0 & \omega L_q & \omega kM_Q & \omega kM_G \\
0 & r_F & 0 & 0 & 0 & 0 \\
0 & 0 & r_D & 0 & 0 & 0 \\
-\omega L_d & -\omega kM_F & -\omega kM_D & r & 0 & 0 \\
0 & 0 & 0 & 0 & r_Q & 0
\end{bmatrix}
\begin{bmatrix}
i_d \\
i_F \\
i_D \\
i_q \\
i_Q \\
i_G
\end{bmatrix}
\]

\[
\begin{bmatrix}
L_d & kM_F & kM_D & 0 & 0 & 0 \\
kM_F & L_F & M_R & 0 & 0 & 0 \\
kM_D & M_R & L_D & 0 & 0 & 0 \\
0 & 0 & 0 & L_q & kM_Q & kM_G \\
0 & 0 & 0 & kM_Q & L_Q & M_Y \\
0 & 0 & 0 & kM_G & M_Y & L_G
\end{bmatrix}
\begin{bmatrix}
i_d \\
i_F \\
i_D \\
i_q \\
i_Q \\
i_G
\end{bmatrix}
\]

(eq. 4.74')

Note that in the above equation,

- The “u” subscript was dropped; however, all parameters are in per-unit.
- We have dropped the zero-sequence voltage equation since we will be interested in balanced conditions for stability studies. (A system having a three-phase fault, considered to be, usually, the most severe, is still a balanced system. This does not mean that we cannot analyze unbalanced faults using stability programs. It is possible to analyze the effects of unbalanced faults on the positive sequence network represented in stability programs – see Kimbark Vol I, pp. 220-221). Otherwise, eq. (4.74’) is precisely the same as eqt. 4.39 in your text (see the same
equation as 4.39, except for the “G” winding included, on page 27 of the notes called “macheqts”– we called it there, eq. 4.39’).

- The equations are rearranged to better display the coupling and decoupling between the various circuits. This coupling can be well illustrated by a figure similar to Fig. 4.3 in your text, given below as Fig. 4.3’. Notice that the coupling between the F and D windings is captured by $M_X$. We have called this mutual inductance $M_R$ in our work above to remain consistent with A&F.

Fig 4.3’
Now let’s make some definitions:

\[
\begin{bmatrix}
  r & 0 & 0 & 0 & 0 & 0 \\
  0 & r_F & 0 & 0 & 0 & 0 \\
  0 & 0 & r_D & 0 & 0 & 0 \\
  0 & 0 & 0 & r & 0 & 0 \\
  0 & 0 & 0 & 0 & r_Q & 0 \\
  0 & 0 & 0 & 0 & 0 & r_G
\end{bmatrix}; \quad \begin{bmatrix}
  0 & 0 & 0 & L_q & kM_Q & kM_G \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  -L_d & -kM_F & -kM_D & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
  L_d & kM_F & kM_D & 0 & 0 & 0 \\
  kM_F & L_F & M_R & 0 & 0 & 0 \\
  kM_D & M_R & L_D & 0 & 0 & 0 \\
  0 & 0 & 0 & L_q & kM_Q & kM_G \\
  0 & 0 & 0 & kM_Q & L_Q & M_Y \\
  0 & 0 & 0 & kM_G & M_Y & L_G
\end{bmatrix}
\]

\[
\begin{bmatrix}
  v_d \\
  -v_F \\
  0 \\
  v_q \\
  0
\end{bmatrix}; \quad \begin{bmatrix}
  i_d \\
  i_F \\
  i_D \\
  i_q \\
  i_Q \\
  i_G
\end{bmatrix}
\]

With these definitions, we rewrite eqt. (4.74’) in compact notation:

\[
v = -(R + \omega N)i - Li
\]

(eq. 4.75)

We may solve eq. (4.75) for \(\frac{di}{dt}\) so that it is in state-space form:

\[
i = -L^{-1}(R + \omega N)i - L^{-1}v
\]

(eq. 4.76)
3.0 Per-unit mutuals (See Section 4.11)

A useful observation regarding per-unit values of $M_F$, $M_D$, and $M_R$:

Recall our definitions of the D-axis k-factors:

$$k_F = \frac{I_B}{I_{FB}} = \frac{V_{FB}}{V_B}$$

$$k_D = \frac{I_B}{I_{DB}} = \frac{V_{DB}}{V_B}$$

and that we developed (see eqs. 4.54, pg 12-13 of these notes):

$$L_{md}I_B^2 = L_{mf}I_{FB}^2 = L_{mb}I_{DB}^2 = kM_F I_B I_{FB} = kM_D I_B I_{DB} = M_R I_{FB} I_{DB}$$ (17)

From the first and fourth expression in eq (17), we have:

$$L_{md}I_B^2 = kM_F I_B I_{FB}$$

Thus,

$$\frac{I_B}{I_{FB}} = \frac{kM_F}{L_{md}}$$ (18)

Likewise, from the first and fifth, and from the fourth and sixth expressions in eq (17), we have:

$$L_{md}I_B^2 = kM_D I_B I_{DB}$$

Thus,

$$\frac{I_B}{I_{DB}} = \frac{kM_D}{L_{md}}$$ and $$\frac{I_B}{I_{DB}} = \frac{M_R}{kM_F}$$ (19)

From the definitions of the k-factors, and eqs (18) and (19), we have:

$$k_F = \frac{kM_F}{L_{md}}$$ and $$k_D = \frac{kM_D}{L_{md}} = \frac{M_R}{kM_F}$$ (20)
Also, from eq. 4.57 in text (also see p13-14 of these notes), we find
\[ M_{FB} = k_F L_B, \quad M_{DB} = k_D L_B, \quad M_{RB} = k_F k_D L_B \] (21)
which we got by using the fact that base mutuals must be the geometric mean of the base self-inductances (see prob 4.18).

Now, recall the elements in the per-unitized voltage equations as given by eq. 4.74' (see page 24 of these notes).

\[
\begin{bmatrix}
  v_d \\
  -v_F \\
  0 \\
  v_q \\
  0 \\
\end{bmatrix} =
\begin{bmatrix}
  r & 0 & 0 & \omega L_q & \omega kM_Q & \omega kM_G & 0 & 0 & i_d \\
  0 & r_F & 0 & 0 & 0 & 0 & 0 & i_F \\
  0 & 0 & r_D & 0 & 0 & 0 & 0 & i_D \\
  -\omega L_d & -\omega kM_F & -\omega kM_D & r & 0 & 0 & 0 & i_q \\
  0 & 0 & 0 & 0 & 0 & 0 & r_Q & i_Q \\
  0 & 0 & 0 & 0 & 0 & 0 & r_G & i_G \\
\end{bmatrix}
\]

In particular, consider the mutual terms in the last matrix for the d-axis. This would be the upper left-hand 3x3 block. These terms, in pu, are by definition the ratio of the term in MKS to the appropriate base. Therefore:

- **Stator-field mutual:** \( kM_{Fu} = \frac{kM_F}{M_{FB}} \).

Substituting for \( M_{FB} \) from eq. (21) and then \( k_F \) from eq. (20) results in:
\[
\begin{align*}
km_{Fu} &= \frac{km_F}{M_{FB}} = \frac{km_F}{k_F L_B} = \frac{km_F L_{md}}{km_F L_B} = \frac{L_{md}}{L_B} \equiv L_{mdu} \\
km_{Du} &= \frac{km_D}{M_{DB}} = \frac{km_D}{k_D L_B} = \frac{km_D L_{md}}{km_D L_B} = \frac{L_{md}}{L_B} \equiv L_{mdu}
\end{align*}
\]

- **Stator-D-winding damper mutual:** \( km_{Du} = \frac{km_D}{M_{DB}} \).

Substituting for \( M_{DB} \) from eq. (21) and then \( k_D \) from eq. (20) results in:

\[
\begin{align*}
km_{Du} &= \frac{km_D}{M_{DB}} = \frac{km_D}{k_D L_B} = \frac{km_D L_{md}}{km_D L_B} = \frac{L_{md}}{L_B} \equiv L_{mdu}
\end{align*}
\]

- **Field-D-winding damper mutual:** \( M_{Ru} = \frac{M_R}{M_{RB}} \).

Substituting for \( M_{RB} \) from eq. (21) and then \( k_F \) and \( k_D \) from eq. (20) (using the 2\textsuperscript{nd} expression for \( k_D \) in eq. (4)) results in:

\[
\begin{align*}
M_{Ru} &= \frac{M_R}{M_{RB}} = \frac{M_R}{k_F k_D L_B} = \frac{M_R L_{md} km_F}{km_F M_R L_B} = \frac{L_{md}}{L_B} \equiv L_{mdu}
\end{align*}
\]

Important fact: In per-unit, all d-axis mutuals are numerically equal! We will define a new term for them, \( L_{AD} \), as the per-unit value of any d-axis mutual inductance, so that:

\[
L_{AD} \equiv L_{mdu} = km_{Fu} = km_{Du} = M_{Ru}
\]

Also note that, since the mutual is the difference between the self and the leakage, this implies

\[
L_{du} - l_{du} = L_{Du} - l_{Du} = L_{Fu} - l_{Fu} = L_{AD}
\]

The above relations are given in eqs. 4.107 and 4.108 in your text.
We can go through a similar process for the q-axis mutuals (from 4.74', we see that these are the terms in the lower right-hand block of the matrix, $kM_Q$, $kM_G$, and $M_Y$). I will leave this for you to do. The result is:

$$L_{AQ} = L_{mqu} = kM_{Qu} = kM_{Gu} = M_{Yu}$$

$L_{qu}l_{qu} = L_{Qu}l_{Qu} = L_{Gu}l_{Gu} = L_{AQ}$

The above relations are given by eq. 4.109 in your text, except for the addition of the G-term and the Y-term.

$L_{AD}$ and $L_{AQ}$ are very important for drawing the equivalent circuits.

They are also important in dealing with saturation because they provide for the definition of the per-unit mutual flux (we will see this in our development of the flux-linkage state-space model).

### 4.0 Equivalent Circuits (See Section 4.11)

Let’s return to the voltage equations that we had before we folded in the speed voltage terms. They were:

\[
\begin{bmatrix}
    v_0 \\
    v_d \\
    v_q \\
    -v_p \\
    0 \\
    0 \\
    0
\end{bmatrix} =
\begin{bmatrix}
    r_g & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & r_b & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & r_c & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & r_r & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & r_p & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & r_q & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & r_o
\end{bmatrix}
\begin{bmatrix}
    i_o \\
    i_d \\
    i_q \\
    i_r \\
    i_p \\
    i_q \\
    i_o
\end{bmatrix}
\]

\[
\begin{bmatrix}
    v_0 \\
    v_d \\
    v_q \\
    -v_p \\
    0 \\
    0 \\
    0
\end{bmatrix} =
\begin{bmatrix}
    L_o & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & L_d & 0 & \frac{3\sqrt{2}M_F}{2} & \frac{3\sqrt{2}M_Q}{2} & 0 & 0 \\
    0 & 0 & L_q & 0 & \frac{3\sqrt{2}M_G}{2} & \frac{3\sqrt{2}M_G}{2} & 0 \\
    0 & 0 & \frac{3\sqrt{2}M_R}{2} & 0 & L_F & M_R & 0 \\
    0 & 0 & \frac{3\sqrt{2}M_R}{2} & 0 & M_R & L_Q & 0 \\
    0 & 0 & \frac{3\sqrt{2}M_R}{2} & 0 & 0 & L_Q & M_F \\
    0 & 0 & \frac{3\sqrt{2}M_R}{2} & 0 & 0 & 0 & M_R
\end{bmatrix}
\begin{bmatrix}
    i_o \\
    i_d \\
    i_q \\
    i_r \\
    i_p \\
    i_q \\
    i_o
\end{bmatrix}
\]

\[
\begin{bmatrix}
    0 & 0 & 0 & 0 & 3r_p l_o - 3L_o i_o & 0 & 0 \\
    -\omega_l q & 0 & 0 & 0 & 0 & 0 & 0 \\
    \omega_l q & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

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Assume all of the above is in per-unit (but we have dropped the \( u \)-subscript).

There is some advantage to re-writing these equations in terms of \( L_{\text{AD}} \) and \( L_{\text{AQ}} \). For example, consider the d-axis equation. It is:

\[
v_d = -r_i d - L_d \dot{i}_d - kM_F \dot{i}_F - kM_D \dot{i}_D - \omega \lambda_q
\]

Recall that \( L_d = L_{\text{md}} + l_d \)\( \Rightarrow L_{\text{md}} = L_d - l_d \)

Let’s modify the d-axis voltage equation by adding and subtracting \( l_d \) (\( di_d/dt \)):

\[
v_d = -r_i d - L_d \dot{i}_d + l_d \dot{i}_d - l_d \dot{i}_d - kM_F \dot{i}_F - kM_D \dot{i}_D - \omega \lambda_q,
\]

which can be written as:

\[
v_d = -r_i d - l_d \dot{i}_d - [(L_d - l_d) \dot{i}_d + kM_F \dot{i}_F + kM_D \dot{i}_D] - \omega \lambda_q
\]

The advantage to this is that, in per-unit, we recall that \( L_d - l_d = kM_F = kM_D = L_{\text{AD}} \). Therefore,

\[
v_d = -r_i d - l_d \dot{i}_d - L_{\text{AD}}[\dot{i}_d + \dot{i}_F + \dot{i}_D] - \omega \lambda_q
\]

Let’s repeat this for the G-axis equation, which is, from the matrix equation at the beginning of this section:

\[
v_G = 0 = -r_G i_G - kM_G \dot{i}_q - M_Y \dot{i}_Q - L_G \dot{i}_G
\]

Let’s modify the G-winding voltage equation by adding and subtracting \( l_G \) (\( di_G/dt \)):

\[
v_G = 0 = -r_G i_G - kM_G \dot{i}_q - M_Y \dot{i}_Q - L_G \dot{i}_G + l_G \dot{i}_G - l_G \dot{i}_G
\]

which can be written as:

\[
v_G = 0 = -r_G i_G - kM_G \dot{i}_q - M_Y \dot{i}_Q - (L_G - l_G) \dot{i}_G - l_G \dot{i}_G
\]

The advantage to this is that, in pu, we have \( L_G - l_G = kM_G = M_Y = L_{\text{AQ}} \).
Therefore
\[ v_G = 0 = -r_G i_G - l_G \dot{i}_G - L_{AQ} (\dot{i}_q + \dot{i}_Q + \dot{i}_G) \]
Repeating this procedure for the F, D, and q equations, and then summarizing, we obtain:

**D-axis relations:**
\[
\begin{align*}
v_d &= -r_d \dot{i}_d - l_d \dot{i}_d - L_{AD} [\dot{i}_d + i_F + \dot{i}_D] - \omega \lambda_q \\
v_F &= -r_F i_F - l_F \dot{i}_F - L_{AD} [\dot{i}_d + i_F + \dot{i}_D] \\
v_D &= 0 = -r_D \dot{i}_D - l_D \dot{i}_D - L_{AD} [\dot{i}_d + i_F + \dot{i}_D]
\end{align*}
\]

**Q-axis relations:**
\[
\begin{align*}
v_q &= -r_q \dot{i}_q - l_q \dot{i}_q - L_{AQ} (\dot{i}_q + \dot{i}_Q + \dot{i}_G) + \omega \lambda_d \\
v_Q &= 0 = -r_Q \dot{i}_Q - l_Q \dot{i}_Q - L_{AQ} (\dot{i}_q + \dot{i}_Q + \dot{i}_G) \\
v_G &= 0 = -r_G \dot{i}_G - l_G \dot{i}_G - L_{AQ} (\dot{i}_q + \dot{i}_Q + \dot{i}_G)
\end{align*}
\]
We desire to draw circuits that are characterized by these equations.

**Note:**
- The D-axis relations are coupled through the \( L_{AD} \) terms.
- This term, for each equation, may be represented by a single “center” branch.
- The other terms, for each equation, may be represented as single branches which feed the center branch.
This results in the circuit of Fig 4.5 in your text.

Similar reasoning results in the circuit of Fig. 4.6 in your text.
We redraw these circuits below.
Direct-axis equivalent circuit:
The above is the same as Fig. 4.5 in your text

\[
 v_d = -r_i d - l_d i_d - L_A D [i_d + i_F + i_D] - \omega \lambda_q \\
 - v_F = -r_F i_F - l_F i_F - L_A D [i_d + i_F + i_D] \\
 v_D = 0 = -r_D i_D - l_D i_D - L_A D [i_d + i_F + i_D]
\]

Quadrature-axis equivalent circuit:
The above is the same as Fig. 4.6 in your text, except we have included the G-circuit

\[
 v_q = -r_i q - l_q i_q - L_A Q (i_q + i_Q + i_G) + \omega \lambda_d \\
 v_Q = 0 = -r_Q i_Q - l_Q i_Q - L_A Q (i_q + i_Q + i_G) \\
 v_G = 0 = -r_G i_G - l_G i_G - L_A Q (i_q + i_Q + i_G)
\]
The ability to draw these circuits is a direct result of the $L_{AD}$ and $L_{AQ}$ relations that occur only in per-unit. Therefore, it is important to be in the per-unit system when utilizing these circuits.

These equivalent circuits are useful for:
- Remembering the voltage relations
- Gaining physical understanding of relations between d-q-F-D-Q-G quantities.
- Reading the literature, where you will see them often.