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Two-Reaction Theory of Synchronous Machines

Generalized Method of Analysis—Part I

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Synopsis.—Starting with the basic assumption of no saturation or hysteresis, and with distribution of armature phase m. m. f. effectively sinusoidal as far as regards phenomena dependent upon rotor position, general formulas are developed for current, voltage, power, and torque under steady and transient load conditions. Special detailed formulas are also developed which permit the determination of current and torque on three-phase short circuit, during starting, and when only small deviations from an average operating angle are involved.

In addition, new and more accurate equivalent circuits are developed for synchronous and asynchronous machines operating in parallel, and the domain of validity of such circuits is established.

Throughout, the treatment has been generalized to include salient poles and an arbitrary number of rotor circuits. The analysis is thus adapted to machines equipped with field pole collars, or with amortisseur windings of any arbitrary construction.

It is proposed to continue the analysis in a subsequent paper.

* * * * *

THIS paper presents a generalization and extension of the work of Blondel, Dreyfus, and Doherty and Nickle, and establishes new and general methods of calculating current power and torque in salient and non-salient pole synchronous machines, under both transient and steady load conditions.

Attention is restricted to symmetrical three-phase† machines with field structure symmetrical about the axes of the field winding and interpolar space, but salient poles and an arbitrary number of rotor‡ circuits is considered.

Idealization is resorted to, to the extent that saturation and hysteresis in every magnetic circuit and eddy currents in the armature iron are neglected, and in the assumption that, as far as concerns effects depending on the position of the rotor, each armature winding may be regarded as, in effect, sinusoidally distributed.[3]

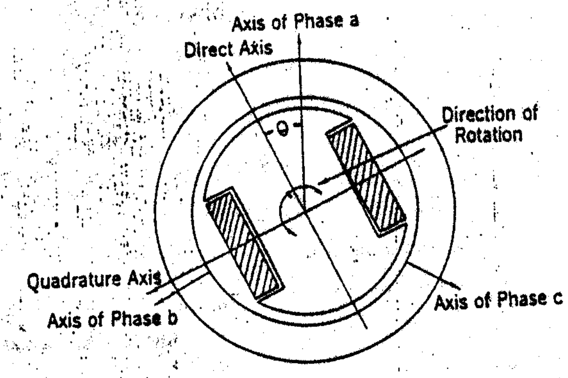

FIG. 1

A. Fundamental Circuit Equations

Consider the ideal synchronous machine of Fig. 1, and let

i_a, i_b, i_c = per unit instantaneous phase currents

e_a, e_b, e_c = per unit instantaneous phase voltages

ψ_a, ψ_b, ψ_c = per unit instantaneous phase linkages

t = time in electrical radians

$$p = \frac{d}{d\,t}$$

Then there is

$$\begin{aligned} e_a &= p\,\psi_a - r\,i_a \\ e_b &= p\,\psi_b - r\,i_b \\ e_c &= p\,\psi_c - r\,i_c \end{aligned} \qquad (1)$$

It has been shown previously[1] that

$$\begin{aligned} \psi_a = {} & I_d \cos\theta - I_q \sin\theta \\ & - \frac{x_0}{3}[i_a + i_b + i_c] - \frac{x_d + x_q}{3}\left[i_a - \frac{i_b + i_c}{2}\right] \\ & - \frac{x_d - x_q}{3}[i_a \cos 2\theta + i_b \cos(2\theta - 120) \\ & \qquad + i_c \cos(2\theta + 120)] \\ \psi_b = {} & I_d \cos(\theta - 120) - I_q \sin(\theta - 120) \\ & - x_0 \frac{i_a + i_b + i_c}{3} - \frac{x_d + x_q}{3}\left[i_b - \frac{i_c + i_a}{2}\right] \\ & - \frac{x_d - x_q}{3}[i_a \cos(2\theta - 120) + i_b \cos(2\theta + 120) \\ & \qquad + i_c \cos 2\theta] \end{aligned} \qquad (2)$$

$$\begin{aligned} \psi_c = {} & I_d \cos(\theta + 120) \\ & - I_q \sin(\theta + 120) - x_0 \frac{i_a + i_b + i_c}{3} \\ & - \frac{x_d + x_c}{3}\left[i_c - \frac{i_a + i_b}{2}\right] \\ & - \frac{x_d - x_q}{3}[i_a \cos(2\theta + 120) + i_b \cos 2\theta \\ & \qquad + i_c \cos(2\theta - 120)] \end{aligned}$$

where,

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†Single-phase machines may be regarded as three-phase machines with one phase open circuited.

‡Stator for a machine with stationary field structure.

[3]For numbered references see Bibliography.

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I_d = per-unit excitation in direct axis
I_q = per-unit excitation in quadrature axis
x_d = direct synchronous reactance
x_q = quadrature synchronous reactance
x_0 = zero phase-sequence reactance

As shown in the Appendix, if normal linkages in the field circuit are defined as those obtaining at no load* there is in the case of no rotor circuits in the direct axis in addition to the field,

Φ = per-unit instantaneous field linkages
$= I - (x_d - x_d')\, i_d$

where,

I = per-unit instantaneous field current

$$i_d = \frac{2}{3}\{i_a \cos\theta + i_b \cos(\theta - 120) + i_c \cos(\theta + 120)\} \tag{3}$$

On the other hand, if n additional rotor circuits exist in the direct axis there is,

$$\Phi = I + X_{f1d} I_{1d} + X_{f2d} I_{2d} + \ldots + X_{fnd} I_{nd} - (x_d - x_d')\, i_d$$

where,

I_{1d}, I_{2d}, . . . etc., are the per-unit instantaneous currents in circuits 1, 2, etc., of the direct axis, X_{f1d}, X_{f2d}, . . . etc., are per-unit mutual coefficients between the field and circuits 1, 2, etc., of the direct axis.

Similar relations exist for the linkages in each of the additional rotor circuits except $x_d - x_d'$ is to be replaced by a term x_m. However, since all of these additional circuits are closed, it follows that there is an operational result

$$I_d = I + I_{1d} + I_{2d} + \ldots + I_{nd} = G(p)\,E + H(p)\,i_d \tag{4}$$

where E is the per-unit value of the instantaneous field voltage, and $G(p)$ and $H(p)$ are operators such that

$$G(o) = 1 \qquad G(\infty) = 0$$
$$H(o) = 0 \qquad H(\infty) = x_d - x_d''$$

x_d'' = the subtransient reactance[2]

It will be convenient to write $H(p) = x_d - x_d(p)$ and to rewrite (4) in the form,

$$I_d = G(p)\,E + [x_d - x_d(p)]\, i_d \tag{4a}$$

If there are no additional rotor circuits, there is, as shown in Appendix I,

$$\Psi = I - (x_d - x_d')\, i_d$$
$$E = T_0\, p\, \Psi + I$$

where T_0 is the open circuit time constant of the field in radians.

There is then,

$$G(p) = \frac{1}{T_0 p + 1}$$

$$x_d(p) = \frac{x_d' T_0 p + x_d}{T_0 p + 1}$$

*This definition is somewhat different from that given in reference 2.

If there is one additional rotor circuit in the direct axis there is,

$$\Psi = I + X_{f1d} I_{1d} - (x_d - x_d')\, i_d = \frac{E - I}{T_0\, p}$$

$$\Psi_{1d} = X_{11d} I_{1d} + X_{f1d} I - x_{m1d}\, i_d = \frac{-I_{1d}}{T_{01d}\, p}$$

which gives,

$$G(p) = \frac{[X_{11d} - X_{f1d}]\, T_{01d}\, p + 1}{A(p)}$$

$$x_d(p) = x_d - \frac{T_0 T_{01d} [X_{11d}(x_d - x_d') - X_{f1d} x_{m1d}]\, p^2 + [(x_d - x'_d)\, T_{01d} + x_{m1d} T_0]\, p}{A(p)}$$

where,

$$A(p) = [X_{11d} - X_{f1d}^2]\, T_0 T_{01d}\, p^2 + [X_{11d} T_0 + T_{01d}]\, p + 1$$

If there is more than one additional rotor circuit the operators $G(p)$ and $x_d(p)$ will be more complicated but may be found in the same way. The effects of external field resistance may be found by changing the term I in the field voltage equation to $R\,I$. Open circuited field corresponds to R equal to infinity.

Similarly, there will be

$$I_q = [x_q - x_q(p)]\, i_q \tag{5}$$

where,

$$i_q = -\frac{2}{3}\{i_a \sin\theta + i_b \sin(\theta - 120) + i_c \sin(\theta + 120)\} \tag{3a}$$

$$x_q(0) = x_q,\quad x_q(\infty) = x_q''$$

So far, 10 equations have been established relating the 15 quantities e_a, e_b, e_c, i_a, i_b, i_c, ψ_a, ψ_b, ψ_c, i_d, i_q, I_d, I_q, E, θ in a general way. It follows that when any five of the quantities are known the remaining 10 may be determined. Their determination is very much facilitated, however, by the introduction of certain auxiliary quantities e_d, e_q, e_0, i_0, ψ_d, ψ_q, ψ_0.

Thus, let

$$i_0 = \frac{1}{3}\{i_a + i_b + i_c\} \tag{3b}$$

$$e_d = \frac{2}{3}\{e_a \cos\theta + e_b \cos(\theta - 120) + e_c \cos(\theta + 120)\}$$

$$e_q = -\frac{2}{3}\{e_a \sin\theta + e_b \sin(\theta - 120) + e_c \sin(\theta + 120)\} \tag{6}$$

$$e_0 = \frac{1}{3}\{e_a + e_b + e_c\}$$

$$\psi_d = \frac{2}{3}\{\psi_a \cos\theta + \psi_b \cos(\theta - 120) + \psi_c \cos(\theta + 120)\}$$

$$\psi_q = -\frac{2}{3}\{\psi_a \sin\theta + \psi_b \sin(\theta - 120) + \psi_c \sin(\theta + 120)\} \tag{7}$$

$$\psi_0 = \frac{1}{3}\{\psi_a + \psi_b + \psi_c\}$$

then from Equation (1) there is

$$e_d = \frac{2}{3}\{\cos\theta\, p\,\psi_a + \cos(\theta-120)\, p\,\psi_b + \cos(\theta+120)\, p\,\psi_c\} - r\,i_d$$

$$e_q = -\frac{2}{3}\{\sin\theta\, p\,\psi_a + \sin(\theta-120)\, p\,\psi_b + \sin(\theta+120)\, p\,\psi_c\} - r\,i_q$$

$$e_0 = p\,\psi_0 - r\,i_0$$

but,

$$p\,\psi_d = \frac{2}{3}\{\cos\theta\, p\,\psi_a + \cos(\theta-120)\, p\,\psi_b + \cos(\theta+120)\, p\,\psi_c\}$$
$$-\frac{2}{3}\{\sin\theta\,\psi_a + \sin(\theta-120)\, p\,\psi_b + \sin(\theta+120)\, p\,\psi_c\}\, p\,\theta$$
$$= e_d + r\,i_d + \psi_q\, p\,\theta$$

$$p\,\psi_q = -\frac{2}{3}\{\sin\theta\, p\,\psi_a + \sin(\theta-120)\, p\,\psi_b + \sin(\theta+120)\, p\,\psi_c\}$$
$$-\frac{2}{3}\{\cos\theta\,\psi_a + \cos(\theta-120)\,\psi_b + \cos(\theta+120)\,\psi_c\}\, p\,\theta$$
$$= e_q + r\,i_q - \psi_d\, p\,\theta$$

hence there is

$$e_d = p\,\psi_d - r\,i_d - \psi_q\, p\,\theta \qquad (8)$$
$$e_q = p\,\psi_q - r\,i_q + \psi_d\, p\,\theta \qquad (9)$$
$$e_0 = p\,\psi_0 - r\,i_0 \qquad (10)$$

Also it may be readily verified that

$$\psi_d = I_d - x_d\, i_d = G(p)\,E - x_d(p)\, i_d \qquad (11)$$
$$\psi_q = I_q - x_q\, i_q = -x_q(p)\, i_q \qquad (12)$$
$$\psi_0 = -x_0\, i_0 \qquad (13)$$

Equations (8) to (13) establish six relatively simple relations between the 11 quantities e_d, e_q, e_0, i_d, i_q, i_0, ψ_d, ψ_q, ψ_0, E, θ. In practise it is usually possible to determine five of these quantities directly from the terminal conditions, after which the remaining six may be calculated with relative simplicity. After the direct, quadrature, and zero quantities are known the phase quantities may be determined from the identical relations

$$\begin{aligned} i_a &= i_d\cos\theta - i_q\sin\theta + i_0 \\ i_b &= i_d\cos(\theta-120) - i_q\sin(\theta-120) + i_0 \qquad (14) \\ i_c &= i_d\cos(\theta+120) - i_q\sin(\theta+120) + i_0 \end{aligned}$$

$$\begin{aligned} \psi_a &= \psi_d\cos\theta - \psi_q\sin\theta + \psi_0 \\ \psi_b &= \psi_d\cos(\theta-120) - \psi_q\sin(\theta-120) + \psi_0 \qquad (15) \\ \psi_c &= \psi_d\cos(\theta+120) - \psi_q\sin(\theta+120) + \psi_0 \end{aligned}$$

$$\begin{aligned} e_a &= e_d\cos\theta - e_q\sin\theta + e_0 \\ e_b &= e_d\cos(\theta-120) - e_q\sin(\theta-120) + e_0 \qquad (16) \\ e_c &= e_d\cos(\theta+120) - e_q\sin(\theta+120) + e_0 \end{aligned}$$

Referring to Fig. 2, it may be seen that when there are no zero quantities, that is, when $e_0 = \psi_0 = i_0 = 0$, the phase quantities may be regarded as the projection of vectors $\bar{e}$, $\bar{\psi}$, and $\bar{i}$ on axes lagging the direct axis by

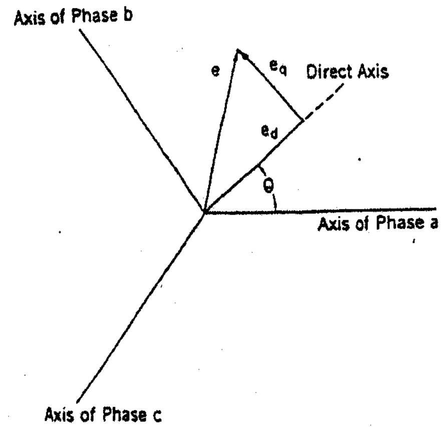

FIG. 2

angles θ, $\theta - 120$ and $\theta + 120$, where taking the direct axis as the axis of reals,

$$\bar{e} = e_d + j\,e_q$$
$$\bar{\psi} = \psi_d + j\,\psi_q$$
$$\bar{i} = i_d + j\,i_q$$

If we introduce in addition the vector quantity,

$$\bar{I} = I_d + j\,I_q$$

the circuit equations previously obtained may be

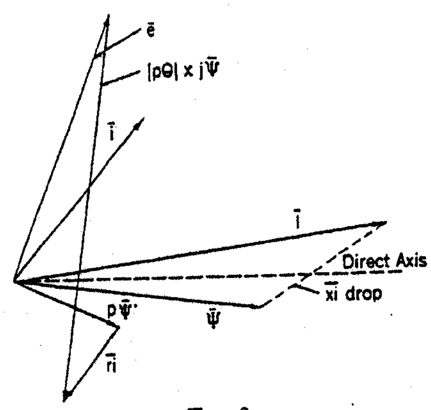

FIG. 3

transferred into the corresponding vector forms,

$$\bar{e} = p\,\bar{\psi} - \overline{ri} + [p\,\theta]\,j\,\bar{\psi}$$
$$\bar{\psi} = \bar{I} - \overline{xi}$$

where, $\overline{xi} = x_d\, i_d + j\,x_q\, i_q$

Fig. 3 shows these relations graphically.

B. Armature Power Output

The per-unit instantaneous power output from the armature is necessarily proportional to the sum

$e_a\, i_a + e_b\, i_b + e_c\, i_c$. By consideration of any instant during normal operation at unity power factor it may be seen that the factor of proportionality must be 2/3. That is,

$$P = \text{per-unit instantaneous power output}$$
$$= 2/3\ \{ e_a\, i_a + e_b\, i_b + e_c\, i_c \}$$

Substituting from Equations (14) and (16) there results the useful relation,

$$P = e_d\, i_d + e_q\, i_q + e_0\, i_0 \tag{17}$$

C. Electrical Torque on Rotor

It is possible to determine the electrical torque on the rotor directly from the general relation,

{Total power output} =

{mechanical power transferred across gap}

+ {rate of decrease of total stored magnetic energy}

− {total ohmic losses} (18)

However, since this torque depends uniquely only on the magnitudes of the currents in every circuit of the machine, it follows that a general formula for torque may be derived by considering any special case in which arbitrary conditions are imposed as to the way in which these currents are changing as the rotor moves.

The simplest conditions to impose are that I_d, I_q, i_d, i_q, and i_0 remain constant as the rotor moves. In this case there will be no change in the stored magnetic energy of the machine as the rotor moves, and the power output of the rotor will be just equal in magnitude and opposite in sign to the rotor losses. It follows that under the special conditions assumed, Equation (18) becomes simply,

{armature power output} =

{mechanical power across gap} − {armature losses}

$$\text{or, } P = T\, p\, \theta - \frac{2\, r}{3}\ \{ i_a^2 + i_b^2 + i_c^2 \}$$

$$= T\, p\, \theta - r\ \{ i_d^2 + i_q^2 + i_0^2 \}$$

Then,

T = per-unit instantaneous electrical torque

$$= \frac{e_d\, i_d + e_q\, i_q + e_0\, i_0 + r\ \{i_d^2 + i_q^2 + i_0^2\}}{p\, \theta}$$

but subject to the conditions imposed,

$$e_d = -\ \psi_q\, p\, \theta - r\, i_d$$
$$e_q = \psi_d\, p\, \theta - r\, i_q$$
$$e_0 = -\ r\, i_0$$

It therefore follows that,

$$T = i_q\, \psi_d - i_d\, \psi_q \tag{19}$$
$$= \text{vector product of } \bar{\psi} \text{ and } \bar{i}$$
$$= \bar{\psi} \times \bar{i} \tag{19a}$$

a result which could have been established directly by physical reasoning. Formula (19) is employed by Dreyfus in his treatment of self-excited oscillations of synchronous machines.[14]

D. Constant Rotor Speed

Suppose that the constant slip of the rotor is s. Then there is,

$$e_d = p\, \psi_d - r\, i_d - (1 - s)\, \psi_q$$
$$e_q = p\, \psi_q - r\, i_q + (1 - s)\, \psi_d$$

but,

$$\psi_d = G\,(p)\, E - x_d\,(p)\, i_d$$
$$\psi_q = -\ x_q\,(p)\, i_q$$

Putting

$$p\, x_d\,(p) + r = z_d\,(p)$$
$$p\, x_q\,(p) + r = z_q\,(p)$$

there is

$$e_d = p\, G\,(p)\, E - z_d\,(p)\, i_d + (1 - s)\, x_q\,(p)\, i_q \tag{20}$$
$$e_q = (1 - s)\, [G\,(p)\, E - x_d\,(p)\, i_d] - z_q\,(p)\, i_q \tag{21}$$

Solving gives,

$$i_d = \{[p\, z_q\,(p) + (1 - s)^2 x_q\,(p)]\, G\,(p)\, E - z_q\,(p)\, e_d - (1 - s)\, x_q\,(p)\, e_q\ \} \div D\,(p) \tag{22}$$

$$i_q = \frac{(1 - s)\, r\, G\,(p)\, E - z_d\,(p)\, e_q + (1 - s)\, x_d\,(p)\, e_d}{D\,(p)} \tag{23}$$

where, $D\,(p) = z_d\,(p)\, z_q\,(p) + (1 - s)^2\, x_d\,(p)\, x_q\,(p)$

E. Two Machines Connected Together

Suppose that two machines which we will designate respectively by the subscripts g and h, are connected together, but not to any other machines or circuits, and assume in addition that there are no zero quantities. In this case the voltages of each machine will be equal

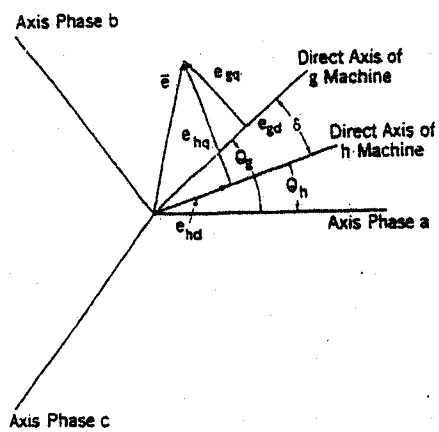

FIG. 4

phase for phase, and it therefore follows that the voltage vectors of each machine must coincide, as shown in Fig. 4.

Referring to the figure it will be seen that the direct and quadrature components of voltage of the two machines are subject to the mutual relations,

$$e_{hd} = e_{gd} \cos \delta - e_{gq} \sin \delta$$
$$e_{hq} = e_{gd} \sin \delta + e_{gq} \cos \delta \tag{24}$$
$$e_{gd} = e_{hd} \cos \delta + e_{hq} \sin \delta$$
$$e_{gq} = -\ e_{hd} \sin \delta + e_{hq} \cos \delta \tag{25}$$

On the other hand, for currents there will be

$$i_{hd} = -\{i_{gd}\cos\delta - i_{gq}\sin\delta\}$$
$$i_{hq} = -\{i_{gd}\sin\delta + i_{gq}\cos\delta\} \qquad (26)$$
$$i_{gd} = -\{i_{hd}\cos\delta + i_{hq}\sin\delta\}$$
$$i_{gq} = -\{-i_{hd}\sin\delta + i_{hq}\cos\delta\} \qquad (27)$$

F. One Machine on an Infinite Bus

In (*E*), if machine h has zero impedance, it follows from (20) and (21) that $e_{hd} = 0$, e_{hq} = bus voltage say $= e$.

Then for machine g there is,

$$e_d = e\sin\delta$$
$$e_q = e\cos\delta \qquad (28)$$

G. Torque Angle Relations

From Equations (11), (12), and (19), there is,

$$T = \frac{I_q\,\psi_d}{x_q} - \frac{I_d\,\psi_q}{x_d} - \frac{x_d - x_q}{x_d\,x_q}\,\psi_d\,\psi_q$$

Then if the rotor leads the vector $\bar{\psi}$ by an angle $\bar{\delta}$ there is

$$\psi_q = -\psi\sin\delta$$
$$\psi_d = \psi\cos\delta$$
$$T = \frac{I_q\,\psi}{x_q}\cos\delta + \frac{I_d\,\psi\sin\delta}{x_d} + \frac{x_d - x_q}{2\,x_d\,x_q}\,\psi^2\sin 2\delta \qquad (29)$$

A derivation of this formula for steady load conditions has been previously given by Doherty and Nickle.[9]

H. Three-Phase Short Circuit with Constant Rotor Speed Maintained

Since a three-phase short circuit causes e_d and e_q to vanish suddenly, its effect with constant rotor speed maintained may be found by impressing $e_d = -e_{d0}$, $e_q = -e_{q0}$ in (22) and (23) where e_{d0} and e_{q0} are the values of e_d and e_q before the short circuit. The initial currents existing before the short circuit must be added to the currents found in this way in order to obtain the resultant current after the short circuit.

With $s = 0$ and E constant there is in detail,

$$i_d = \frac{x_q(p)\,e_{d0} + x_q(p)\,e_{q0}}{D(p)}\cdot 1 + \frac{x_q E - r\,e_{d0} - x_q\,e_{q0}}{r^2 + x_d\,x_q}$$

$$i_q = \frac{x_d(p)\,e_{q0} - x_d(p)\,e_{d0}}{D(p)}\cdot 1 + \frac{r\,E - r\,e_{q0} + x_d\,e_{d0}}{r^2 + x_d\,x_q} \qquad (30)$$

The working out of the formulas may be illustrated by consideration of the simple case of a machine with no rotor circuits in addition to the field. In this case there is

$$x_q(p) = x_q$$
$$x_d(p) = \frac{x_d'\,T_0\,p + x_d}{T_0\,p + 1}$$
$$D(p) = \left\langle \frac{x_d'\,T_0\,p + x_d}{T_0\,p + 1}\,p + r \right\rangle (x_q\,p + r)$$
$$+ \frac{x_d'\,T_0\,p + x_d}{T_0\,p + 1}\,x_q$$

$$= \frac{\begin{aligned} &x_d'\,x_q\,T_0\,p^3 \\ &+ [x_d'\,r\,T_0 + (x_d + r\,T_0)\,x_q]\,p^2 \\ &+ [r\,(x_d + x_q + r\,T_0) + x_d'\,x_q\,T_0]\,p \\ &+ r^2 + x_d\,x_q \end{aligned}}{T_0\,p + 1}$$

$$= \frac{d(p)}{T_0\,p + 1} \qquad (31)$$

By the expansion theorem there is, finally,

$$i_d = \frac{x_q\,E}{r^2 + x_d\,x_q} + \sum_1^3 \frac{(T_0\,\alpha_n + 1)\,\langle (x_q\,\alpha_n + r)\,e_{d0} + x_q\,e_{q0}\rangle\,\epsilon^{-\alpha_n t}}{\alpha_n\,d'(\alpha_n)}$$

$$i_q = \frac{r\,E}{r^2 + x_d\,x_q} + \sum_1^3 \frac{(x_d'\,T_0\,\alpha_n^2 + (x_d + r\,T_0)\,\alpha_n + r)\,e_{q0} - (T_0\,\alpha_n\,x'_d + x_d)\,e_{d0}}{\alpha_n\,d'(\alpha_n)}\,\epsilon^{-\alpha_n t} \qquad (32)$$

where the summation is extended over the roots of

$$d(\alpha) = 0 \text{ and } d'(p) = \frac{d}{d\,p}\,d(p)$$

The phase currents may, of course, be found from

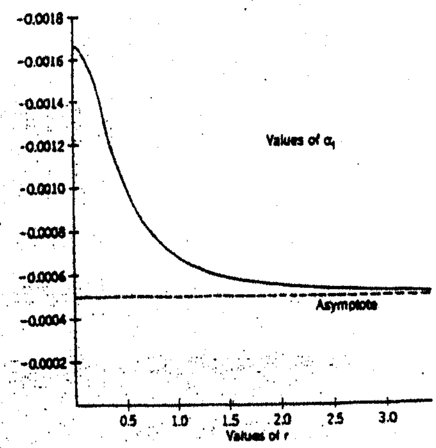

FIG. 5

Equations (32) by the application of Equations (14). For the particular case

$$T_0 = 2{,}000,\ x_d = 1.00,\ x_q = 0.60,\ x_d' = 0.30$$

the roots α_1, α_2, α_3 of the equation $d(p) = 0$, were found to be as shown in Figs. 5, 6, and 7, where

$$\alpha_2 = \alpha_a + \alpha_b$$
$$\alpha_3 = \alpha_a - \alpha_b$$

It will be noted that, as would necessarily be the case, where $r = 0$, α_1 is equal to the reciprocal of the short circuit time constant of the machine, *i. e.*, for $r = 0$,

$$\alpha_1 = -\frac{x_d}{x_d'}\frac{1}{T_0} = -0.001667$$

while for $r = \infty$

$$\alpha_1 = -\frac{1}{T_0} = -0.000500$$

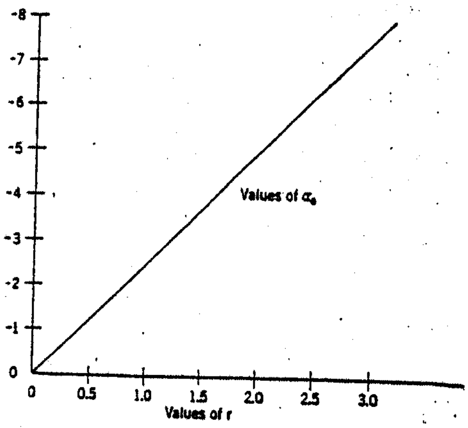

FIG. 6

The root α_a is found to be almost exactly equal to the value which it would have were $T_0 = \infty$, *i. e.*,

$$\alpha_a = \frac{r\,(x_d' + x_q)}{2\,x_d'\,x_q} \text{ approximately}$$

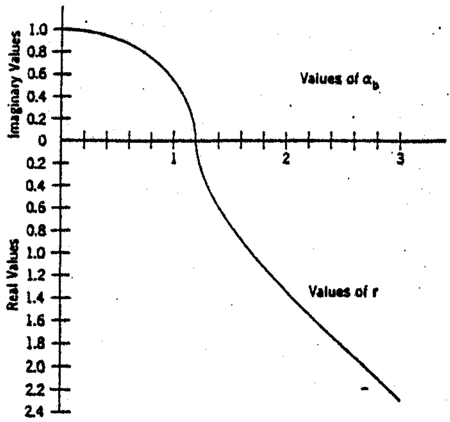

FIG. 7

Thus, in the special case considered this approximate formula gives

$$\alpha_a = \frac{(0.30 + 0.60)\,r}{2 \times 0.30 \times 0.60} = 2.50\,r$$

which checks the result found by the exact solution of the cubic.

I. Starting Torque

On infinite bus and with slip s, there will be, choosing $\delta = \frac{\pi}{2} - s\,t$, and referring to Equation (28),

$$e_d = \cos s\,t$$
$$e_q = \sin s\,t$$

If we now introduce a system of vectors rotating at s per-unit angular velocity there is

$$e_d = 1.0$$
$$e_q = -j$$
$$p = j\,s \qquad (33)$$

Then from (22) and (23),

$$i_d = \{ j\,s\,x_q(j\,s) + r - j\,(1-s)\,x_q(j\,s) \} \div \{ [j\,s\,x_d(j\,s) + r]\,[j\,s\,x_q(j\,s) + r] + (1-s)^2\,x_d(j\,s)\,x_q(j\,s) \}$$

$$= \frac{j\,(1-2\,s)\,x_q(j\,s) - r}{r^2 + (1-2\,s)\,x_d'(j\,s)\,x_q'(j\,s) + j\,s\,r[x_d'(j\,s) + x_q'(j\,s)]}$$

$$= \left\{ j\,x_q(j\,s) - \frac{r}{1-2\,s} \right\} \div \left\{ x_d(j\,s)\,x_q(j\,s) + \frac{r}{1-2\,s}\,[r + j\,s\,(x_d(j\,s) + x_q(j\,s))] \right\} \qquad (34)$$

$$i_q = -\frac{[j\,s\,x_d(j\,s) + r]\,(-j) - (1-s)\,x_d(j\,s)}{r^2 + (1-2\,s)\,x_d(j\,s)\,x_q(j\,s) + j\,s\,r[x_d(j\,s) + x_q(j\,s)]}$$

$$= \left\{ x_d(j\,s) + \frac{j\,r}{1-2\,s} \right\} \div \left\{ x_d(j\,s)\,x_q(j\,s) + \frac{r}{1-2\,s}\,[r + j\,s\,(x_d(j\,s) + x_q(j\,s))] \right\} \qquad (35)$$

The expressions for average power and torque then become,

$$P_{av} = 1/2\,[e_d \cdot i_d + e_q \cdot i_q]$$
$$T_{av} = 1/2\,[i_q \cdot \psi_d - i_d \cdot \psi_q]$$

where the dot indicates the scalar product, or

$$P_{av} = 1/2\,[1 \cdot i_d - j \cdot i_q]$$
$$= 1/2\,[\text{Real of } i_d - \text{Imaginary of } i_q] \qquad (36)$$

There is in general,

$$e_d + r\,i_d = p\,\psi_d - (1-s)\,\psi_q$$
$$e_q + r\,i_q = (1-s)\,\psi_d + p\,\psi_q$$

$$\psi_d = \frac{\begin{vmatrix} e_d + r\,i_d & -(1-s) \\ e_q + r\,i_q & p \end{vmatrix}}{\begin{vmatrix} p & -(1-s) \\ (1-s) & p \end{vmatrix}}$$

$$= \frac{p\,(e_d + r\,i_d) + (1-s)\,(e_q + r\,i_q)}{p^2 + (1-s)^2} \qquad (37)$$

$$\psi_q = \frac{p\,(e_q + r\,i_q) - (1-s)\,(e_d + r\,i_d)}{p^2 + (1-s)^2} \qquad (38)$$

$$\psi_d = \frac{j\,s\,(e_d + r\,i_d) + (1-s)\,(e_q + r\,i_q)}{1-2\,s}$$

$$\psi_q = \frac{j\,s\,(e_q + r\,i_q) - (1-s)\,(e_d + r\,i_d)}{1-2\,s}$$

with $e_d = 1.0,\ e_q = -j$

$$\psi_d = \frac{j\,s + j\,s\,r\,i_d + (1-s)\,(-j) + (1-s)\,r\,i_q}{1-2\,s}$$

$$= \frac{-(1-2s)\,j + r\,[j\,s\,i_d + (1-s)\,i_q]}{1-2\,s}$$

$$= -j + \frac{r}{1-2\,s}\,[j\,s\,i_d + (1-s)\,i_q] \qquad (39)$$

$$\psi_q = \frac{j\,s\,(-j + r\,i_q) - (1-s) - r\,(1-s)\,i_d}{1-2\,s}$$

$$= \frac{-(1-2\,s) + r\,[j\,s\,i_q - (1-s)\,i_d]}{1-2\,s}$$

$$= -1 + \frac{r}{1-2\,s}\,[j\,s\,i_q - (1-s)\,i_d] \qquad (40)$$

Thus,

$$T_{av} = 1/2 \begin{bmatrix} i_q\,.\,(-j) + i_q\,.\,\dfrac{r}{1-2\,s}\,\langle j\,s\,i_d + (1-s)\,i_q\rangle \\ -\,i_d\,.\,(-1) - i_d\,.\,\dfrac{r}{1-2\,s}\,\langle j\,s\,i_q - (1-s)\,i_d\rangle \end{bmatrix}$$

$$= P_{av} + \frac{r}{2(1-2\,s)} \begin{bmatrix} (1-s)\,\langle i_q^2 + i_d^2\rangle \\ +\,2\,s\,i_q\,.\,j\,i_d \end{bmatrix}$$

$$= P_{av} + \frac{r}{2}\,(i_q^2 + i_d^2) + \frac{r\,s}{2(1-2\,s)} \begin{bmatrix} i_q^2 + i_d^2 \\ +\,2\,i_q\,.\,j\,i_d \end{bmatrix}$$

$$= P_{av} + r\,\frac{i_q^2 + i_d^2}{2} + \frac{r\,s}{2(1-2\,s)}\,(i_q + j\,i_d)^2 \qquad (41)$$

Mr. Ralph Hammar, who has been engaged in the application of the general method of calculation outlined above, to the predetermination of the starting torque of practical synchronous motors, has suggested an interesting modification of formulas (36) and (41), based upon the fact that, since the total m. m. f. consists of direct and quadrature components pulsating at slip frequency, it may be resolved into two components, one moving forward at a per-unit speed $1 - s + s = 1.0$, and the other moving backward at a per-unit speed $1 - s - s = 1 - 2\,s$. Thus from this standpoint half of both the direct and quadrature components will move forward, and half backward. Since the quadrature axis is ahead of the direct it follows that as far as concerns the forward component the quadrature current i_q is equivalent to a d-c. $j\,i_q$, while as regards backward component it is equivalent to a direct component $-j\,i_q$. It follows that the vector amounts of forward and backward m. m. f. or current are

$$\text{forward current} = i_f = \frac{1}{2}\,(i_d + j\,i_q)$$

$$\text{backward current} = i_b = \frac{1}{2}\,(i_d - j\,i_q) \qquad (42)$$

If we define by analogy,

$$\text{forward voltage} = \frac{1}{2}\,(e_d + j\,e_q)$$

$$\text{backward voltage} = \frac{1}{2}\,(e_d - j\,e_q) \qquad (43)$$

There is,

$$i_f = \frac{1}{2}\left\{\frac{-2\,r}{1-2\,s} + j\,[x_d\,(j\,s) + x_q\,(j\,s)]\right\} \div \left\{x_d\,(j\,s)\,x_q\,(j\,s) + \frac{r}{1-2\,s}\,\langle r + j\,s\,[x_d\,(j\,s) + x_q\,(j\,s)]\rangle\right\}$$

$$i_b = \frac{1}{2}\left\{j\,[x_q\,(j\,s) - x_d\,(j\,s)]\right\} \div \left\{x_d\,(j\,s)\,x_q\,(j\,s) + \frac{r}{1-2\,s}\,\langle r + j\,s\,[x_d\,(j\,s) + x_q\,(j\,s)]\rangle\right\}$$

$$e_f = 1.0 \qquad (44)$$

$$e_b = 0 \qquad (45)$$

$$P_{av} = e_f\,.\,i_f = \text{real of } i_f \qquad (46)$$

$$T_{av} = P_{av} + r\,i_f^2 + \frac{r}{1-2\,s}\,i_b^2 \qquad (47)$$

J. *Zero Armature Resistance, One Machine Connected to an Infinite Bus*

Assume that a machine of negligible armature resistance is operating from an infinite bus of per-unit voltage e, at synchronous speed, with a steady excitation voltage E_0, and displacement angle δ_0. At the instant $t = 0$, let δ and E change.

There is,

$$i_d = \frac{E_0 - \psi_{d0}}{x_d} - \frac{1}{x_d\,(p)}\,\Delta\,\psi_d + \frac{G\,(p)}{x_d\,(p)}\,\Delta\,E$$

$$i_q = -\frac{\psi_{q0}}{x_q} - \frac{1}{x_q\,(p)}\,\Delta\,\psi_q$$

$$\psi_d = e\cos\delta$$

$$\psi_q = -e\sin\delta$$

From which there is, by obvious re-arrangement,

$$i_d = \frac{E - e\cos\delta}{x_d} + e\,\frac{x_d - x_d(p)}{x_d\,x_d(p)}(\cos\delta_0 - \cos\delta) - \frac{x_d(p) - G(p)\,x_d}{x_d\,x_d(p)}\,\Delta E$$

$$i_q = \frac{e\sin\delta}{x_q} + e\,\frac{x_q - x_q(p)}{x_q\,x_q(p)}(\sin\delta - \sin\delta_0) \qquad (48)$$

Then,

$$T = \frac{E\,e\sin\delta}{x_d} + \frac{x_d - x_q}{2\,x_d\,x_q}\,e^2\sin 2\delta + e^2\cos\delta\,\frac{x_q - x_q(p)}{x_q\,x_q(p)}(\sin\delta - \sin\delta_0) \qquad (49)$$

$$+ e^2\sin\delta\,\frac{x_d - x_d(p)}{x_d\,x_d(p)}(\cos\delta_0 - \cos\delta) - e\sin\delta\,\frac{x_d(p) - x_d\,G(p)}{x_d\,x_d(p)}\,.\,\Delta E$$

But quantities a_{dn}, a_{qn}, α_{dn}, α_{qn}, b_n, β_n may be found such that

$$\frac{x_q - x_q(p)}{x_q(p)}\,.\,1 = \frac{x_q - x_q''}{x_q''}\sum a_{dn}\,\epsilon^{-\alpha_{dn}t} \qquad (50)$$

$$\frac{x_d - x_d(p)}{x_d(p)}\,.\,1 = \frac{x_d - x_d''}{x_d''}\sum a_{qn}\,\epsilon^{-\alpha_{qn}t}$$

$$\frac{x_d(p) - x_d\,G(p)}{x_d(p)}\,.\,1 = \sum b_n\,\epsilon^{-\beta_n t}$$

$$x_q'' = x_q(\infty)$$
$$x_d'' = x_d(\infty)$$
$$\Sigma\,a_{dn} = 1.0$$
$$\Sigma\,a_{qn} = 1.0$$
$$\Sigma\,b_n = 1.0$$

It therefore follows from the operational rule that,

$$f(p)\,F(t) = F(o)\,\phi(t) + \int_0^t \phi(t-u)\,F'(u)\,du \qquad (51)$$

where,

$$\phi(t) = f(p)\,.\,1$$

that if

$$\delta = \delta(t)$$
$$p\,\delta = \delta'(t)$$
$$\Delta E = \Delta E(t)$$
$$p\,\Delta E = \Delta E'(t)$$

Equations (48) and (49) may be rewritten in the form,

$$i_d = \frac{E - e\cos\delta}{x_d} + e\,\frac{x_d - x_d''}{x_d\,x_d''}\sum a_{dn}\,\epsilon^{-\alpha_{dn}t}\int_0^t \epsilon^{\alpha_{dn}u}\sin\delta(u)\,\delta'(u)\,du - \frac{1}{x_d}\sum b_n\,\epsilon^{-\beta_n t}\int_0^t \epsilon^{\beta_n u}\,\Delta E'(u)\,du \qquad (48a)$$

$$i_q = \frac{e\sin\delta}{x_q} + e\,\frac{x_q - x_q''}{x_q\,x_q''}\sum a_{qn}\,\epsilon^{-\alpha_{qn}t}\int_0^t \epsilon^{\alpha_{qn}u}\cos\delta(u)\,\delta'(u)\,du$$

$$T = \frac{E\,e\sin\delta}{x_d} + \frac{e^2(x_d - x_q)}{2\,x_d\,x_q}\sin 2\delta + e^2\frac{x_d - x_d''}{x_d\,x_d''}\sin\delta\sum a_{dn}\,\epsilon^{-\alpha_{dn}t}\int_0^t \epsilon^{\alpha_{dn}u}\sin\delta(u)\,\delta'(u)\,du$$

$$+ e^2\frac{x_q - x_q''}{x_q\,x_q''}\cos\delta\sum a_{qn}\,\epsilon^{-\alpha_{qn}t}\int_0^t \epsilon^{\alpha_{qn}u}\cos\delta(u)\,\delta'(u)\,du - \frac{e\sin\delta}{x_d}\sum b_n\,\epsilon^{-\beta_n t}\int_0^t \epsilon^{\beta_n u}\,\Delta E'(u)\,du \qquad (49a)$$

Formula (49a) may be used to determine starting torque and current with zero armature resistance, by introducing $\delta(t) = s\,t$, $\delta'(t) = s$. Thus the average component of torque is found to be,

$$T_{av} = \frac{1}{2}\,\frac{x_d - x_d''}{x_d\,x_d''}\sum a_{dn}\,\frac{\alpha_{dn}\,s}{\alpha_{dn}^2 + s^2} + \frac{1}{2}\,\frac{x_q - x_q''}{x_q\,x_q''}\sum a_{qn}\,\frac{\alpha_{qn}\,s}{\alpha_{qn}^2 + s^2} \qquad (52)$$

Since

$\dfrac{\alpha\,s}{\alpha^2 + s^2}$ is never greater than ½, and

$$\Sigma\,a_{dn} = \Sigma\,a_{qn} = 1.0$$

it follows that T_{av} is never greater than

$$\frac{1}{4}\left\{\frac{x_d - x_d''}{x_d\,x_d''} + \frac{x_q - x_q''}{x_q\,x_q''}\right\} \qquad (53)$$

Equation (53) thus provides a very simple criterion of the maximum possible starting torque of a synchronous motor of given dimensions, when armature resistance is neglected.

The same formula may also be used to obtain a simple expression for the damping and synchronizing components of pulsating torque due to a given small angular pulsation of the rotor.

Thus if the angular pulsation is

$$\Delta\,\delta = [\Delta\,\delta]\sin(s\,t)$$

and if the pulsation of torque is expressed in the form

$$\Delta T = T_s \Delta\delta + T_d \frac{d}{dt}\Delta\delta$$

there results,

$$T_s = T_{s0} + e^2 \sin^2\delta_0 \frac{x_d - x_d''}{x_d x_d''} \sum \frac{a_{dn} s^2}{(\alpha_{dn})^2 + s^2}$$

$$+ e^2 \cos^2\delta_0 \frac{x_q - x_q''}{x_q x_q''} \sum \frac{a_{qn} s^2}{(\alpha_{qn})^2 + s^2} \qquad (54)$$

$$s T_d = e^2 \sin^2\delta_0 \frac{x_d - x_d''}{x_d x_d''} \sum \frac{a_{dn}\, s\, \alpha_{dn}}{(\alpha_{dn})^2 + s^2}$$

$$+ e^2 \cos^2\delta_0 \frac{x_q - x_q''}{x_q x_q''} \sum \frac{a_{qn}\, s\, \alpha_{qn}}{(\alpha_{qn})^2 + s^2}$$

where,

$$T_{s0} = \frac{e\, I_{d0} \cos\delta_0}{x_d} + \frac{e^2 (x_d - x_q)}{x_d x_q} \cos 2\delta_0$$

δ_0 = average angular displacement, *i. e.*, total angle $= \delta = \delta_0 + \Delta\delta$.

It can be shown that for the case of no additional rotor circuits, Equations (54) are exactly equivalent to Equations (24) and (25) in Doherty and Nickle's paper, *Synchronous Machines III*. The new formulas herein developed are, however, very much simpler in form, especially since in the case which Doherty and Nickle have treated, there is only one term in the summation; that is, $n = 1$, and α is merely the reciprocal of the short circuit time constant of the machine, expressed in radians.

K. The Equivalent Circuit of Synchronous Machines Operating in Parallel at No Load, Neglecting the Effect of Armature Resistance

Let, δ_a = angle of rotor a and bus
θ_0 = angle of rotor a in space

In general, the shaft torque of a machine depends on its acceleration and speed in space, and the magnitude and rate of change of the bus voltage as a vector. If all of the machines are operating at no load and if there is no armature resistance, a small displacement of any one machine will change the magnitude of the bus voltage only by a second order quantity; consequently for small displacements the magnitude of the bus voltage may be regarded as fixed, and only the angle of the bus and rotor need be considered. Furthermore, the electrical torque may be found in terms of (δ) by employing an infinite bus formula. But Equation (**49a**) implies the alternative general operational form,

$$T = \frac{e\, I_{d0} \sin\delta}{x_d} + \frac{e^2 (x_d - x_q) \sin 2\delta}{2 x_d x_q}$$

$$- \frac{x_d - x_d''}{x_d x_d'} e^2 \sin\delta \sum \frac{a_{nd}\, p}{p + \alpha_{nd}} \cdot \cos\delta \quad (49b)$$

$$+ \frac{x_q - x_q''}{x_q x_q''} e^2 \cos\delta \sum \frac{a_{nq}\, p}{p + \alpha_{nq}} \cdot \sin\delta$$

Therefore in the case under consideration there is for machine a,

$$T_a = \left[\frac{e\, I_a}{x_{da}} + e^2 \frac{(x_{da} - x_{qa})}{x_{da} x_{qa}} \right] \delta_a$$

$$+ \frac{x_{qa} - x_{qa}''}{x_{qa} x_{qa}''} e^2 \sum a_{nqa} \frac{p}{p + \alpha_{nqa}} \cdot \delta_a \qquad (55)$$

where: e = per-unit bus voltage
I_a = per-unit excitation of machine a, etc.

This equation can be represented by Fig. 8, in which the charge through the circuit represents (δ_a) and the

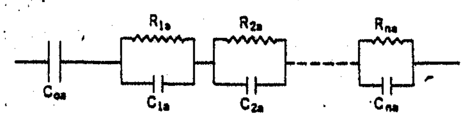

Fig. 8

voltage across the circuit represents the electrical torque of the machine (T_a).

The capacitances and resistances must be chosen so that

$$C_{0a} = \frac{x_{da} x_{qa}}{e\, I_a x_{qa} + e^2 (x_{da} - x_{qa})} \qquad (56)$$

$$C_{na} = \frac{x_{qa} x_{qa}''}{e^2 a_{nqa} (x_{qa} - x_{qa}'')}$$

$$R_{na} = \frac{1}{C_{na}\, \alpha_{nqa}}$$

The equation for the mechanical torque is

$$T_{sa} = T_a + M_a\, p\, s_a \qquad (57)$$

where:

M_a = inertia factor of machine a in radians

$$= \frac{2 \times \text{stored mech. energy at normal speed}}{\text{base power}}$$

$$= 2\pi f \frac{0.462 \; W R^2 \left(\frac{\text{rev. per min.}}{1000}\right)^2}{\text{base kw.}}$$

s_a = per-unit speed of machine a

t = time in seconds $\left(p = \frac{d}{dt}\right)$

But, $s_a = p\, \theta_a$

Thus there is

$$T_{sa} = T_a + M_a\, p^2\, \theta_a \qquad (57a)$$

which corresponds to the equivalent circuit of Fig. 9, in which change $= \theta_a$

$$L_a = M_a$$

The machine operating on an infinite bus can be represented by the equivalent circuit of Fig. 10, since the condition

$$\theta_a = \delta_a = 0$$

is fulfilled.

Several machines in parallel on the same bus may be

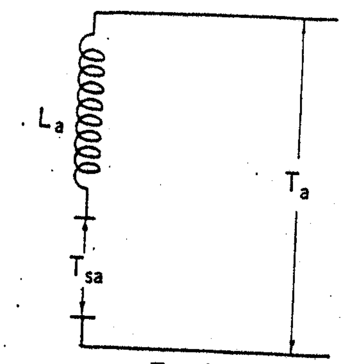

Fig. 9

represented by the diagram of Fig. 11, since the conditions

$$\theta_a - \delta_a = \theta_b - \delta_b = \ldots (= \text{bus angle in space})$$

$$T_a + T_b + T_c, \text{etc.} = \text{bus power output} = 0$$

A transmission line may be represented by a condenser.

Thus two machines connected by a line of reactance (x) would be represented by the circuit of Fig. 12, where

$$C = \frac{x}{e^2} \qquad (58)$$

Shaft torques are, of course, represented by voltages.

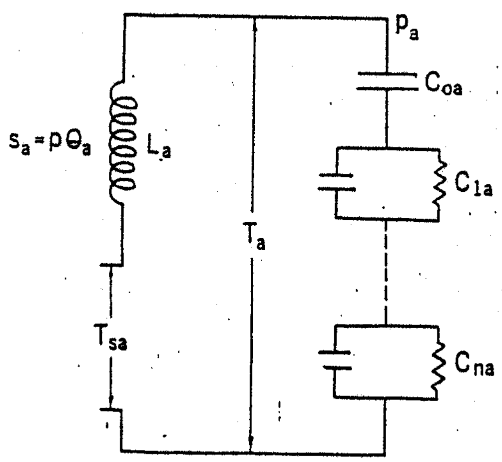

Fig. 10

Mechanical damping, such as that due to a fan on a motor shaft or that due to the prime mover, is represented by resistance in series with the inductance (L) as in Fig. 13. (R) must be chosen equal to the rate of decrease in available driving torque with increase in speed.

Governors and other prime mover characteristics may also be represented by connecting their circuits in the inductive branch of the circuit. Thus a governor which acts through a single time constant may be represented by the circuit of Fig. 14, where

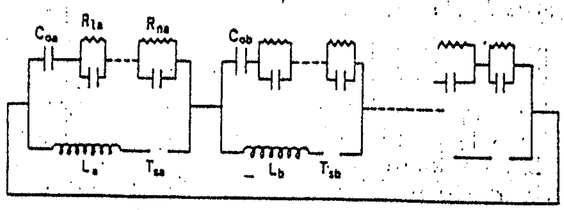

Fig. 11

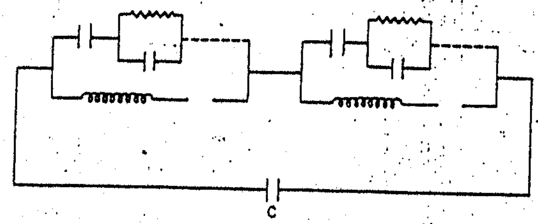

Fig. 12

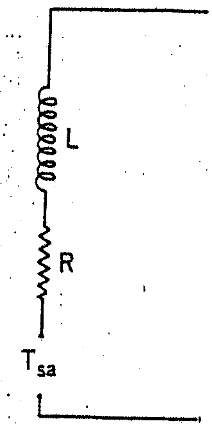

Fig. 13

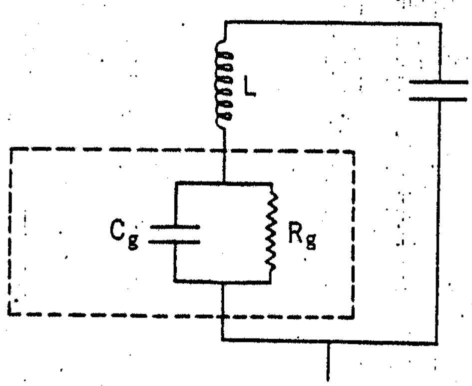

Fig. 14

$$R_g = \frac{1}{\text{regulation}}$$

$$C_g = \frac{\text{time constant of governor in elec. radians}}{R_g} \qquad (59)$$

An induction motor is represented by the simple circuit of Fig. 15 and is precisely the circuit of a synchronous machine with only one time constant and $C_0 = \infty$ on account of $I = 0$.

Results similar to these have been previously shown by Arnold, Nickle,[10] and others, but simpler and more approximate circuits were used, the branches of the several circuits were not directly evaluated in terms of machine constants, and the derivation was incomplete in that the limitation to no load and zero resistance was not appreciated.

L. Torque Angle Relations of a Synchronous Machine Connected to an Infinite Bus, for Small Angular Deviations from an Average Operating Angle

There is, in general,

$$T = T_0 + \Delta T = (\psi_{d0} + \Delta \psi_d)(i_{q0} + \Delta i_q) - (i_{d0} + \Delta i_d)(\psi_{q0} + \Delta \psi_q)$$

For small angular deviations,

$$\Delta T = i_{q0} \Delta \psi_d + \psi_{d0} \Delta i_q - i_{d0} \Delta \psi_q - \psi_{q0} \Delta i_d = \{\psi_{d0} + i_{d0} x_q(p)\} \Delta i_q - \{\psi_{q0} + i_{q0} x_d(p)\} \Delta i_d \quad (60)$$

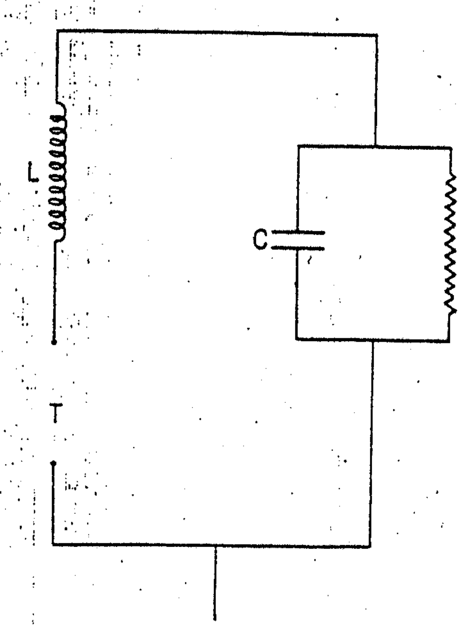

FIG. 15

$$e_{d0} + \Delta e_d = p \Delta \psi_d - r(i_{d0} + \Delta i_d) - (\psi_{q0} + \Delta \psi_q)(1 + p \Delta \delta)$$
$$e_{q0} + \Delta e_q = p \Delta \psi_q - r(i_{q0} + \Delta i_q) + (\psi_{d0} + \Delta \psi_d)(1 + p \Delta \delta)$$
$$\Delta e_d = p \Delta \psi_d - r \Delta i_d - \psi_{q0} p \Delta \delta - \Delta \psi_q$$
$$\Delta e_q = p \Delta \psi_q - r \Delta i_q + \psi_{d0} p \Delta \delta + \Delta \psi_d$$

from which there is

$$z_d(p) \Delta i_d - x_q(p) \Delta i_q = - \Delta e_d - \psi_{q0} p \Delta \delta$$
$$z_q(p) \Delta i_q + x_d(p) \Delta i_d = - \Delta e_q + \psi_{d0} p \Delta \delta$$

$$\Delta i_d = \frac{z_q(p)(-\Delta e_d - \psi_{q0} p \Delta \delta) + x_q(p)(-\Delta e_q + \psi_{d0} p \Delta \delta)}{D(p)} \quad (61)$$

$$\Delta i_q = \frac{z_d(p)(-\Delta e_q + \psi_{d0} p \Delta \delta) - x_d(p)(-\Delta e_d - \psi_{q0} p \Delta}{D(p)}$$

where,

$$D(p) = z_d(p)\, z_q(p) + x_d(p)\, x_q(p)$$

but from Equations (28),

$$e_{d0} + \Delta e_d = e \sin(\delta_0 + \Delta \delta)$$
$$e_{q0} + \Delta e_q = e \cos(\delta_0 + \Delta \delta)$$
$$\Delta e_d = e \cos \delta_0 \Delta \delta$$
$$\Delta e_q = - e \sin \delta_0 \Delta \delta$$

(6

$$\Delta i_d = \frac{-(e \cos \delta_0 + \psi_{q0} p)\, z_q(p) + (e \sin \delta_0 + \psi_{d0} p)\, x_q(p)}{D(p)}$$

$$\Delta i_q = \frac{(e \sin \delta_0 + \psi_{d0} p)\, z_d(p) + (e \cos \delta_0 + \psi_{q0} p)\, x_d(p)}{D(p)}$$

(6

$$\Delta T = \frac{[\psi_{d0} + i_{d0} x_q(p)] \left\{ \begin{array}{l} (e \sin \delta_0 + \psi_{d0} p)\, z_d(p) \\ + (e \cos \delta_0 + \psi_{q0} p)\, x_d(p) \end{array} \right\} + [\psi_{q0} + i_{q0} x_d(p)] \left\{ \begin{array}{l} (e \cos \delta_0 + \psi_{q0} p)\, z_q(p) \\ - (e \sin \delta_0 + \psi_{d0} p)\, x_q(p) \end{array} \right\}}{D(p)}$$

(6

say,

$$\Delta T = f(p) \,.\, \Delta \delta$$

From (57a) the equation for shaft torque becomes

$$\Delta T_s = (M p^2 + f(p)) \,.\, \Delta \delta$$

Thus,

$$\Delta \delta = \frac{1}{M p^2 + f(p)} \,.\, \Delta T_s$$

(6

Appendix

Formula for Linkages and Voltage in Field Circuit w no Additional Rotor Circuits

In this case the per-unit field linkages will depe linearly on the armature and field currents. That in general,

$$\Psi = a I - b i_d$$

Then if normal linkages are defined as those existi at no load there must be $a = 1.0$.

The quantity b may be found by suddenly impressi terminal linkages ψ_d with no initial currents in t machines and $E = 0$.

By definition there is, initially

$$i_d = - \frac{\psi_d}{x_d'}$$

but also there must be from the definition of $x_d{}^2$

$$i_d = \frac{I - \psi_d}{x_d}$$

hence there must be an initial induced field current of amount

$$I = \psi_d \left(1 - \frac{x_d}{x_d{}'} \right).$$

But, initially the field linkages are zero, thus

$$\Psi = \psi_d 1 - \left[\frac{x_d}{x_d{}'} + \frac{b}{x_d{}'} \right] = 0$$

hence $b = x_d - x_d{}'$

Similarly, there will be

E = per-unit field voltage

$= c\,p\,\Psi + d\,I$

Normal field voltage will be here defined as those existing at no load and normal voltage. This requires that $d = 1$. The quantity c may then be recognized as the time constant of the field in radians when the armature is open circuited, since with the field shorted under these conditions there is

$$(T_0\, p + 1)\, I = 0$$

$$c\,p\,\Psi + I = 0$$

$$\psi = I$$

$c = T_0$ = time constant of field with armature open circuited.

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Discussion

H. C. Specht: I should think Mr. Park's theory could be applied just as well to the so-called synchronous induction motor, that is an induction motor in which the rotor teeth between the poles are cut out for a distance of about one-third of the pole pitch. Such a motor runs at synchronous speed. However, the pull-out torque is much less than that of an induction motor with the full number of teeth.

C. MacMillan: There was one statement in the first page of Mr. Park's paper to the effect that "Idealization is resorted to, to the extent that saturation and hysteresis in every magnetic circuit and eddy currents in the armature iron are neglected . . .". And with regard to Fig. 5, Mr. Park remarked that it represented a rigorous solution. Perhaps Mr. Park could give us a little more insight into the effect of taking into account saturation, and give other cases in which certain elements have been neglected with more or less effect upon the final results.

W. J. Lyon: In a paper of this description, certain premises should be chosen and, with these always in mind, the mathematical development should be rigorous. The paper may then be criticized either because of insufficient premises or because of incorrect mathematical development. I believe that the former is the kinder method; it is the one I shall employ.

The premises that Mr. Park chooses are that the field and armature windings are symmetrical, that saturation and hysteresis are neglected and that the armature windings are in effect sinusoidally distributed. I take this last to mean that the air-gap flux due to the armature currents is sinusoidally distributed, for if the armature windings themselves were sinusoidally distributed, there would be produced space harmonics in the air-gap flux distribution due to the saliency of the poles, which, as we all know, would complicate the problem tremendously. In order that the mathematical method used by Mr. Park shall be rigorous, I believe it is necessary to make one further assumption. I think I can best explain this by asking you to consider the result of supplying the field winding with a sinusoidal current while the armature rotates at some speed which may be called synchronous. Under these conditions, there will first be produced in the armature windings two sets of balanced currents each of which will produce 3 component flux distributions in the gap. The first of these is what would be produced if the air-gap were uniform, and is proportional to $1/2\,(x_d + x_q - x_a)$, where x_a equals the armature leakage reactance. The second of these components is proportional to $1/2\,(x_d - x_q)$. The third component is of the same size as the second. Using the values that Mr. Park gives under Section H of his paper, the relative magnitudes of these components would be $(0.8 - x_a)$ and 0.2. The first and second components react on the field, and produce in it a current of the impressed field frequency. These are the components that Mr. Park has recognized, but the third component produces an entirely different frequency in the field, which will then be reflected into the armature and the process will be repeated. That is, in this respect, it is similar to the condition that exists in a single-phase alternator. As far as I am aware, the Heaviside operational method cannot be used to obtain a rigorous solution for the single-phase alternator. In spite of this, I think the objection that I have raised is of no more importance than the effect of neglecting saturation or hysteresis or the space harmonics in the air-gap flux that are actually produced, but I do think that Mr. Park should have mentioned this point in order that the mathematical work which he builds up should be rigorous. I discovered the necessity of this additional premise about a year ago while working on the problem of the transient short-circuit currents in a salient-pole alternator.

There is another problem of considerable interest that Mr. Park has not mentioned. About two years ago I became aware that Mr. Fortescue's historic 1918 paper could be extended to the transient case by using an operational method. That is to say, any of the problems involving unbalanced circuits that Mr. Fortescue rigorously solved in the steady state can also be solved in the transient state by an operational method.

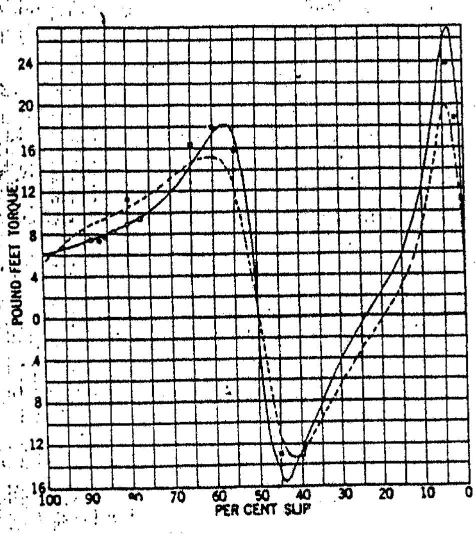

FIG. 1—ELECTRICAL TORQUE OF A H I

Solid Line—Calculated
Dotted Line—Experimental

Note: The six points plotted above were taken experimentally at a temperature of 25 deg. cent.

The balance of the dotted curve was taken at varying higher temperatures

The solid curve was calculated for a temperature of 25 deg. cent., and the points in squares for 100 deg. cent.

W. B. Heinz: (communicated after adjournment) Since the completion of Mr. Park's paper as presented, experimental work has been done which illustrates the worth of the new theory developed.

A small synchronous motor rated 15 hp., 220 volts, 1800 rev. per min. was selected for study. Its squirrel-cage winding was removed in order to simplify the mechanical work of calculation.

This motor was set up with an electric dynamometer, and its torque-slip curve was taken at 110 volts over the entire range of slip from 100 per cent to zero. Power was supplied to it by a special 220-volt sine-wave generator of 375-kv-a. capacity. The constants of this machine as well as of the connecting line and switches were determined, and these constants were included in the calculation of torque in order to simulate as closely as possible infinite-bus conditions.

The torque as indicated by the dynamometer was corrected for windage and friction and for dynamometer errors, the net electrical torque exerted on the rotor thus being obtained.

By the use of measured constants of the test machine and its circuit, a mathematical evaluation of torque was made for a temperature of 25 deg. cent. These constants as determined, on a per unit base of 24.75 kv-a. at 220 volts, are as follows:

$x_d = 1.0$
$x_d' = 0.178$
$x_q = 0.442$
$x_q' = 0.442$
$r = 0.0325$
$T_0 = 159$ radians
Normal torque = 96.8 lb-ft.

The calculated curve appears as a solid line in Fig. 1 herewith and near it is the dotted curve obtained experimentally.

During the progress of the test run with the dynamometer no control or regular measurement of the synchronous motor temperature was made. It fluctuated widely with changes in slip, and the only points taken at a known temperature are the ones near the beginning of the run. For the first six, those surrounded by circles, the temperature probably had not risen far from ambient of 25 deg. cent. They check quite closely the calculated curve.

In order to evaluate the effect of temperature change eight points were calculated on the basis of 100 deg. cent. These are shown within squares. They show a better coincidence of slips at which maximums occur, but little improvement in the maximum error. Previous theories have usually neglected the effect of salient poles. The solid line in Fig. 2 shows the calculated curve in this case, that is, for $x_q = x_d$. A comparison of the solid curves in Figs. 1 and 2 shows the large differences between the two calculations.

I. H. Summers: (communicated after adjournment) One outstanding feature of this paper is that the equations are expressed in "per-unit" quantities. This feature presents the important advantage that the quantities involved are those which are known directly from test or from the design of the machine. Furthermore, machines are more readily compared with others of different rating when their constants are known in "per-unit" values.

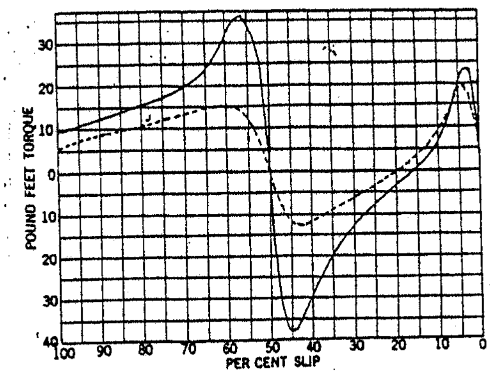

FIG. 2—TORQUE CURVES FOR A H I

Solid Line—Calculated
Dotted Line—Experimental

However, these equations may not look familiar at first sight to those who are more used to dealing in the physical quantities such as ohms, amperes, volts, henrys, and farads. It therefore seems appropriate to point out the correlation between this new method and the more usual one. Consider, for example, Equations (4a) in the paper, together with the two immediately following.

Ordinarily we should have written for a machine with no rotor circuits in addition to the field,

$$(R_f + L_f\, p)\, I_f - M\, p\,[j_a \cos\theta + j_b \cos(\theta - 120°) + j_c \cos(\theta + 120°)] = E_f \qquad (1)$$

In this equation R_f is field resistance, L_f is field inductance, M is mutual inductance between field and any one armature phase in the position of maximum mutual inductance, j_a, j_b, and j_c

are armature phase currents, and E_f is field voltage. All these quantities are in practical units.

Let

$$j_d = \frac{2}{3}\left[j_a \cos\theta + j_b \cos(\theta - 120°) + j_c \cos(\theta + 120°)\right] \qquad (2)$$

Divide through by $I_n R_f$ where I_n is that value of field current that causes normal voltage to be generated in the armature at no load.

$$\frac{E_f}{I_n R_f} = (1 + T_0 p)\frac{I_f}{I_n} - \frac{3}{2}\frac{L_f}{R_f}\frac{M_p}{I_n L_f} j_d \qquad (3)$$

where $\qquad T_0 = L_f/R_f \qquad (4)$

Let

$$\left.\begin{aligned} E_f/I_n R_f &= E \\ I_f/I_n &= I \\ j_d/i_n &= i_d \end{aligned}\right\} \qquad (5)$$

Where i_n is normal armature current, maximum value. Then

$$E = T_0 p\left(I - \frac{3}{2}\frac{M}{L_f}\frac{i_n}{I_n} i_d\right) + I \qquad (6)$$

This equation may be put into the form:

$$\left.\begin{aligned} E &= T_0 p \Psi + I \\ \Psi &= I - \left(\frac{3}{2}\frac{M_{in}}{L_f I_n}\right) i_d \end{aligned}\right\} \qquad (7)$$

These results may be compared with those immediately following (4a) in the paper. It is easy to show by methods similar to those used by Mr. Park that

$$\frac{3}{2}\frac{M_{in}}{L_f I_n} = x_d - x_d' \qquad (8)$$

where x_d and x_d' are the per-unit direct axis synchronous and transient reactance respectively. Then the equations are the same as those given by the author.

Without this last relation we should doubtless have written the equations as in (7) or even as in (1). But both of these contain cumbersome quantities involving mutual and self inductances. The expression used by Mr. Park is simpler and more expressive and contains quantities which are familiar to most engineers who have to do with synchronous machines.

Throughout this paper, although the problem is inherently complex and some of the demonstrations appear long, the results are relatively simple and in terms adapted for immediate use.

R. H. Park: Mr. Specht is entirely correct in his thought that the present theory could be applied to synchronous induction motors such as those to which he refers. In principle this type of motor is not greatly different from a salient-pole synchronous motor with an amortisseur winding.

In reply to Mr. MacMillan's question, I would point out that it is always necessary to make some assumptions; in fact, even in the simplest problems there invariably exists an enormous number of assumptions most of which are not recognized as such. Therefore a rigorous solution invariably means only "rigorous on the basis of the assumptions." The solutions presented in the paper are rigorously correct in this sense.

Professor Lyon has criticized the assumptions made in the paper. This is due to a very definite misconception as to their character. Thus, referring to the assumption "that the armature windings are in effect sinusoidally distributed," Professor Lyon states that it is necessary to interpret the statement to mean something other than it says. I believe, however, that this is not necessary. Thus he states that were the armature windings themselves sinusoidally distributed, there would be produced space harmonics in the air-gap flux distribution due to the saliency of the poles, which "would complicate the problem tremendously." The reason that it does not complicate the problem is that with a sinusoidal distribution of turns only space fundamental flux produces any armature flux linkages.

This follows from the fact that

$$\int_0^{2\pi} \cos n\theta \cos m\theta \, d\theta = \int_0^{2\pi} \cos n\theta \sin m\theta \, d\theta = 0$$

if n and m are integers and $n = m$.

Although this answers Professor Lyon's objection the impression should not be derived that the theory developed strictly presupposes an exactly sinusoidally distributed winding. The full statement of the assumption referred to is that the armature windings are in effect sinusoidally distributed "as far as concerns effects depending on rotor position." Interpreted in terms of self and mutual inductive coefficients this statement is exactly equivalent to the two assumptions,

(a) that the self inductance of the armature circuits is expressible by an equation of the form,

$$L = L_0 + L_2 \cos 2\theta$$

(b) that the mutual inductance between the armature and any rotor circuit is expressible by an equation of the form,

$$M = M_1 \cos\theta$$

The assumptions which have been made in previous studies, such as those which Mr. Ku in his paper "Transient Analysis of A-C. Machinery" has referred to as "exact," have been precisely the same except that the second harmonic term in the armature self inductance equation has not been considered, while amortisseur and other circuits in addition to the field have been neglected.

Actually the expressions for both armature and rotor self inductance will involve all of the even harmonics of angular position, and the expression for armature mutual inductance will involve all of the odd harmonics of angular position. However, tests on actual machines have shown that in most cases only the zero and second harmonics of armature self inductance, the first harmonic of mutual inductance, and the zero harmonic of rotor circuit self inductance are of primary importance. About the only case in which a consideration of any other harmonics would be of value appears to be in the study of locking torques. With proper design, however, the tendency to lock at other than half speed is slight.

Professor Lyon is, in my opinion, also in error when he endeavors to point out the inadequacy of the theory by means of an example in which alternating current is supplied to a rotating field in a salient-pole machine. In this connection he merely makes certain dogmatic statements without proof, which would not agree with the writer's findings. However, such mere statements or opinions do not constitute a proof, and the writer continues to disagree with them.

As I understand the example given, it corresponds to the application of alternating current to a synchronous machine field winding with a three-phase short circuit across the armature terminals.

From Equations (11) and (22) of the paper there is in this case

$$I_d = \left[1 + \frac{\langle x_d - x_d(p)\rangle \langle p z_q(p) + (1 - s^2) x_q(p)\rangle}{D(p)}\right] G(p) E$$

= total per unit rotor excitation.

Evidently this is of the same frequency as that of the field voltage E. That is, there are no harmonics. Mr. C. A. Nickle has submitted a physical interpretation of this result. Thus he states

"Let $I \cos t$ be impressed on the field which is running at a speed S expressed as a fraction of synchronous speed for the frequency of the current impressed on the field winding. The current $I \cos t$ produces a space fundamental flux $I \cos t$ in the direct axis and space harmonics alternating in time. But since the armature winding is assumed to have sine distribution, these harmonic fluxes can induce no voltage and may be neglected. The fundamental flux, $I \cos t$, may be resolved into two rotating components $I/2$ rotating forward at normal speed with reference to the pole structure and $I/2$ rotating backward at the same speed. The first component rotates forward with respect to the armature at a speed $1 + S$ and since the armature is short circuited polyphase, a positive phase sequence current of frequency $1 + S$ will flow in the armature winding. Similarly a negative phase sequence current of frequency $1 - S$ will flow in the armature winding due to the backward rotating component of field flux. The polyphase current i_{1+s} will produce a rotating m. m. f. also i_{1+s} and this m. m. f. rotates forward relative to the pole structure at normal speed. Two fluxes are produced due to dissymmetry of the direct and quadrature axes. These are

$$\frac{i_{1+s}}{2}(x_d' + x_q')$$

rotating forward with respect to the poles at normal speed or $1 + S$ with respect to the armature, and

$$\frac{i_{1+s}}{2}(x_d' - x_q')$$

rotating backward with respect to the poles at normal speed or at $1 - S$ backward with respect to the armature. The first will induce an armature voltage of frequency $1 + S$, and the second a voltage of frequency $1 - S$.

The armature current of frequency i_{1-s} produces an m. m. f. i_{1-s} rotating backward with respect to the poles at normal speed and produces two rotating fluxes. One,

$$\frac{i_{1-s}}{2}(x_d' + x_q')$$

rotates backward with respect to the armature at a speed $1 - S$ and

$$\frac{i_{1-s}}{2}(x_d' - x_q')$$

rotates forward with respect to the armature at a speed $1 + S$. Thus, this current also produces armature voltages of frequencies $1 + S$ and $1 - S$. Therefore, no frequencies other than $(1 + S)$ and $(1 - S)$ need be assumed, in the armature winding and therefore, these are the only frequencies that exist there. Moreover, this implies that only fundamental frequency can exist in the rotor."