Flux-linkage equations for 7-winding representation (eq. 4.11 in VMAF)

$$\begin{bmatrix} \lambda_a \\ \lambda_b \\ \lambda_c \\ \lambda_F \\ \lambda_G \\ \lambda_D \\ \lambda_Q \end{bmatrix} = \begin{bmatrix} L_{aa} & L_{ab} & L_{ac} & L_{aF} & L_{aG} & L_{aD} & L_{aQ} \\ L_{ba} & L_{bb} & L_{bc} & L_{bF} & L_{bG} & L_{bD} & L_{bQ} \\ L_{ca} & L_{bc} & L_{cc} & L_{cF} & L_{cG} & L_{cD} & L_{cQ} \\ L_{Fa} & L_{Fb} & L_{Fc} & L_{FF} & L_{FG} & L_{FD} & L_{FQ} \\ L_{Ga} & L_{Gb} & L_{Gc} & L_{GF} & L_{GG} & L_{GD} & L_{GQ} \\ L_{Da} & L_{Db} & L_{Dc} & L_{DF} & L_{DG} & L_{DD} & L_{DQ} \\ L_{Qa} & L_{Qb} & L_{Qc} & L_{QF} & L_{QG} & L_{QD} & L_{QQ} \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_c \\ i_F \\ i_G \\ i_D \\ i_Q \end{bmatrix}$$

The above terms are defined as follows:

Stator-stator terms:

$L_{aa}=L_s+L_m\cos 2\theta$
$L_{ab}=-[M_s+L_m\cos 2(\theta+30°)]$
$L_{ac}=-[M_s+L_m\cos 2(\theta+150°)]$

$L_{ba}=-[M_s+L_m\cos 2(\theta+30°)]$
$L_{bb}=L_s+L_m\cos 2(\theta-120°)$
$L_{bc}=-[M_s+L_m\cos 2(\theta-90°)]$

$L_{ca}=-[M_s+L_m\cos 2(\theta+150°)]$
$L_{cb}=-[M_s+L_m\cos 2(\theta-90°)]$
$L_{cc}=L_s+L_m\cos 2(\theta-240°)$

Rotor-rotor terms:

$L_{FF}=L_F$
$L_{FD}=M_R$
$L_{FQ}= L_{FG} =0$

$L_{DF}=M_R$	$L_{QF}=L_{QD}=0$	$L_{GF}=L_{GD}=0$
$L_{DD}=L_D$	$L_{QQ}=L_Q$	$L_{GQ}=M_Y$
$L_{DQ}=L_{DG}=0$	$L_{QG}=M_Y$	$L_{GG}=L_G$

Stator-rotor terms:

$L_{aF}=M_F\cos\theta$
$L_{aD}=M_D\cos\theta$
$L_{aQ}=M_Q\sin\theta$
$L_{aG}=M_G\sin\theta$

$L_{bF}=M_F\cos(\theta-120°)$
$L_{bD}=M_D\cos(\theta-120°)$
$L_{bQ}=M_Q\sin(\theta-120°)$
$L_{bG}=M_G\sin(\theta-120°)$

$L_{cF}=M_F\cos(\theta-240°)$
$L_{cD}=M_D\cos(\theta-240°)$
$L_{cQ}=M_Q\sin(\theta-240°)$
$L_{cG}=M_G\sin(\theta-240°)$

Rotor-stator terms:

$L_{Fa}=M_F\cos\theta$
$L_{Fb}=M_F\cos(\theta-120°)$
$L_{Fc}=M_F\cos(\theta-240°)$

$L_{Da}=M_D\cos\theta$
$L_{Db}=M_D\cos(\theta-120°)$
$L_{Dc}=M_D\cos(\theta-240°)$

$L_{Qa}=M_Q\sin\theta$
$L_{Qb}=M_Q\sin(\theta-120°)$
$L_{Qc}=M_Q\sin(\theta-240°)$

$L_{Ga}=M_G\sin\theta$
$L_{Gb}=M_G\sin(\theta-120°)$
$L_{Gc}=M_G\sin(\theta-240°)$

So the compact form of the flux linkage equations are

$$\begin{bmatrix} \underline{\lambda}_{abc} \\ \underline{\lambda}_{FGDQ} \end{bmatrix} = [\underline{L}] \begin{bmatrix} \underline{i}_{abc} \\ \underline{i}_{FGDQ} \end{bmatrix} = \begin{bmatrix} \underline{L}_{aa} & \underline{L}_{aR} \\ \underline{L}_{Ra} & \underline{L}_{RR} \end{bmatrix} \begin{bmatrix} \underline{i}_{abc} \\ \underline{i}_{FGDQ} \end{bmatrix} \qquad \text{(eq. L)}$$

which, when expanded with the expressions for self and mutual inductances, become:

$$\begin{bmatrix} \lambda_a \\ \lambda_b \\ \lambda_c \\ \lambda_F \\ \lambda_G \\ \lambda_D \\ \lambda_Q \end{bmatrix} = \begin{bmatrix} L_S + L_m \cos 2\theta & -[M_S + L_m \cos 2(\theta + 30°)] & -[M_S + L_m \cos 2(\theta + 150°)] & M_F \cos\theta & M_G \sin\theta & M_D \cos\theta & M_Q \sin\theta \\ -[M_S + L_m \cos 2(\theta + 30°)] & L_S + L_m \cos 2(\theta - 120°) & -[M_S + L_m \cos 2(\theta - 90°)] & M_F \cos(\theta - 120°) & M_G \sin(\theta - 120°) & M_D \cos(\theta - 120°) & M_Q \sin(\theta - 120°) \\ -[M_S + L_m \cos 2(\theta + 150°)] & -[M_S + L_m \cos 2(\theta - 90°)] & L_S + L_m \cos 2(\theta - 240°) & M_F \cos(\theta - 240°) & M_G \sin(\theta - 240°) & M_D \cos(\theta - 240°) & M_Q \sin(\theta - 240°) \\ M_F \cos\theta & M_F \cos(\theta - 120°) & M_F \cos(\theta - 240°) & L_F & 0 & M_R & 0 \\ M_G \sin\theta & M_G \sin(\theta - 120°) & M_G \sin(\theta - 240°) & 0 & L_G & 0 & M_Y \\ M_D \cos\theta & M_D \cos(\theta - 120°) & M_D \cos(\theta - 240°) & M_R & 0 & L_D & 0 \\ M_Q \sin\theta & M_Q \sin(\theta - 120°) & M_Q \sin(\theta - 240°) & 0 & M_Y & 0 & L_Q \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_b \\ i_F \\ i_G \\ i_D \\ i_Q \end{bmatrix}$$

(eq. L-ex)

Voltage equations

The voltage equations developed here will characterize the electromagnetic dynamics of the synchronous machine.

Consider the stator circuit as it appears as in Fig. 1 (Fig. 4.2 in text):

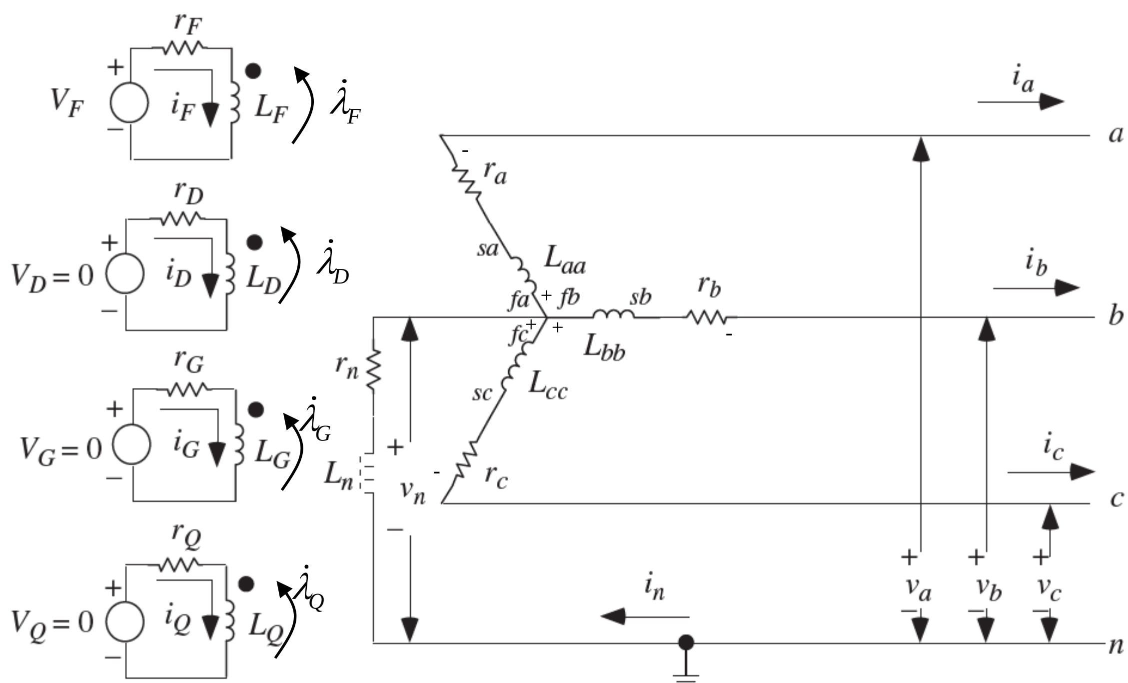


Fig. 1

The defined positive voltage polarity across each phase winding (for phase a, fa is positive wrpst sa) is opposite to the polarity associated with that of Fig. 4.1a in the text) and therefore the induced voltage, when expressed within the phase's voltage equation, will be negative.

We assume that the neutral conductor is not magnetically coupled with any other circuit.

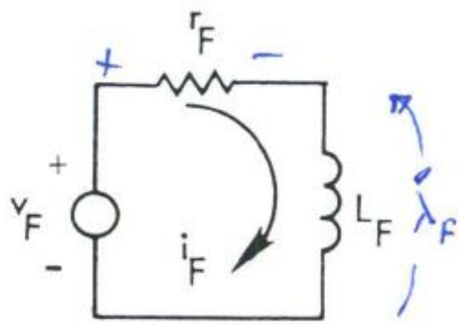
With this information, and following the polarities of the circuit diagram, we can write a voltage equation for each of the phase windings as follows:

$$v_a = -i_a r_a - \dot{\lambda}_a + v_n$$
$$v_b = -i_b r_b - \dot{\lambda}_b + v_n$$
$$v_c = -i_c r_c - \dot{\lambda}_c + v_n$$

We may also write a voltage equation for the neutral circuit as follows:

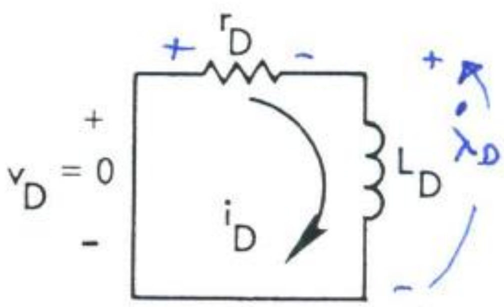
$$v_n = -i_n r_n - L_n \dot{i}_n = -(i_a + i_b + i_c) r_n - L_n (\dot{i}_a + \dot{i}_b + \dot{i}_c)$$

Now let's look at the rotor circuits. There are four of them.



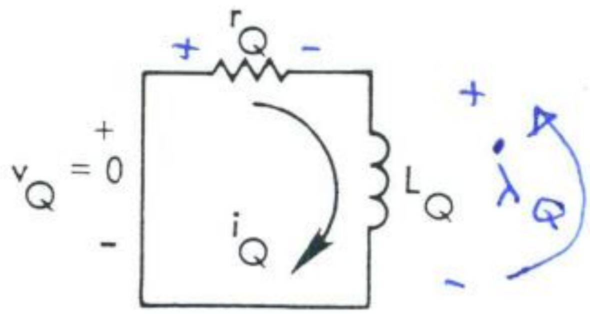
$$v_F = r_F i_F + \dot{\lambda}_F$$
$$\Rightarrow -v_F = -r_F i_F - \dot{\lambda}_F$$

Fig. 2: D-Axis Field



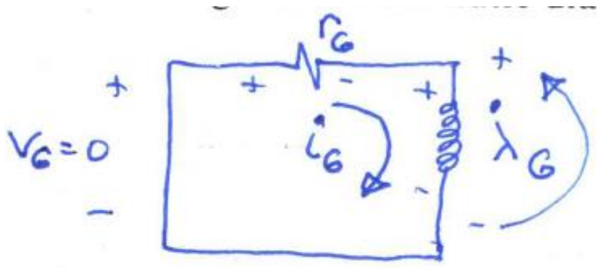
$$0 = -r_D i_D - \dot{\lambda}_D$$

Fig. 3: D-Axis Damper



$$0 = -r_Q i_Q - \dot{\lambda}_Q$$

Fig. 4: Q-Axis Damper



$$0 = -r_G i_G - \dot{\lambda}_G$$

Fig. 5: Q-Axis Field

Putting all of these equations together in matrix form, we have that:

$$\begin{bmatrix} v_a \\ v_b \\ v_c \\ -v_F \\ 0 \\ 0 \\ 0 \end{bmatrix} = -\begin{bmatrix} r_a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & r_b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r_c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r_F & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & r_G & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & r_D & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & r_Q \end{bmatrix}\begin{bmatrix} i_a \\ i_b \\ i_c \\ i_F \\ i_G \\ i_D \\ i_Q \end{bmatrix} - \begin{bmatrix} \dot{\lambda}_a \\ \dot{\lambda}_b \\ \dot{\lambda}_c \\ \dot{\lambda}_F \\ \dot{\lambda}_G \\ \dot{\lambda}_D \\ \dot{\lambda}_Q \end{bmatrix} + \begin{bmatrix} v_n \\ v_n \\ v_n \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{(eq 4.23')}$$

The primed notation on equation numbers indicates the given equation corresponds to the equation identified in the A&F Second edition text by that number, with some modification. Generally, the modification is the addition of the voltage equation for the "G"-circuit. There should be no difference with equations here and those in the third edition (VMAF).

We can write this more compactly, similar to eq. 4.26 in text:

$$\begin{bmatrix} \underline{v}_{abc} \\ \underline{v}_{FGDQ} \end{bmatrix} = -\begin{bmatrix} \underline{R}_{abc} & \underline{0} \\ \underline{0} & \underline{R}_{FGDQ} \end{bmatrix}\begin{bmatrix} \underline{i}_{abc} \\ \underline{i}_{FGDQ} \end{bmatrix} - \begin{bmatrix} \dot{\underline{\lambda}}_{abc} \\ \dot{\underline{\lambda}}_{FGDQ} \end{bmatrix} + \begin{bmatrix} \underline{v}_n \\ \underline{0} \end{bmatrix} \quad \text{(eq. 4.26')}$$

Motivation for Park's Transformation

We desire to get the above equation into state-space form ($\dot{\underline{x}} = \underline{A}\underline{x}$) so that we can combine it with our inertial equations and then apply numerical integration and solve them together. ***Do not lose sight that this is our objective***!!!!!

We notice, however, that we have two types of related state variables in the above equations: flux linkages (λ) and currents (i). We can eliminate one of them, and this is not hard since flux linkages can be expressed as functions of the currents that produce them. For example, for a single conductor, we write that λ=Li (see also eq. (L) at the beginning of this document).

But eq. 4.26' has derivatives on λ. Again, no problem, since dλ/dt=d(Li)/dt.

It is here that we run into trouble, since the inductances that we are dealing with are, in general, functions of θ, which is itself a function of time. Therefore the inductances are functions of time, and differentiation of flux linkages (using the product rule of calculus) results in expressions like:

$$\frac{d\lambda}{dt} = \frac{dL}{dt} i + \frac{di}{dt} L$$

The differentiation with respect to L, dL/dt, will result in a time-varying coefficient on the state variable. When we replace, in eq. 4.26', the derivatives on λ with the derivatives on i, and then solve for the derivatives on i (in order to obtain $\dot{\underline{x}} = \underline{A}\underline{x}$), we will obtain current variables on the right-hand-side that have *time* *varying coefficients*, i.e., the coefficient matrix A will not be constant. This means that we will have to deal with differential equations with time varying coefficients, which are generally more difficult to solve than differential equations with constant coefficients.

This presents some significant difficulties, in terms of solution, that we prefer to avoid. We look for a different approach. We will find the different approach not only solves this problem but offers a simpler view and understanding of synchronous machine electromagnetic dynamics.

The different approach is based on the observation that our trouble comes from the inductances related to the stator (phase windings):

- Stator self inductances
- Stator-stator mutual inductances
- Stator-rotor mutual inductances

i.e., all of these have time-varying inductances.

To alleviate the trouble, we ***project*** a-b-c currents onto a rotating pair of axes which we call the d and q axes or the d-q axes which comprise a rotating coordinate frame of reference. Although we may specify the speed of these axes to be any speed convenient to us, we choose it to be rotor speed[1] to remain consistent with Park.

[1] VMAF, at the top of p. 93, emphasize that the frame of reference "moves with the rotor." This is in contrast to moving at "synchronous speed," since moving "with the rotor" is not necessarily moving at synchronous speed during disturbance conditions. Indeed, this distinction is made in Krause's book (p. 147) and in C. O'Rourke, J.Kirtley, et al. "A Geometric Interpretation of Reference Frames and Transformations: dq0, Clarke, and Park." IEEE Transactions on Energy Conversion 34, 4, (December 2019): 2070 - 2083 © 2019 IEEE, available at https://dspace.mit.edu/bitstream/handle/1721.1/123557/Final_Submission__Open_Access.pdf?sequence=1&isAllowed=y. It is further discussed in VMAF, p. 206, in applying Park's transformation for an induction machine, which requires the angle θ be given relative to a synchronous rotating reference (and not the rotor).

In making these projections, we want to obtain expressions for the components of the stator currents that are in phase with the d and q axes.

One can visualize the projection by thinking of the a-b-c currents, each having direction the same as the flux it produces, as having sinusoidal variation IN TIME along their respective axes. The picture below illustrates for the a-phase.

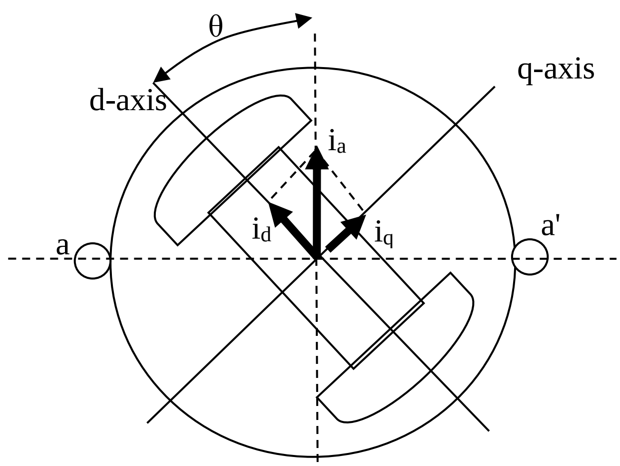


Fig. 6 (Fig. 4.1b in text).

It is important to understand that Fig. 6 implies two kinds of time variation:
(i) the time variation of the a-phase stator current i_a. The mental image of this variation should have the i_a vector in the form shown, then decreasing to zero, then increasing in the negative (downward) direction, and then back to zero, etc.
(ii) the time variation of the d-q axis as it rotates CCW with the rotor.

It is useful to conceptualize the impact on the "i_a to d-q axis projection" of each of the above time variations (i) and (ii) by themselves. Park's transformation is a result of both effects simultaneously.

We observe from Fig. 6 that i_a will have a component in the d-axis direction of $i_a\cos\theta$ and a component in the q-axis direction of $i_a\sin\theta$.

Decomposing the b-phase current i_b and the c-phase current i_c in the same way, and then adding them up, provides us with:

$$i_d = k_d\left(i_a \cos\theta + i_b \cos(\theta - 120°) + i_c \cos(\theta + 120°)\right)$$
$$i_q = k_q\left(i_a \sin\theta + i_b \sin(\theta - 120°) + i_c \sin(\theta + 120°)\right)$$

Here, the constants k_d and k_q are chosen so as to simplify the numerical coefficients in the generalized KVL equations that we will get.

We have transformed 3 variables i_a, i_b, and i_c into two variables i_d and i_q. This yields an under-determined system, meaning

- We can uniquely transform i_a, i_b, and i_c to i_d and i_q (if we specify i_a, i_b, and i_c, then we obtain unique values of i_d and i_q).
- We cannot uniquely transform i_d and i_q to i_a, i_b, and i_c (unless there is another constraint such as $i_a+i_b+i_c=0$).

So we need a third current. We take this current proportional to the zero-sequence current:

$$i_0 = k_0\left(i_a + i_b + i_c\right) \qquad \text{(i-zero)}$$

We note that, under balanced conditions, i_0 is zero, and therefore produces no flux. [In fact, it is possible to show that i_0 produces no flux linking the rotor windings at all (see Concordia's book, p. 14; Kimbark V. III, p. 60; Bergan&Vittal, p. 468, pp. 597-599; Kothari&Nagrath, p. 384; Glover,Sarma,Overbye, p. 417), an idea that is apparent if we consider that zero sequence currents (which flow in all 3 stator windings) produce MMFs that are in time-phase but distributed in space by 120°.] The implication is that under all conditions, since i_d and i_q are equivalent to i_a, i_b, and i_c, and since flux from i_0 does not link with other circuits, then, **i_d and i_q produce the exact same flux linkage as i_a, i_b, and i_c.**

We write our transformation more compactly as:

$$\underbrace{\begin{bmatrix} i_0 \\ i_d \\ i_q \end{bmatrix}}_{\underline{i}_{odq}} = \underbrace{\begin{bmatrix} k_0 & k_0 & k_0 \\ k_d \cos\theta & k_d \cos(\theta - 120) & k_d \cos(\theta + 120) \\ k_q \sin\theta & k_q \sin(\theta - 120) & k_q \sin(\theta + 120) \end{bmatrix}}_{\underline{P}} \underbrace{\begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix}}_{\underline{i}_{abc}}$$

$$\underline{i}_{0dq} = \underline{P}\underline{i}_{abc} \qquad \text{(eq. 4.3)}$$

We may also operate on voltages and fluxes in the same way:

$$\underline{v}_{0dq} = \underline{P}\underline{v}_{abc}, \qquad \underline{\lambda}_{0dq} = \underline{P}\underline{\lambda}_{abc} \qquad \text{(eq. 4.7)}$$

This transformation resulted from the work done by Blondel (1923), Doherty and Nickle (1926), and Park (1929, 1933), and as a result, is usually called "Park's transformation," and the transformation matrix *P* is usually called "Park's transformation matrix" or just "Park's matrix."

In 2000, Park's 1929 paper was voted the 2nd most important paper of the last 100 years (behind Fortescue's paper on symmetrical components).

- R. Park, "Two reaction theory of synchronous machines," Transactions of the AIEE, v. 48, p. 716-730, 1929.
- G. Heydt, S. Venkata, and N. Balijepalli, "High impact papers in power engineering, 1900-1999, NAPS, 2000.

Robert H. Park, 1902-1994

See www.nap.edu/openbook.php?record_id=5427&page=175 for an interesting biography on Park, written by Charles Concordia (himself one of the most famous power system engineers ever!), replicated below, together with a statement that was posted to the PowerGlobe a few years ago.

Charlie Concordia (1908-2003)

ROBERT H. PARK

1902–1994

BY CHARLES CONCORDIA

ROBERT H. PARK will long be remembered by electric power system engineers and electrical machine designers as the originator of what are universally known as "Park's equations." These were given in an American Institute of Electrical Engineers technical paper in 1929. Essentially, they provided a set of relations that made practical and simple the calculation of the dynamic performance of electric (ac) generators (and motors). Such a tool was necessary, but not yet available, for the calculation of the dynamic performance of electric power systems to ensure stable and reliable operation in the face of possible disturbances. This seminal paper has been the basis not only for an enormous flood of useful work in the field but also for many careers in the field. It was, and still is, unmatched in that respect. By itself it would have been enough to make Park famous among power system engineers worldwide.

Before Park's work, several papers had been written on electric generator equations. However, they were so complex as to be of little practical use. David M. Jones, for whom Park then worked at General Electric, recognized this and also recognized that Park was the person who could bring order out of chaos. So he assigned the job to Park, with world-shaking results. Incidentally, it is ironic that the resulting paper did not elicit any discussion when it was presented.

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Although he fully recognized the significance of his contribution, Park was equally interested in many other things. About the same time, he had made contributions to the determination of switching transient voltages and was a major influence in promoting the importance of, and showing how to produce switch gear with, very much smaller interrupting times than were then thought possible.

During World War II, he served in the Naval Ordnance Laboratory in charge of mine development, resulting in seventeen patents (assigned to the U.S. government).

In the 1950s and 1960s he manufactured plastic bottles, inventing the machinery to automate the process.

Later, his interest returned to electric power. He formed a company, Fast Load Control, Inc., to promote the idea of fast control of turbine valves as a means for improving power system stability, and developed several means for accomplishing this.

Rather late in his life, he was recognized by the Institute of Electrical and Electronics Engineers as a fellow in 1965 and was awarded the Lamme Medal in 1972 "for outstanding contributions to the analysis of a-c machines and systems." He had received (in 1945) the Navy's highest civilian award "for distinguished service to the U.S. Navy in time of war in the designing of magnetic mines." In 1986 he was elected to the National Academy of Engineering.

Perhaps the lateness in recognition by "the establishment" was due to the nature of his contribution. It was not a new machine, nor yet a new method of analysis. It was a new structure particularly well suited to facilitate analysis and application to new problems. It has been said that it was a ladder that others could climb and that it was the opening of a gate so that others could enter and cultivate the garden. Thus, it was appreciated immediately by the young engineers at the bottom of the ladder long before those at the top realized what was going on. "Park" was a household word among the young engineers and students long before any awards came. Even at the Lamme Medal award ceremony in 1972, his contribution was compared with that of two other engineers as being similar, apparently without realization of the difference: their papers remained on the shelf, Park's paper took fire and traveled around the world.

Robert Park was an original thinker, a prolific innovator, and a forceful advocate of his ideas. This was his forte. He did not spend time thinking about his past accomplishments but was more interested in his new projects. He was an inventor and proud that he did not require an attorney to help him prepare his later patents. (He had 64.)

Robert Park was a clear thinker, sure in his opinions (which stood well the test of time) and was neither very diplomatic or sentimental. And he was a valued friend, whose counsel was always sound as well as illuminating.

Park was born in Strassburg, Germany, while his father, the sociologist Robert Erza Park, was studying and teaching at Heidelberg University. He grew up in Wollaston, Massachusetts, and graduated from the Massachusetts Institute of Technology in 1923 in electrical engineering. He did postgraduate work at the Royal Technical Institute in Stockholm, Sweden. He worked on a wide variety of subjects in a wide variety of companies and organizations, among which are General Electric, American Cyanamid, the Naval Ordnance Laboratory, the Bureau of Ordnance, Emhart Manufacturing Company, R. H. Park Company, and Fast Load Control, Inc. He was a private consultant to the end of his life. He is survived by a daughter, three sons, and a nephew.

From a recent "PowerGlobe" discussion:

"The real foundation of most of the synchronous machine theory talked today was laid in a paper by a French Engineer, Blondel, who was the first to propose "two reaction theory" in 1895. Then Doherty and Nickle published extensive analysis of synchronous machines using two reaction theory in a number of papers between 1923 and 1928. At the behest of Charlie Concordia (as told by Charlie himself), Park published three papers in 1928 to 1933 and organized the work of Doherty and Nickle in a matrix form and that is what is best known today in terms of Park's Transformation. Concordia and Park were colleagues in GE at that time."

- OM Malik, Professor of Electrical and Computer Engineering, U. of Calgary

In Park's original paper, he used k_0=1/3, k_d=2/3, and k_q=-2/3 (he assumed the q-axis as leading the d-axis; if he would have assumed the q-axis as lagging the d-axis, as we have done, then he would have had k_q=2/3). However, there are two main disadvantages with this choice:

1. The transformation is not orthogonal. This means that $\underline{P}^{-1} \neq \underline{P}^T$. If the transformation were orthogonal ($\underline{P}^{-1}=\underline{P}^T$), then the power calculation, which is $p = \underline{v}_{abc}^T \underline{i}_{abc}$, is also given by $p = \underline{v}_{0dq}^T \underline{i}_{0dq}$ (and is therefore called "power invariant" by VMAF). This can be proven (see eq. 4.10 in VMAF) as follows. From above eqs. 4.3, 4.7, $\underline{P}^{-1}\underline{v}_{0dq} = \underline{v}_{abc}$ and $\underline{P}^{-1}\underline{i}_{0dq} = \underline{i}_{abc}$, we may write:
$$p = \underline{v}_{abc}^T \underline{i}_{abc} = \left(\underline{P}^{-1}\underline{v}_{0dq}\right)^T \left(\underline{P}^{-1}\underline{i}_{0dq}\right)$$
Recalling that $(ab)^T=b^Ta^T$, the above is:
$$p = \underline{v}_{abc}^T \underline{i}_{abc} = \underline{v}_{odq}^T \left(\underline{P}^{-1}\right)^T \left(\underline{P}^{-1}\underline{i}_{0dq}\right) = \underline{v}_{odq}^T \left(\underline{P}\right)\left(\underline{P}^{-1}\underline{i}_{0dq}\right)$$
$$= \underline{v}_{odq}^T \underline{i}_{0dq}$$
2. The transformed mutual inductances, when per-unitized, do not provide that $M_{jk}=M_{kj}$, implying that the per-unit inductance matrix is not symmetric. This prevents us from finding a real physical circuit to use in modeling the transformed system. See text, pg. 97 for more on this.

In order to overcome these problems, VMAF makes a different choice of constants, according to:

$$k_0 = \frac{1}{\sqrt{3}}, \qquad k_d = k_q = \sqrt{\frac{2}{3}}$$

The choice of k_0, when applied to eq. (i-zero) above, results in:

$$i_0 = \frac{1}{\sqrt{3}}\left(i_a + i_b + i_c\right) = \sqrt{\frac{2}{3}}\left(\frac{1}{\sqrt{2}}i_a + \frac{1}{\sqrt{2}}i_b + \frac{1}{\sqrt{2}}i_c\right)$$

So we see that the factor $\sqrt{\frac{2}{3}}$ is the multiplier on all three equations, resulting in a Park's transformation (and the one that we will use) as:

$$\underline{P} = \sqrt{\frac{2}{3}}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \cos\theta & \cos(\theta - 120) & \cos(\theta + 120) \\ \sin\theta & \sin(\theta - 120) & \sin(\theta + 120) \end{bmatrix} \qquad (4.5)$$

Another choice of coefficients is to choose them as 1/3, 2/3, and 2/3, respectively, which causes the magnitude of the d-q quantities to be equal to that of the three-phase quantities. This choice, used by Kimbark Vol III, eqs. (106) (with negation for i_q coefficient due to use of leading q-axis), is referred to as "magnitude invariance," which we prove below for the i_q equation only (but this causes a 3/2 multiplier in front of the power expression and so is not power-invariant).

$$i_q = k_d\left(i_a \cos\theta + i_b \cos(\theta - 120°) + i_c \cos(\theta + 120°)\right)$$

PROOF: Let i_a=Acos(ωt); i_b=Acos(ωt-120); i_c=Acos(ωt-240) and substitute into i_q equation (it is similar for i_d equation):

$$i_q = k_d\left(A\cos\omega t\cos\theta + A\cos(\omega t-120)\cos(\theta-120°) + A\cos(\omega t+120)\cos(\theta+120°)\right)$$

$$= k_d A\left(\cos\omega t\cos\theta + \cos(\omega t-120)\cos(\theta-120°) + \cos(\omega t+120)\cos(\theta+120°)\right)$$

Now use trig identity: cos(u)cos(v)=(1/2)[cos(u-v)+cos(u+v)]

$$i_q = \frac{k_d A}{2}\{\cos(\omega t-\theta)+\cos(\omega t+\theta) + \cos(\omega t-120-\theta+120)+\cos(\omega t-120+\theta-120) + \cos(\omega t+120-\theta-120)+\cos(\omega t+120+\theta+120)\}$$

$$\Rightarrow i_q = \frac{k_d A}{2}\{\cos(\omega t-\theta)+\cos(\omega t+\theta) + \cos(\omega t-\theta)+\cos(\omega t+\theta-240) + \cos(\omega t-\theta)+\cos(\omega t+\theta+240)\}$$

Now collect terms in ωt-θ and place brackets around what is left:

$$i_q = \frac{k_d A}{2}\{3\cos(\omega t-\theta)+[\cos(\omega t+\theta)+\cos(\omega t+\theta-240)+\cos(\omega t+\theta+240)]\}$$

Observe that what is in the brackets is zero! Therefore:

$$i_q = \frac{k_d A}{2}\{3\cos(\omega t-\theta)\} = \frac{3k_d A}{2}3\cos(\omega t-\theta)$$

Now note that for $3k_dA/2=A$ (thus achieving magnitude invariance for the q current), we must have $k_d=2/3$. QED.

We make 3 more comments about Park's transformation. First, because it is orthogonal[2], the inverse is easy to obtain – it is just $\underline{P}^T$, given explicitly as follows:

$$\underline{P}^{-1} = \sqrt{\frac{2}{3}}\begin{bmatrix} \frac{1}{\sqrt{2}} & cos\theta & sin\theta \\ \frac{1}{\sqrt{2}} & cos\left(\theta-\frac{2\pi}{3}\right) & sin\left(\theta-\frac{2\pi}{3}\right) \\ \frac{1}{\sqrt{2}} & cos\left(\theta+\frac{2\pi}{3}\right) & sin\left(\theta+\frac{2\pi}{3}\right) \end{bmatrix} \quad (4.9)$$

[2] Sometimes people confuse unitary with orthogonal, and for good reason, as they are closely related. *Unitary* is the complex equivalent of *orthogonal*. A complex square matrix **A** is unitary if ($\mathbf{A}^{T*}$)**A**=**I**. A real square matrix **A** is orthogonal if $\mathbf{A}^T\mathbf{A}=\mathbf{I}$.

Second, the angle θ in P can be generalized by choosing any initial angle and any speed, resulting in

$$\theta = \int_0^t \omega(\gamma)d\gamma + \theta(0)$$

where γ is a dummy variable of integration. Although Park chose the speed to be the rotor speed (and so will we), it can be any constant or varying angular velocity or it may remain stationary. You will often hear of the "arbitrary reference frame." The phrase "arbitrary" stems from the fact that the angular velocity of the generalized transformation is unspecified and can be selected arbitrarily to expedite the solution of the equations or to satisfy the system constraints [see Krause's book, Chapt 3, "Reference Frame Theory" for ***excellent treatment*** on generalized reference frame theory.].

Third (and we will repeat these at the end of these notes):

1. i_d and i_q are currents in a fictitious pair of windings *fixed on the rotor*.
2. These currents produce the same flux as do the a,b,c currents.
3. For balanced steady-state operating conditions, we can use $\underline{i}_{0dq} = \underline{P}\underline{i}_{abc}$ to show that the currents in the d and q windings are dc! The implication of this is that:
   - The a,b,c currents fixed in space, varying in time, **produce the same synchronously rotating magnetic field as**
   - The d,q currents, varying in space, fixed in time!

Park's Transformation Applied toVoltage equations for 7-winding representation

Now perform the Park's transformation on both sides of the voltage equation (eq. 4.23' or 4.26'). Note that we apply $\underline{P}$ to only the a-b-c quantities, i.e., we leave the F-G-D-Q quantities alone since these quantities are already on the rotor (and the rotor-rotor inductances are already constants). This means we need to multiply eq. (4.23' or 4.26') through by a matrix

$\begin{bmatrix} \underline{P} & \underline{0} \\ \underline{0} & \underline{U}_4 \end{bmatrix}$ where $\underline{U}_4$ is a 4x4 identity matrix.

Recall (4.26') is:

$$\begin{bmatrix} \underline{v}_{abc} \\ \underline{v}_{FGDQ} \end{bmatrix} = -\begin{bmatrix} \underline{R}_{abc} & \underline{0} \\ \underline{0} & \underline{R}_{FGDQ} \end{bmatrix} \begin{bmatrix} \underline{i}_{abc} \\ \underline{i}_{FGDQ} \end{bmatrix} - \begin{bmatrix} \dot{\underline{\lambda}}_{abc} \\ \dot{\underline{\lambda}}_{FGDQ} \end{bmatrix} + \begin{bmatrix} \underline{v}_n \\ \underline{0} \end{bmatrix} \quad \text{(eq. 4.26')}$$

Multiplying through by our matrix, we obtain:

$$\underbrace{\begin{bmatrix} \underline{P} & \underline{0} \\ \underline{0} & \underline{U}_4 \end{bmatrix} \begin{bmatrix} \underline{v}_{abc} \\ \underline{v}_{FGDQ} \end{bmatrix}}_{\text{term 1}} = -\underbrace{\begin{bmatrix} \underline{P} & \underline{0} \\ \underline{0} & \underline{U}_4 \end{bmatrix} \begin{bmatrix} \underline{R}_{abc} & \underline{0} \\ \underline{0} & \underline{R}_{FGDQ} \end{bmatrix} \begin{bmatrix} \underline{i}_{abc} \\ \underline{i}_{FGDQ} \end{bmatrix}}_{\text{term 2}} - \underbrace{\begin{bmatrix} \underline{P} & \underline{0} \\ \underline{0} & \underline{U}_4 \end{bmatrix} \begin{bmatrix} \dot{\underline{\lambda}}_{abc} \\ \dot{\underline{\lambda}}_{FGDQ} \end{bmatrix}}_{\text{term 3}} + \underbrace{\begin{bmatrix} \underline{P} & \underline{0} \\ \underline{0} & \underline{U}_4 \end{bmatrix} \begin{bmatrix} \underline{v}_n \\ \underline{0} \end{bmatrix}}_{\text{term 4}}$$

(eq. tve1)

We need to express eq. (tve1) in terms of 0-d-q quantities. In what follows, we do this one term at a time. Our general procedure will be to replace the a-b-c quantities with 0-d-q quantities and then simplify.

The easiest one is term 1, so we begin with it.

Term 1:

$$\begin{bmatrix} \underline{P} & \underline{0} \\ \underline{0} & \underline{U}_4 \end{bmatrix} \begin{bmatrix} \underline{v}_{abc} \\ \underline{v}_{FGDQ} \end{bmatrix} = \begin{bmatrix} \underline{P}\underline{v}_{abc} \\ \underline{v}_{FGDQ} \end{bmatrix} = \begin{bmatrix} \underline{v}_{0dq} \\ \underline{v}_{FGDQ} \end{bmatrix}$$

Term 2:

$$\begin{bmatrix} \underline{P} & \underline{0} \\ \underline{0} & \underline{U}_4 \end{bmatrix} \begin{bmatrix} \underline{R}_{abc} & \underline{0} \\ \underline{0} & \underline{R}_{FGDQ} \end{bmatrix} \begin{bmatrix} \underline{i}_{abc} \\ \underline{i}_{FGDQ} \end{bmatrix}$$

Note that

$$\begin{bmatrix} \underline{i}_{0dq} \\ \underline{i}_{FGDQ} \end{bmatrix} = \begin{bmatrix} \underline{P} & \underline{0} \\ \underline{0} & \underline{U}_4 \end{bmatrix} \begin{bmatrix} \underline{i}_{abc} \\ \underline{i}_{FGDQ} \end{bmatrix} \Rightarrow \begin{bmatrix} \underline{i}_{abc} \\ \underline{i}_{FGDQ} \end{bmatrix} = \begin{bmatrix} \underline{P}^{-1} & \underline{0} \\ \underline{0} & \underline{U}_4 \end{bmatrix} \begin{bmatrix} \underline{i}_{0dq} \\ \underline{i}_{FGDQ} \end{bmatrix}$$

Substitution (corresponding to above arrow) yields:

$$\begin{bmatrix} \underline{P} & \underline{0} \\ \underline{0} & \underline{U}_4 \end{bmatrix} \begin{bmatrix} \underline{R}_{abc} & \underline{0} \\ \underline{0} & \underline{R}_{FGDQ} \end{bmatrix} \begin{bmatrix} \underline{P}^{-1} & \underline{0} \\ \underline{0} & \underline{U}_4 \end{bmatrix} \begin{bmatrix} \underline{i}_{0dq} \\ \underline{i}_{FGDQ} \end{bmatrix}$$

$$= \begin{bmatrix} \underline{P}\underline{R}_{abc} & \underline{0} \\ \underline{0} & \underline{R}_{FGDQ} \end{bmatrix} \begin{bmatrix} \underline{P}^{-1} & \underline{0} \\ \underline{0} & \underline{U}_4 \end{bmatrix} \begin{bmatrix} \underline{i}_{0dq} \\ \underline{i}_{FGDQ} \end{bmatrix} = \begin{bmatrix} \underline{P}\underline{R}_{abc}\underline{P}^{-1} & \underline{0} \\ \underline{0} & \underline{R}_{FGDQ} \end{bmatrix} \begin{bmatrix} \underline{i}_{0dq} \\ \underline{i}_{FGDQ} \end{bmatrix}$$

Note that the upper left-hand element (circled) has a diagonal matrix in the middle of two orthogonal matrices.

Fact: If $\underline{P}$ is orthogonal, then $\underline{P}\underline{R}_{abc}\underline{P}^{-1} = \underline{R}_{abc}$ if $\underline{R}_{abc}$ is diagonal having equal elements on the diagonal.

You can test this as follows. Let

$$\underline{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

It is easy to show this is orthogonal using $\underline{A}\ \underline{A}^T=\underline{U}$.

Then try multiplying $\underline{A}\ \underline{R}\ \underline{A}^T$ where $\underline{R} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, and

you should obtain $\underline{R}$.

It is easy to prove as follows. If $\underline{R}$ is a diagonal matrix with all of its diagonal elements the same, call them r, then $\underline{R}=r\underline{U}$. Then

$$\underline{ARA}^T= \underline{A}r\underline{UA}^T= r\underline{AUA}^T=r\underline{AA}^T=r\underline{U}=\underline{R}.$$

Here, we will assume $r_a=r_b=r_c$ which is standard for synchronous machines and simply implies that all phase windings are equal length with the same type of conductor, which is always the case.

Therefore term 2 is just:

$$\begin{bmatrix} \underline{P} & \underline{0} \\ \underline{0} & \underline{U}_4 \end{bmatrix} \begin{bmatrix} \underline{R}_{abc} & \underline{0} \\ \underline{0} & \underline{R}_{FGDQ} \end{bmatrix} \begin{bmatrix} \underline{i}_{abc} \\ \underline{i}_{FGDQ} \end{bmatrix}$$

$$= \begin{bmatrix} \underline{P}\,\underline{R}_{abc}\,\underline{P}^{-1} & \underline{0} \\ \underline{0} & \underline{R}_{FGDQ} \end{bmatrix} \begin{bmatrix} \underline{i}_{0dq} \\ \underline{i}_{FGDQ} \end{bmatrix} = \begin{bmatrix} \underline{R}_{abc} & \underline{0} \\ \underline{0} & \underline{R}_{FGDQ} \end{bmatrix} \begin{bmatrix} \underline{i}_{0dq} \\ \underline{i}_{FGDQ} \end{bmatrix}$$

Repeating our equation (tve1) here for convenience….

$$\underbrace{\begin{bmatrix}\underline{P} & \underline{0}\\ \underline{0} & \underline{U}_4\end{bmatrix}\begin{bmatrix}\underline{v}_{abc}\\ \underline{v}_{FGDQ}\end{bmatrix}}_{\text{term 1}} = -\underbrace{\begin{bmatrix}\underline{P} & \underline{0}\\ \underline{0} & \underline{U}_4\end{bmatrix}\begin{bmatrix}\underline{R}_{abc} & \underline{0}\\ \underline{0} & \underline{R}_{FGDQ}\end{bmatrix}\begin{bmatrix}\underline{i}_{abc}\\ \underline{i}_{FGDQ}\end{bmatrix}}_{\text{term 2}} - \underbrace{\begin{bmatrix}\underline{P} & \underline{0}\\ \underline{0} & \underline{U}_4\end{bmatrix}\begin{bmatrix}\dot{\underline{\lambda}}_{abc}\\ \dot{\underline{\lambda}}_{FGDQ}\end{bmatrix}}_{\text{term 3}} + \underbrace{\begin{bmatrix}\underline{P} & \underline{0}\\ \underline{0} & \underline{U}_4\end{bmatrix}\begin{bmatrix}\underline{v}_n\\ \underline{0}\end{bmatrix}}_{\text{term 4}}$$

and recalling what we have done so far:

TERM 1: $\begin{bmatrix}\underline{P} & \underline{0}\\ \underline{0} & \underline{U}_4\end{bmatrix}\begin{bmatrix}\underline{v}_{abc}\\ \underline{v}_{FGDQ}\end{bmatrix} = \begin{bmatrix}\underline{P}\underline{v}_{abc}\\ \underline{v}_{FGDQ}\end{bmatrix} = \begin{bmatrix}\underline{v}_{0dq}\\ \underline{v}_{FGDQ}\end{bmatrix}$

TERM 2:

$$\begin{bmatrix}\underline{P} & \underline{0}\\ \underline{0} & \underline{U}_4\end{bmatrix}\begin{bmatrix}\underline{R}_{abc} & \underline{0}\\ \underline{0} & \underline{R}_{FGDQ}\end{bmatrix}\begin{bmatrix}\underline{i}_{abc}\\ \underline{i}_{FGDQ}\end{bmatrix}$$

$$= \begin{bmatrix}\underline{P}\underline{R}_{abc}\underline{P}^{-1} & \underline{0}\\ \underline{0} & \underline{R}_{FGDQ}\end{bmatrix}\begin{bmatrix}\underline{i}_{0dq}\\ \underline{i}_{FDQG}\end{bmatrix} = \begin{bmatrix}\underline{R}_{abc} & \underline{0}\\ \underline{0} & \underline{R}_{FGDQ}\end{bmatrix}\begin{bmatrix}\underline{i}_{0dq}\\ \underline{i}_{FGDQ}\end{bmatrix}$$

Substituting, we obtain:

$$\underbrace{\begin{bmatrix}\underline{v}_{0dq}\\ \underline{v}_{FGDQ}\end{bmatrix}}_{\text{term 1}} = -\underbrace{\begin{bmatrix}\underline{R}_{abc} & \underline{0}\\ \underline{0} & \underline{R}_{FGDQ}\end{bmatrix}\begin{bmatrix}\underline{i}_{0dq}\\ \underline{i}_{FGDQ}\end{bmatrix}}_{\text{term 2}} - \underbrace{\begin{bmatrix}\underline{P} & \underline{0}\\ \underline{0} & \underline{U}_4\end{bmatrix}\begin{bmatrix}\dot{\underline{\lambda}}_{abc}\\ \dot{\underline{\lambda}}_{FGDQ}\end{bmatrix}}_{\text{term 3}} + \underbrace{\begin{bmatrix}\underline{P} & \underline{0}\\ \underline{0} & \underline{U}_4\end{bmatrix}\begin{bmatrix}\underline{v}_n\\ \underline{0}\end{bmatrix}}_{\text{term 4}}$$

eq. (tve2)

Now we observe that terms 3 and 4 have variables not in terms of 0-d-q quantities. We work on term 4 next (before term 3) because it is easier.

Term 4:

Observe that $\underline{v}_n=[v_n \quad v_n \quad v_n]^T$. Therefore, when we multiply $\underline{P}\underline{v}_n$, we get elements in the second and third rows of $\underline{P}$ being scaled by the same constant (v_n) and then summed. Consider these elements in the second and third rows of $\underline{P}$, below.

$$\underline{P}\underline{v}_n = \sqrt{\frac{2}{3}}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \cos\theta & \cos(\theta-120) & \cos(\theta+120) \\ \sin\theta & \sin(\theta-120) & \sin(\theta+120) \end{bmatrix}\begin{bmatrix} v_n \\ v_n \\ v_n \end{bmatrix}$$

So the product of the second row and $\underline{v}_n$, or of the third row and $\underline{v}_n$, will include a summation of symmetrical components, *which will be zero*! So the only non-zero element in $\underline{Pv}_n$ will be the product of the first row of $\underline{P}$ and $\underline{v}_n$, i.e., the first element of the term 4 vector, which is

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}\begin{bmatrix} v_n \\ v_n \\ v_n \end{bmatrix} = \frac{3v_n}{\sqrt{3}} \qquad (*)$$

But recall from our circuit the voltage equation indicates that:

$$v_n = -i_n r_n - L_n \dot{i}_n = -(i_a + i_b + i_c)r_n - L_n(\dot{i}_a + \dot{i}_b + \dot{i}_c) \quad (**)$$

Also, recall that from the Park's transformation $\underline{i}_{odq}=\underline{P}\underline{i}_{abc}$ that the i_0 current is (pg. 14 of these notes):

$$i_0 = \frac{1}{\sqrt{3}}(i_a + i_b + i_c) \Rightarrow i_a + i_b + i_c = \sqrt{3}i_0 \qquad (***)$$

Substitution of (***) into (**) yields:

$$v_n = -(\sqrt{3}i_0)r_n - L_n(\sqrt{3}\dot{i}_0)$$

and replacing v_n in (*) with this, we have:

$$\begin{bmatrix} \underline{P} & \underline{0} \\ \underline{0} & \underline{U}_4 \end{bmatrix} \begin{bmatrix} \underline{v}_n \\ \underline{0} \end{bmatrix} = \begin{bmatrix} \underline{Pv}_n \\ \underline{0} \end{bmatrix} = \begin{bmatrix} 3r_n i_0 - 3L_n \dot{i}_0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \underline{n}_{0dq} \\ \underline{0} \end{bmatrix} \qquad (*\#)$$

where $\underline{n}_{0dq}$ is the first 3 elements and $\underline{0}$ is the last 4 elements.

Now recall eqt. (tve2) p. 20, repeated here for convenience:

$$\underbrace{\begin{bmatrix} \underline{v}_{0dq} \\ \underline{v}_{FGDQ} \end{bmatrix}}_{\text{term 1}} = -\underbrace{\begin{bmatrix} \underline{R}_{abc} & \underline{0} \\ \underline{0} & \underline{R}_{FGDQ} \end{bmatrix} \begin{bmatrix} \underline{i}_{0dq} \\ \underline{i}_{FGDQ} \end{bmatrix}}_{\text{term 2}} - \underbrace{\begin{bmatrix} \underline{P} & \underline{0} \\ \underline{0} & \underline{U}_4 \end{bmatrix} \begin{bmatrix} \dot{\underline{\lambda}}_{abc} \\ \dot{\underline{\lambda}}_{FGDQ} \end{bmatrix}}_{\text{term 3}} + \underbrace{\begin{bmatrix} \underline{P} & \underline{0} \\ \underline{0} & \underline{U}_4 \end{bmatrix} \begin{bmatrix} \underline{v}_n \\ \underline{0} \end{bmatrix}}_{\text{term 4}}$$

and substitute in eqt. (*#) to obtain

$$\underbrace{\begin{bmatrix} \underline{v}_{0dq} \\ \underline{v}_{FGDQ} \end{bmatrix}}_{\text{term 1}} = -\underbrace{\begin{bmatrix} \underline{R}_{abc} & \underline{0} \\ \underline{0} & \underline{R}_{FGDQ} \end{bmatrix} \begin{bmatrix} \underline{i}_{0dq} \\ \underline{i}_{FGDQ} \end{bmatrix}}_{\text{term 2}} - \underbrace{\begin{bmatrix} \underline{P} & \underline{0} \\ \underline{0} & \underline{U}_4 \end{bmatrix} \begin{bmatrix} \dot{\underline{\lambda}}_{abc} \\ \dot{\underline{\lambda}}_{FGDQ} \end{bmatrix}}_{\text{term 3}} + \underbrace{\begin{bmatrix} \underline{n}_{0dq} \\ \underline{0} \end{bmatrix}}_{\text{term 4}}$$

eq. (tve3)

And so now the only a-b-c variables remaining are in term 3. So let's work on term 3.

Term 3:

Term 3 is:

$$\begin{bmatrix} \underline{P} & \underline{0} \\ \underline{0} & \underline{U}_4 \end{bmatrix} \begin{bmatrix} \underline{\dot{\lambda}}_{abc} \\ \underline{\dot{\lambda}}_{FGDQ} \end{bmatrix} = \begin{bmatrix} \underline{P}\underline{\dot{\lambda}}_{abc} \\ \underline{\dot{\lambda}}_{FGDQ} \end{bmatrix} \qquad (4.30')$$

So we need to do two things:

1. Obtain $\underline{P}\underline{\dot{\lambda}}_{abc}$ in terms of the 0-d-q quantities.
2. Express all of term 3 in terms of currents instead of flux linkages.

To begin this task, recall that $\underline{\lambda}_{0dq} = \underline{P}\underline{\lambda}_{abc}$, and take derivatives of both sides. Note in differentiating the right-hand-side, we need to account for the fact that $\underline{P}$ is time-dependent. Thus:

$$\underline{\dot{\lambda}}_{0dq} = \underline{P}\underline{\dot{\lambda}}_{abc} + \underline{\dot{P}}\underline{\lambda}_{abc}$$

Solving for $\underline{P}\underline{\dot{\lambda}}_{abc}$, we obtain:

$$\underline{P}\underline{\dot{\lambda}}_{abc} = \underline{\dot{\lambda}}_{0dq} - \underline{\dot{P}}\underline{\lambda}_{abc} \qquad (\#)$$

But the right-hand side still has $\underline{\lambda}_{abc}$. We can eliminate this using

$$\underline{\lambda}_{abc} = \underline{P}^{-1}\underline{\lambda}_{odq}$$

Substitution into eq. (#) yields:

$$\underline{P}\underline{\dot{\lambda}}_{abc} = \underline{\dot{\lambda}}_{0dq} - \underline{\dot{P}}\underline{P}^{-1}\underline{\lambda}_{odq} \qquad (4.31)$$

Now we have expressed $\underline{P}\underline{\dot{\lambda}}_{abc}$ in terms of the 0-d-q quantities. Substitution of eq. (4.31) into eq. (4.30') above yields:

$$\begin{bmatrix} \underline{P} & \underline{0} \\ \underline{0} & \underline{U}_4 \end{bmatrix} \begin{bmatrix} \underline{\dot{\lambda}}_{abc} \\ \underline{\dot{\lambda}}_{FDQG} \end{bmatrix} = \begin{bmatrix} \underline{P}\underline{\dot{\lambda}}_{abc} \\ \underline{\dot{\lambda}}_{FDQG} \end{bmatrix} = \underbrace{\begin{bmatrix} \underline{\dot{\lambda}}_{0dq} \\ \underline{\dot{\lambda}}_{FDQG} \end{bmatrix}}_{\text{term 3a}} - \underbrace{\begin{bmatrix} \underline{\dot{P}}\underline{P}^{-1}\underline{\lambda}_{0dq} \\ \underline{0} \end{bmatrix}}_{\text{term 3b}}$$

So we have accomplished our objective 1, which was to obtain $\underline{P}\underline{\dot{\lambda}}_{abc}$ in terms of the 0-d-q quantities. Let's substitute the above equation into eq. (tve3)

$$\underbrace{\begin{bmatrix} \underline{v}_{0dq} \\ \underline{v}_{FDQG} \end{bmatrix}}_{\text{term 1}} = -\underbrace{\begin{bmatrix} \underline{R}_{abc} & \underline{0} \\ \underline{0} & \underline{R}_{FDQG} \end{bmatrix} \begin{bmatrix} \underline{i}_{0dq} \\ \underline{i}_{FDQG} \end{bmatrix}}_{\text{term 2}} - \underbrace{\begin{bmatrix} \underline{P} & \underline{0} \\ \underline{0} & \underline{U}_4 \end{bmatrix} \begin{bmatrix} \underline{\dot{\lambda}}_{abc} \\ \underline{\dot{\lambda}}_{FDQG} \end{bmatrix}}_{\text{term 3}} + \underbrace{\begin{bmatrix} \underline{n}_{0dq} \\ \underline{0} \end{bmatrix}}_{\text{term 4}}$$

eq. (tve3)

to obtain

$$\underbrace{\begin{bmatrix} \underline{v}_{0dq} \\ \underline{v}_{FDQG} \end{bmatrix}}_{\text{term 1}} = -\underbrace{\begin{bmatrix} \underline{R}_{abc} & \underline{0} \\ \underline{0} & \underline{R}_{FDQG} \end{bmatrix} \begin{bmatrix} \underline{i}_{0dq} \\ \underline{i}_{FDQG} \end{bmatrix}}_{\text{term 2}} - \underbrace{\begin{bmatrix} \underline{\dot{\lambda}}_{0dq} \\ \underline{\dot{\lambda}}_{FDQG} \end{bmatrix}}_{\text{term 3a}} + \underbrace{\begin{bmatrix} \underline{\dot{P}}\underline{P}^{-1}\underline{\lambda}_{0dq} \\ \underline{0} \end{bmatrix}}_{\text{term 3b}} + \underbrace{\begin{bmatrix} \underline{n}_{0dq} \\ \underline{0} \end{bmatrix}}_{\text{term 4}}$$

eq. (tve4)

Now we need to accomplish our objective 2, which is to express all of term 3 as currents instead of flux linkages. To do this, let's investigate terms 3a and 3b one at a time. Let's start with term 3a….

Term 3a:

So term 3a is:

$$\begin{bmatrix} \underline{\dot{\lambda}}_{0dq} \\ \underline{\dot{\lambda}}_{FGDQ} \end{bmatrix}$$

Our goal is to see if we can express this in terms of currents, which means we will need to use inductances. Let's start by looking at the same expression but without the derivatives, since we know how to write this using Park's transformation and a-b-c flux linkages. This is:

$$\begin{bmatrix} \underline{\lambda}_{0dq} \\ \underline{\lambda}_{FGDQ} \end{bmatrix} = \begin{bmatrix} \underline{P} & 0 \\ 0 & \underline{U}_4 \end{bmatrix} \begin{bmatrix} \underline{\lambda}_{abc} \\ \underline{\lambda}_{FGDQ} \end{bmatrix} \qquad \text{(eq. 3a-1)}$$

Now to write eq. (3a-1) in terms of the 0dq/FGDQ currents (instead of 0dq/FGDQ flux linkages), recall from eq. (L), pg. 2, repeated here for convenience

$$\begin{bmatrix} \underline{\lambda}_{abc} \\ \underline{\lambda}_{FDQG} \end{bmatrix} = \begin{bmatrix} \underline{L}_{aa} & \underline{L}_{aR} \\ \underline{L}_{Ra} & \underline{L}_{RR} \end{bmatrix} \begin{bmatrix} \underline{i}_{abc} \\ \underline{i}_{FDQG} \end{bmatrix} \qquad \text{(eq. 3a-2)}$$

that the vector of abc/FGDQ flux linkages on the right of (eq. 3a-1) is related through the inductance matrix to the abc/FGDQ currents.

Now recall that the abc/FGDQ currents may be related to the 0dq/FGDQ currents using the inverse Park Transformation according to:

$$\begin{bmatrix} \underline{i}_{abc} \\ \underline{i}_{FGDQ} \end{bmatrix} = \begin{bmatrix} \underline{P}^{-1} & 0 \\ \underline{0} & \underline{U}_4 \end{bmatrix} \begin{bmatrix} \underline{i}_{0dq} \\ \underline{i}_{FGDQ} \end{bmatrix} \qquad \text{(eq. 3a-3)}$$

Substitution of (3a-3) into (3a-2) and then what results into (3a-1), we have

$$\begin{bmatrix} \underline{\lambda}_{0dq} \\ \underline{\lambda}_{FGDQ} \end{bmatrix} = \begin{bmatrix} \underline{P} & 0 \\ 0 & \underline{U}_4 \end{bmatrix} \begin{bmatrix} \underline{L}_{aa} & \underline{L}_{aR} \\ \underline{L}_{Ra} & \underline{L}_{RR} \end{bmatrix} \begin{bmatrix} \underline{P}^{-1} & 0 \\ \underline{0} & \underline{U}_4 \end{bmatrix} \begin{bmatrix} \underline{i}_{0dq} \\ \underline{i}_{FGDQ} \end{bmatrix}$$

Performing the above matrix multiplication, we obtain….

$$\begin{bmatrix} \underline{\lambda}_{0dq} \\ \underline{\lambda}_{FDQG} \end{bmatrix} = \begin{bmatrix} \underline{P}\,\underline{L}_{aa}\underline{P}^{-1} & \underline{P}\,\underline{L}_{aR} \\ \underline{L}_{Ra}\underline{P}^{-1} & \underline{L}_{RR} \end{bmatrix} \begin{bmatrix} \underline{i}_{0dq} \\ \underline{i}_{FDQG} \end{bmatrix}$$

Now we need to go through each of these four matrix multiplications. I will here omit the details and just give the results (note also in what follows the definition of additional nomenclature for each of the four submatrices). But before doing that, let's remind ourselves of what the above inductance terms look like.

$$\begin{bmatrix} \underline{\lambda}_{abc} \\ \underline{\lambda}_{FGDQ} \end{bmatrix} = [\underline{L}] \begin{bmatrix} \underline{i}_{abc} \\ \underline{i}_{FGDQ} \end{bmatrix} = \begin{bmatrix} \underline{L}_{aa} & \underline{L}_{aR} \\ \underline{L}_{Ra} & \underline{L}_{RR} \end{bmatrix} \begin{bmatrix} \underline{i}_{abc} \\ \underline{i}_{FGDQ} \end{bmatrix} \qquad \text{(eq. L)}$$

$$\begin{bmatrix} \lambda_a \\ \lambda_b \\ \lambda_c \\ \lambda_F \\ \lambda_G \\ \lambda_D \\ \lambda_Q \end{bmatrix} = \begin{bmatrix} L_S + L_m \cos 2\theta & -[M_S + L_m \cos 2(\theta + 30°)] & -[M_S + L_m \cos 2(\theta + 150°)] & M_F \cos\theta & M_G \sin\theta & M_D \cos\theta & M_Q \sin\theta \\ -[M_S + L_m \cos 2(\theta + 30°)] & L_S + L_m \cos 2(\theta - 120°) & -[M_S + L_m \cos 2(\theta - 90°)] & M_F \cos(\theta - 120°) & M_G \sin(\theta - 120°) & M_D \cos(\theta - 120°) & M_Q \sin(\theta - 120°) \\ -[M_S + L_m \cos 2(\theta + 150°)] & -[M_S + L_m \cos 2(\theta - 90°)] & L_S + L_m \cos 2(\theta - 240°) & M_F \cos(\theta - 240°) & M_G \sin(\theta - 240°) & M_D \cos(\theta - 240°) & M_Q \sin(\theta - 240°) \\ M_F \cos\theta & M_F \cos(\theta - 120°) & M_F \cos(\theta - 240°) & L_F & 0 & M_R & 0 \\ M_G \sin\theta & M_G \sin(\theta - 120°) & M_G \sin(\theta - 240°) & 0 & L_G & 0 & M_Y \\ M_D \cos\theta & M_D \cos(\theta - 120°) & M_D \cos(\theta - 240°) & M_R & 0 & L_D & 0 \\ M_Q \sin\theta & M_Q \sin(\theta - 120°) & M_Q \sin(\theta - 240°) & 0 & M_Y & 0 & L_Q \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_b \\ i_F \\ i_G \\ i_D \\ i_Q \end{bmatrix}$$

(eq. L-ex)

Back to our matrix multiplications,

$$\begin{bmatrix} \underline{\lambda}_{0dq} \\ \underline{\lambda}_{FDQG} \end{bmatrix} = \begin{bmatrix} \underline{P}\,\underline{L}_{aa}\underline{P}^{-1} & \underline{P}\,\underline{L}_{aR} \\ \underline{L}_{Ra}\underline{P}^{-1} & \underline{L}_{RR} \end{bmatrix} \begin{bmatrix} \underline{i}_{0dq} \\ \underline{i}_{FDQG} \end{bmatrix}$$

where we refer to the four submatrices on the right as submatrix (1,1), submatrix(1,2), submatrix(2,1), and submatrix(2,2).

Submatrix (1,1):

$$\underline{P}\,\underline{L}_{aa}\,\underline{P}^{-1} = \begin{bmatrix} L_0 & 0 & 0 \\ 0 & L_d & 0 \\ 0 & 0 & L_q \end{bmatrix} \equiv \underline{L}_{0dq}$$

where $L_0 = L_S - 2M_S$, $L_d = L_S + M_S + (3/2)L_m$, and $L_q = L_S + M_S - (3/2)L_m$.

Submatrix (1,2):

$$\underline{P}\,\underline{L}_{aR} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \sqrt{\frac{3}{2}}M_F & 0 & \sqrt{\frac{3}{2}}M_D & 0 \\ 0 & \sqrt{\frac{3}{2}}M_G & 0 & \sqrt{\frac{3}{2}}M_Q \end{bmatrix} \equiv \underline{L}_m$$

Submatrix (2,1):

$$\underline{L}_{Ra}\,\underline{P}^{-1} = \begin{bmatrix} 0 & \sqrt{\frac{3}{2}}M_F & 0 \\ 0 & 0 & \sqrt{\frac{3}{2}}M_G \\ 0 & \sqrt{\frac{3}{2}}M_D & 0 \\ 0 & 0 & \sqrt{\frac{3}{2}}M_Q \end{bmatrix} \equiv \underline{L}_m^T$$

Submatrix (2,2) (note that this submatrix is unchanged from the original inductance matrix):

$$\underline{L}_{RR} = \begin{bmatrix} L_F & 0 & M_R & 0 \\ 0 & L_G & 0 & M_Y \\ M_R & 0 & L_D & 0 \\ 0 & M_Y & 0 & L_Q \end{bmatrix} \equiv \underline{L}_{RR}$$

Using the defined nomenclature above for the 4 submatrices, we finally have:

$$\begin{bmatrix} \underline{\lambda}_{0dq} \\ \underline{\lambda}_{FGDQ} \end{bmatrix} = \begin{bmatrix} \underline{L}_{0dq} & \underline{L}_m \\ \underline{L}^T{}_m & \underline{L}_{RR} \end{bmatrix} \begin{bmatrix} \underline{i}_{0dq} \\ \underline{i}_{FGDQ} \end{bmatrix}$$

Expanding…

$$\begin{bmatrix} \lambda_0 \\ \lambda_d \\ \lambda_q \\ \lambda_F \\ \lambda_G \\ \lambda_D \\ \lambda_Q \end{bmatrix} = \begin{bmatrix} L_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & L_d & 0 & kM_F & 0 & kM_D & 0 \\ 0 & 0 & L_q & 0 & kM_G & 0 & kM_Q \\ 0 & kM_F & 0 & L_F & 0 & M_R & 0 \\ 0 & 0 & kM_G & 0 & L_G & 0 & M_Y \\ 0 & kM_D & 0 & M_R & 0 & L_D & 0 \\ 0 & 0 & kM_Q & 0 & M_Y & 0 & L_Q \end{bmatrix} \begin{bmatrix} i_0 \\ i_d \\ i_q \\ i_F \\ i_G \\ i_D \\ i_Q \end{bmatrix} \quad (4.20)$$

where $k = \sqrt{\frac{3}{2}}$. Compare this to eq. (L-ex) on page 27 (and pg. 2) to see a very large improvement in simplicity.

> <u>Aside</u>: It is convenient here to note from the above matrix relation that λ_d and λ_q are given by:
>
> $$\underline{\lambda}_{0dq} = \underline{L}_{0dq}\underline{i}_{0dq} + \underline{L}_m \underline{i}_{FGDQ} \Rightarrow \lambda_d = L_d i_d + \sqrt{\frac{3}{2}} M_F i_F + \sqrt{\frac{3}{2}} M_D i_D$$
>
> $$\Rightarrow \lambda_q = L_q i_q + \sqrt{\frac{3}{2}} M_G i_G + \sqrt{\frac{3}{2}} M_Q i_Q$$
>
> We will use this in developing term 3b below, see p. 33.

One nice surprise from the above is that THE MATRIX IS CONSTANT!!!

As a result of this “nice surprise,” we may differentiate both sides to get:

$$\begin{bmatrix} \dot{\underline{\lambda}}_{0dq} \\ \dot{\underline{\lambda}}_{FGDQ} \end{bmatrix} = \begin{bmatrix} \underline{L}_{0dq} & \underline{L}_m \\ \underline{L}^T{}_m & \underline{L}_{RR} \end{bmatrix} \begin{bmatrix} \dot{\underline{i}}_{0dq} \\ \dot{\underline{i}}_{FGDQ} \end{bmatrix} \qquad (\$)$$

or, when expanded, is:

$$\begin{bmatrix} \dot{\lambda}_0 \\ \dot{\lambda}_d \\ \dot{\lambda}_q \\ \dot{\lambda}_F \\ \dot{\lambda}_G \\ \dot{\lambda}_D \\ \dot{\lambda}_Q \end{bmatrix} = \begin{bmatrix} L_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & L_d & 0 & kM_F & 0 & kM_D & 0 \\ 0 & 0 & L_q & 0 & kM_G & 0 & kM_Q \\ 0 & kM_F & 0 & L_F & 0 & M_R & 0 \\ 0 & 0 & kM_G & 0 & L_G & 0 & M_Y \\ 0 & kM_D & 0 & M_R & 0 & L_D & 0 \\ 0 & 0 & kM_Q & 0 & M_Y & 0 & L_Q \end{bmatrix} \begin{bmatrix} \dot{i}_0 \\ \dot{i}_d \\ \dot{i}_q \\ \dot{i}_F \\ \dot{i}_G \\ \dot{i}_D \\ \dot{i}_Q \end{bmatrix}$$

Substitution of ($) for term 3a into eq. (tve4), repeated here for convenience,

$$\underbrace{\begin{bmatrix} \underline{v}_{0dq} \\ \underline{v}_{FGDQ} \end{bmatrix}}_{\text{term 1}} = -\underbrace{\begin{bmatrix} \underline{R}_{abc} & \underline{0} \\ \underline{0} & \underline{R}_{FGDQ} \end{bmatrix} \begin{bmatrix} \underline{i}_{0dq} \\ \underline{i}_{FGDQ} \end{bmatrix}}_{\text{term 2}} - \underbrace{\begin{bmatrix} \dot{\underline{\lambda}}_{0dq} \\ \dot{\underline{\lambda}}_{FGDQ} \end{bmatrix}}_{\text{term 3a}} + \underbrace{\begin{bmatrix} \dot{\underline{P}}\underline{P}^{-1}\underline{\lambda}_{0dq} \\ \underline{0} \end{bmatrix}}_{\text{term 3b}} + \underbrace{\begin{bmatrix} \underline{n}_{0dq} \\ \underline{0} \end{bmatrix}}_{\text{term 4}}$$

eq. (tve4)

results in

$$\underbrace{\begin{bmatrix} \underline{v}_{0dq} \\ \underline{v}_{FGDQ} \end{bmatrix}}_{\text{term 1}} = -\underbrace{\begin{bmatrix} \underline{R}_{abc} & \underline{0} \\ \underline{0} & \underline{R}_{FGDQ} \end{bmatrix} \begin{bmatrix} \underline{i}_{0dq} \\ \underline{i}_{FGDQ} \end{bmatrix}}_{\text{term 2}} - \underbrace{\begin{bmatrix} \underline{L}_{0dq} & \underline{L}_m \\ \underline{L}^T{}_m & \underline{L}_{RR} \end{bmatrix} \begin{bmatrix} \dot{\underline{i}}_{0dq} \\ \dot{\underline{i}}_{FGDQ} \end{bmatrix}}_{\text{term 3a}} + \underbrace{\begin{bmatrix} \dot{\underline{P}}\underline{P}^{-1}\underline{\lambda}_{0dq} \\ \underline{0} \end{bmatrix}}_{\text{term 3b}} + \underbrace{\begin{bmatrix} \underline{n}_{0dq} \\ \underline{0} \end{bmatrix}}_{\text{term 4}}$$

eq. (tve5)

We are almost done! The only remaining term which contains flux linkages is term 3b.

Term 3b:

Recalling term 3b is: $\begin{bmatrix} \dot{\underline{P}}\underline{P}^{-1}\underline{\lambda}_{0dq} \\ \underline{0} \end{bmatrix}$

we see that we need to expand the product $\dot{\underline{P}}\underline{P}^{-1}$. First, recall that:

$$\underline{P} = \sqrt{\frac{2}{3}} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \cos\theta & \cos(\theta - 120) & \cos(\theta + 120) \\ \sin\theta & \sin(\theta - 120) & \sin(\theta + 120) \end{bmatrix}$$

Also, recall that

$$\theta = \int_0^t \omega(\gamma)d\gamma + \theta(0)$$

$$\Rightarrow \qquad \dot{\theta} = \omega(t)$$

And note carefully that $\underline{P}$ is a function of time because the angle θ is a function of t. Therefore we need to differentiate $\underline{P}$; we do so using chain rule. This is not hard and results in:

$$\dot{\underline{P}} = \frac{d\underline{P}}{dt} = \sqrt{\frac{2}{3}}\omega \begin{bmatrix} 0 & 0 & 0 \\ -\sin\theta & -\sin(\theta - 120) & -\sin(\theta + 120) \\ \cos\theta & \cos(\theta - 120) & \cos(\theta + 120) \end{bmatrix}$$

Now taking the product $\dot{\underline{P}}\underline{P}^{-1}$, we obtain:

$$\underline{\dot{P}}\underline{P}^{-1} = \sqrt{\frac{2}{3}}\sqrt{\frac{2}{3}}\omega \begin{bmatrix} 0 & 0 & 0 \\ -\sin\theta & -\sin(\theta-120) & -\sin(\theta+120) \\ \cos\theta & \cos(\theta-120) & \cos(\theta+120) \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \cos\theta & \sin\theta \\ \frac{1}{\sqrt{2}} & \cos(\theta-120) & \sin(\theta-120) \\ \frac{1}{\sqrt{2}} & \cos(\theta+120) & \sin(\theta+120) \end{bmatrix}$$

$$= \frac{2}{3}\omega \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -3/2 \\ 0 & 3/2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\omega \\ 0 & \omega & 0 \end{bmatrix}$$

Note in the above that row 1 is all zeros because row 1 in $\dot{P}$ is all zeros. On the other hand, column 1 is all zeros because the multiplication of rows 2 and 3 in $\dot{P}$ by column 1 of $\underline{P}^{-1}$ yield a sum of symmetrical terms.

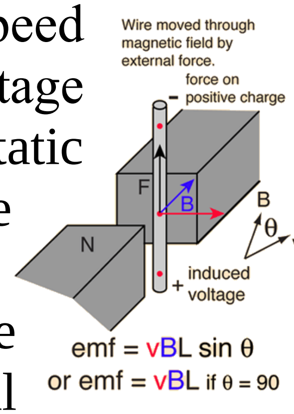
This provides that:

$$\underline{\dot{P}}\underline{P}^{-1}\underline{\lambda}_{0dq} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\omega \\ 0 & \omega & 0 \end{bmatrix} \begin{bmatrix} \lambda_0 \\ \lambda_d \\ \lambda_q \end{bmatrix} = \begin{bmatrix} 0 \\ -\omega\lambda_q \\ \omega\lambda_d \end{bmatrix} \qquad (\$\&)$$

The terms $-\omega\lambda_q$ and $\omega\lambda_d$ are called *speed voltages*; comments:

- These *speed voltage s* together account for the voltages induced in the (fixed) phase windings as a result of the spatially-moving constant magnetic field from the rotor DC current.
- They represent the fact that a constant (in time) flux wave rotating in synchronism with the rotor will create voltages in the stationary armature coils.

- Speed voltages (flux change in space), are so named to contrast them from what may be called *transformer voltages* (flux change in time) which are induced as a result of a time varying magnetic field.
- You may have run across the concept of "speed voltages" in Physics, where you computed a voltage induced in a coil of wire as it moved through a static magnetic field, in which case, you may have used the equation *Blv* where *B* is flux density, *l* is conductor length, and *v* is the component of the velocity of the moving conductor (or moving field) that is normal with respect to the field flux direction (or conductor).



- The first speed voltage term, $-\omega\lambda_q$, appears in the v_d equation. The second speed voltage term, $\omega\lambda_d$, appears in the v_q equation. Thus, we see that the q-axis flux causes a speed voltage in the d-axis winding, and the d-axis flux causes a speed voltage in the q-axis winding.
- Fitzgerald and Kingsley in their book "Electric Machinery" provide a good discussion of speed voltages in Chapter 2; Bergan & Vittal discuss it on pg. 216; Kundur on pg. 71).

Now we are in a position to obtain term 3b. Recall the expressions for λ_d and λ_q obtained in the "Aside" box, p. 29:

$$\lambda_d = L_d i_d + \sqrt{\frac{3}{2}} M_F i_F + \sqrt{\frac{3}{2}} M_D i_D$$

$$\lambda_q = L_q i_q + \sqrt{\frac{3}{2}} M_G i_G + \sqrt{\frac{3}{2}} M_Q i_Q$$

Using these in ($&) above, we obtain:

$$\begin{bmatrix} \frac{\dot{P}P^{-1}\lambda_{0dq}}{\underline{0}} \end{bmatrix} = \begin{bmatrix} 0 \\ -\omega\lambda_q \\ \omega\lambda_d \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -\omega L_q i_q - \omega\sqrt{\frac{3}{2}}M_G i_G - \omega\sqrt{\frac{3}{2}}M_Q i_Q \\ \omega L_d i_d + \omega\sqrt{\frac{3}{2}}M_F i_F + \omega\sqrt{\frac{3}{2}}M_D i_D \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\underline{speed}}{\underline{0}} \end{bmatrix} \quad (\&)$$

where

$$\underline{speed} = \begin{bmatrix} 0 \\ -\omega L_q i_q - \omega\sqrt{\frac{3}{2}}M_G i_G - \omega\sqrt{\frac{3}{2}}M_Q i_Q \\ \omega L_d i_d + \omega\sqrt{\frac{3}{2}}M_F i_F + \omega\sqrt{\frac{3}{2}}M_D i_D \end{bmatrix}; \qquad \underline{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now recalling eq. (tve5),

$$\underbrace{\begin{bmatrix} \underline{v}_{0dq} \\ \underline{v}_{FDQG} \end{bmatrix}}_{\text{term 1}} = -\underbrace{\begin{bmatrix} \underline{R}_{abc} & \underline{0} \\ \underline{0} & \underline{R}_{FDQG} \end{bmatrix}\begin{bmatrix} \underline{i}_{0dq} \\ \underline{i}_{FDQG} \end{bmatrix}}_{\text{term 2}} - \underbrace{\begin{bmatrix} \underline{L}_{0dq} & \underline{L}_{m} \\ \underline{L}^{T}_{m} & \underline{L}_{RR} \end{bmatrix}\begin{bmatrix} \dot{\underline{i}}_{0dq} \\ \dot{\underline{i}}_{FDQG} \end{bmatrix}}_{\text{term 3a}} + \underbrace{\begin{bmatrix} \dot{\underline{P}}\underline{P}^{-1}\underline{\lambda}_{0dq} \\ \underline{0} \end{bmatrix}}_{\text{term 3b}} + \underbrace{\begin{bmatrix} \underline{n}_{0dq} \\ \underline{0} \end{bmatrix}}_{\text{term 4}}$$

eq. (tve5)

we substitute (&) to obtain:

$$\underbrace{\begin{bmatrix} \underline{v}_{0dq} \\ \underline{v}_{FDQG} \end{bmatrix}}_{\text{term 1}} = -\underbrace{\begin{bmatrix} \underline{R}_{abc} & \underline{0} \\ \underline{0} & \underline{R}_{FDQG} \end{bmatrix}\begin{bmatrix} \underline{i}_{0dq} \\ \underline{i}_{FDQG} \end{bmatrix}}_{\text{term 2}} - \underbrace{\begin{bmatrix} \underline{L}_{0dq} & \underline{L}_{m} \\ \underline{L}^{T}_{m} & \underline{L}_{RR} \end{bmatrix}\begin{bmatrix} \dot{\underline{i}}_{0dq} \\ \dot{\underline{i}}_{FDQG} \end{bmatrix}}_{\text{term 3a}} + \underbrace{\begin{bmatrix} \frac{\underline{speed}}{\underline{0}} \end{bmatrix}}_{\text{term 3b}} + \underbrace{\begin{bmatrix} \underline{n}_{0dq} \\ \underline{0} \end{bmatrix}}_{\text{term 4}}$$

eq. (tve6)

Putting it all together:

Let's re-write the voltage equation eq. (tve6) by substituting in complete expressions for all vectors and submatrices in terms 1, 2, 3a, 3b, and 4, as obtained above:

Term 1 Term 2 Term 3a

$$\begin{bmatrix} v_0 \\ v_d \\ v_q \\ -v_F \\ 0 \\ 0 \\ 0 \end{bmatrix} = -\begin{bmatrix} r_a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & r_b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r_c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r_F & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & r_G & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & r_D & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & r_Q \end{bmatrix} \begin{bmatrix} i_0 \\ i_d \\ i_q \\ i_F \\ i_G \\ i_D \\ i_Q \end{bmatrix} - \begin{bmatrix} L_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & L_d & 0 & kM_F & 0 & kM_D & 0 \\ 0 & 0 & L_q & 0 & kM_G & 0 & kM_Q \\ 0 & kM_F & 0 & L_F & 0 & M_R & 0 \\ 0 & 0 & kM_G & 0 & L_G & 0 & M_Y \\ 0 & kM_D & 0 & M_R & 0 & L_D & 0 \\ 0 & 0 & kM_Q & 0 & M_Y & 0 & L_Q \end{bmatrix} \begin{bmatrix} \dot{i}_0 \\ \dot{i}_d \\ \dot{i}_q \\ \dot{i}_F \\ \dot{i}_G \\ \dot{i}_D \\ \dot{i}_Q \end{bmatrix}$$

$$+\begin{bmatrix} 0 \\ -\omega L_q i_q - \omega\sqrt{\frac{3}{2}} M_G i_G - \omega\sqrt{\frac{3}{2}} M_Q i_Q \\ \omega L_d i_d + \omega\sqrt{\frac{3}{2}} M_F i_F + \omega\sqrt{\frac{3}{2}} M_D i_D \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 3r_n i_0 - 3L_n \dot{i}_0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Term 3b Term 4

Now, observe that each of the non-zero elements of term 3b and term 4 is multiplied by a current or current derivative, and that terms 2 and 3a both get multiplied by vectors of currents or current derivatives, respectively. Therefore, we may "fold-in" Term 3b and Term 4 into the Terms 2 and 3a by combining parts of the non-zero term 3b and 4 elements with the appropriate matrix element in terms 2 and 3a.

For example, we may fold in the -$\omega L_q i_q$ term in row 2 of term 3b by including ωL_q in row 2 (since we are dealing with the second equation), column 3 (since we need the term that multiplies i_q) of term 2. Note that since term 2 has a "minus" sign out front, we do not include the "minus" sign of -$\omega L_q i_q$ when we fold it in. The circle and arrow above illustrate this folding-in operation.

The complete results of all fold-in operations are provided in what follows:

$$\begin{bmatrix} v_0 \\ v_d \\ v_q \\ -v_F \\ 0 \\ 0 \\ 0 \end{bmatrix} = -\begin{bmatrix} r_a+3r_n & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & r_b & \omega L_q & 0 & \omega\sqrt{\frac{3}{2}}M_G & 0 & \omega\sqrt{\frac{3}{2}}M_Q \\ 0 & -\omega L_D & r_c & -\omega\sqrt{\frac{3}{2}}M_F & 0 & -\omega\sqrt{\frac{3}{2}}M_D & 0 \\ 0 & 0 & 0 & r_F & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & r_G & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & r_D & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & r_Q \end{bmatrix} \begin{bmatrix} i_0 \\ i_d \\ i_q \\ i_F \\ i_G \\ i_D \\ i_Q \end{bmatrix}$$

$$-\begin{bmatrix} L_0+3L_n & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & L_d & 0 & \sqrt{\frac{3}{2}}M_F & 0 & \sqrt{\frac{3}{2}}M_D & 0 \\ 0 & 0 & L_q & 0 & \sqrt{\frac{3}{2}}M_G & 0 & \sqrt{\frac{3}{2}}M_Q \\ 0 & \sqrt{\frac{3}{2}}M_F & 0 & L_F & 0 & M_R & 0 \\ 0 & 0 & \sqrt{\frac{3}{2}}M_G & 0 & L_G & 0 & M_Y \\ 0 & \sqrt{\frac{3}{2}}M_D & 0 & M_R & 0 & L_D & 0 \\ 0 & 0 & \sqrt{\frac{3}{2}}M_Q & 0 & M_Y & 0 & L_Q \end{bmatrix} \begin{bmatrix} \dot{i}_0 \\ \dot{i}_d \\ \dot{i}_q \\ \dot{i}_F \\ \dot{i}_G \\ \dot{i}_D \\ \dot{i}_Q \end{bmatrix}$$

It is of interest to rearrange the ordering of the variables so that the voltage equations for all d-axis windings are together and the voltage equations for all q-axis windings are together because this will emphasize the presence or absence of the various couplings that we have. The result of this re-ordering of the variables is as follows:

$$\begin{bmatrix} v_0 \\ \hline v_d \\ -v_F \\ v_D=0 \\ \hline v_q \\ v_G=0 \\ v_Q=0 \end{bmatrix} = -\left[\begin{array}{c|ccc|ccc} r+3r_n & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & r & 0 & 0 & \omega L_q & \omega kM_G & \omega kM_Q \\ 0 & 0 & r_F & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r_D & 0 & 0 & 0 \\ \hline 0 & -\omega L_D & -\omega kM_F & -\omega kM_D & r & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & r_G & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & r_Q \end{array}\right] \begin{bmatrix} i_0 \\ \hline i_d \\ i_F \\ i_D \\ \hline i_q \\ i_G \\ i_Q \end{bmatrix}$$

$$-\left[\begin{array}{c|ccc|ccc} L_0+3L_n & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & L_d & kM_F & kM_D & 0 & 0 & 0 \\ 0 & kM_F & L_F & M_R & 0 & 0 & 0 \\ 0 & kM_D & M_R & L_D & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & L_Q & kM_G & kM_Q \\ 0 & 0 & 0 & 0 & kM_Q & M_Y & L_Q \\ 0 & 0 & 0 & 0 & kM_G & L_G & M_Y \end{array}\right] \begin{bmatrix} \dot{i}_0 \\ \dot{i}_d \\ \dot{i}_F \\ \dot{i}_D \\ \dot{i}_q \\ \dot{i}_G \\ \dot{i}_Q \end{bmatrix}$$

(eq. 4.39')

Some observations about the transformed voltage equations:

1. The first matrix gives

a. Resistive voltage drops

b. Speed voltage drops, svd (terms with ω). These svd's

- Occur in the d- and q- circuits, to represent the fact that a flux wave rotating in synchronism with the rotor will create voltages in the stationary armature coils
- Do not occur in circuits physically located on the rotor, since there is no motion between the rotating flux wave and the rotor windings.
- Are caused by currents in the field windings of the "other" axis:

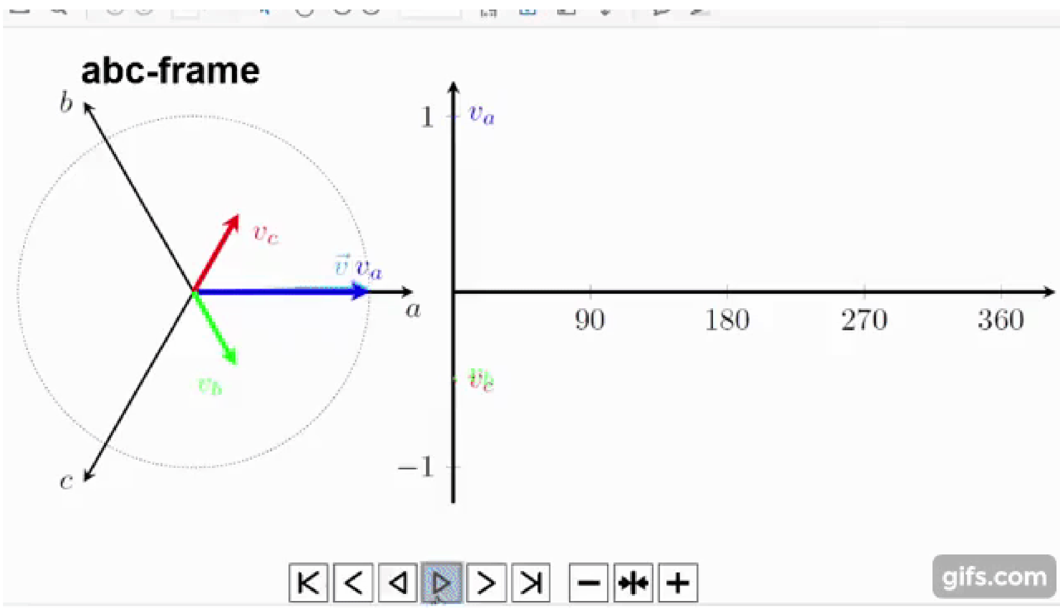
→ the d-circuit svd is caused by i_q, i_Q, and i_G
→ the q-circuit svd is caused by i_d, i_D, and i_F

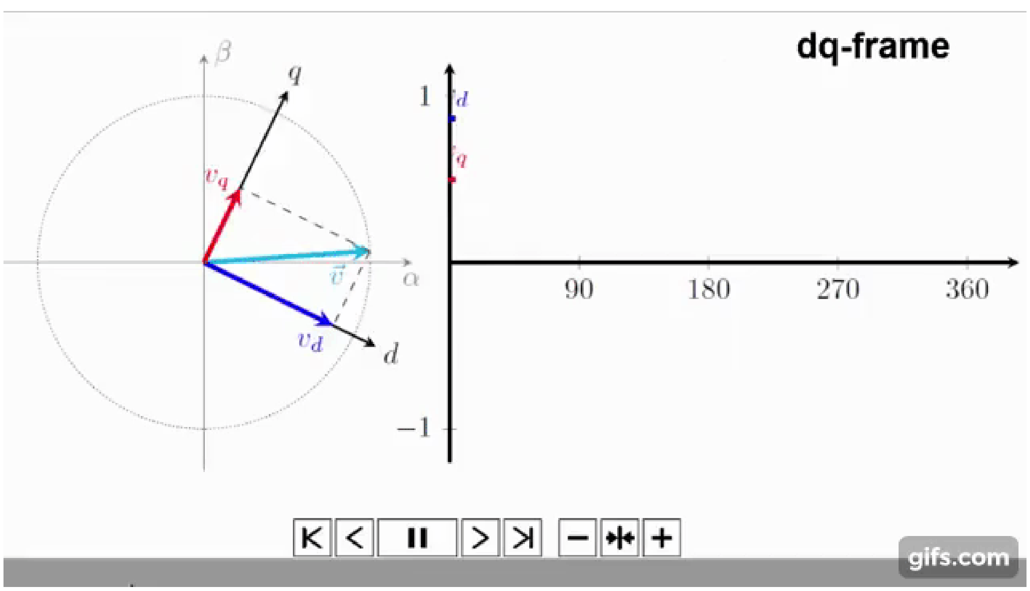
2. The matrices are *almost* constant, except for the svd terms in the first matrix, but even these terms are practically constant since we only see small changes in ω. The constancy of the matrices is the main motivation behind the Park's transformation.
3. The variables have been reorganized so that all d-axis circuits are together and all q-axis circuits are together. This makes it easy to observe any coupling/decoupling between different sets of circuits.
4. The second matrix gives voltage induced by current (or flux) variation. Note that there is no coupling between the d-axis circuits (d, F, D) and the q-axis circuits (q, G, Q). This is because these two sets of circuits are orthogonal.

Finally, some comments about Park's transformation (already made on p. 16):

1. i_d and i_q are currents in a fictitious pair of windings *fixed on the rotor*.
2. These currents produce the same flux as do the a,b,c currents.
3. For balanced steady-state operating conditions, we can use $\underline{i}_{0dq} = \underline{P}\underline{i}_{abc}$ to show that the currents in the d and q windings are dc! The implication of this is that:
    - The a,b,c currents fixed in space, varying in time, **produce the same synchronously rotating magnetic field as**
    - The d,q currents, varying in space, fixed in time!

The below offers a visual comparison of abc quantities vs d-q (fixed on rotor) quantities vs. α-β (fixed on stator) quantities. These illustrations are not animated on the pdf, and so I have also provided a PPT on the website for you to view. These animations were obtained from the excellent youtube video at https://www.youtube.com/watch?v=vdeVVTltr1M.





From Kimbark, Vol. III:

i_d is combined a,b,c MMFs (or currents) projected onto direct axis. i_q is combined a, b, c MMFs (or currents) projected onto quadrature axis. See Fig.6, p.8 of these notes for illustration of i_a projection onto direct & quadrature axes.

Physical interpretation of Park's variables. A physical interpretation of the new variables is now in order. The m.m.f. of each armature phase, being sinusoidally distributed in space, may be represented by a vector the direction of which is that of the phase axis and the magnitude of which is proportional to the instantaneous phase current. The combined m.m.f. of the three phases may likewise be represented by a vector which is the vector sum of the phase-m.m.f. vectors. The projections of the combined-m.m.f. vector on the direct and quadrature axes of the field are equal to the sums of the projections of the phase-m.m.f. vectors on the respective axes as given by the expressions for i_d and i_q, eqs. 106. The constant $\frac{2}{3}$ is arbitrary. Thus i_d may be interpreted as the instantaneous current in a fictitious armature winding which rotates at the same speed as the field winding and remains in such position that its axis always coincides with the direct axis of the field, the value of the current in this winding being such that it gives the same m.m.f. on this axis as do the actual three instantaneous armature phase currents flowing in the actual armature windings. The interpretation of i_q is similar to that of i_d except that it acts in the quadrature axis instead of in the direct axis. The i_0 of the new variables is identical with the usual zero-sequence current except that it is an instantaneous value and is defined in terms of the instantaneous phase currents. This current gives no space-fundamental air-gap flux.

Kimbark, like Park, chose 2/3 as his coefficient in front of the Park matrix, see eq. (4.5), pg. 14 of these notes.

For ss operating conditions,
• The a,b,c currents, fixed in space, varying in time, produce the same synchronously rotating magnetic field as
•The d,q currents, varying in space, fixed in time (DC).

For all operating conditions, i_d and i_q produce the same MMF on their respective axes as i_a, i_b, and i_c.

i_0 produces no air-gap flux.

The flux linkages of the fictitious armature windings in which i_d and i_q flow are ψ_d and ψ_q, respectively.

In view of the foregoing interpretation of i_d and i_q, it is apparent that their m.m.f.'s are stationary with respect to the rotor and therefore act on paths of constant permeance. Hence the corresponding inductances L_d and L_q are independent of rotor position.

We used this same argument in the notes "WindingsAxes" (pp. 10-11) to establish that the self and mutual inductances of rotor-rotor terms in L are constant. Here, Kimbark uses this argument to provide intuition that self and mutual inductances of (fictitious) d- & q- windings are constant.

The fictitious direct-axis stator winding and the field winding are inductively coupled. Each has a self-inductance (L_d and L_{ff}), and there is a mutual inductance between them. It should be noted that the mutual inductance has different values in eqs. 116*a* and *d*, being M_f in one and $\frac{3}{2}M_f$ in the other. The difference could have been avoided by a different choice of the constant coefficients in eqs. 105 and 106; however, we will retain the form of the variables given by Park.

Kimbark's choice of coefficients has the advantage of magnitude-invariance but the disadvantage of unequal mutual (see p. 13 of these notes).

Another interesting paragraph from Kimbark Vol. III

In the interpretation of eqs. 116, it was suggested that i_d and i_q were the currents in fictitious *rotating* stator windings — if that paradoxical expression may be used — which gave the same m.m.f.'s as did the actual armature currents in the actual armature windings. But it is not necessary to have the fictitious windings rotate. The same effect can be achieved by conceiving the armature winding to be stationary (as it actually is) and to be a closed-circuit winding with a commutator on which rest brushes which rotate with the field. The magnetic axis of the stator will always coincide with the brush axis. Thus i_d may be interpreted as the current entering and leaving the armature through a pair of brushes which are aligned with the direct axis of the field. Similarly, i_q may be regarded as the current entering and leaving the armature through a second pair of brushes, aligned with the quadrature axis of the field. In other words, the armature may be thought of as like that of a synchronous converter, having both commutator and slip rings, but having brushes in both axes instead of in the quadrature axis only. The actual phase currents, entering the slip rings, give the same m.m.f.'s as do the substitute currents i_d and i_q entering the commutator brushes.

This physical picture is also correct with respect to the voltages. The terms $-\omega\psi_q$ and $\omega\psi_d$ occurring in eqs. 121 and 122 may be regarded as components of applied voltage required to balance the *speed voltages*. The speed voltage across each pair of brushes is proportional to the flux on the axis 90° ahead of the brush axis.