

## Linearization of Generator Current-State Space Model

These notes parallel sections 8.3-8.4 of VMAF.

We developed a state-space current model for the synchronous machine with the G-circuit represented (see notes on per-unitization), and it was found to be:

$$\begin{bmatrix} v_d \\ -v_F \\ 0 \\ v_q \\ 0 \\ 0 \end{bmatrix} = - \begin{bmatrix} r & 0 & 0 & \omega L_q & \omega kM_G & \omega kM_Q \\ 0 & r_F & 0 & 0 & 0 & 0 \\ 0 & 0 & r_D & 0 & 0 & 0 \\ \hline -\omega L_d & -\omega kM_F & -\omega kM_D & r & 0 & 0 \\ 0 & 0 & 0 & 0 & r_G & 0 \\ 0 & 0 & 0 & 0 & 0 & r_Q \end{bmatrix} \begin{bmatrix} i_d \\ i_F \\ i_D \\ \hline i_q \\ i_G \\ i_Q \end{bmatrix} \tag{L-1}$$

$$- \begin{bmatrix} L_d & kM_F & kM_D & 0 & 0 & 0 \\ kM_F & L_F & M_R & 0 & 0 & 0 \\ kM_D & M_R & L_D & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & L_q & kM_G & kM_Q \\ 0 & 0 & 0 & kM_G & L_G & M_Y \\ 0 & 0 & 0 & kM_Q & M_Y & L_Q \end{bmatrix} \begin{bmatrix} \dot{i}_d \\ \dot{i}_F \\ \dot{i}_D \\ \hline \dot{i}_q \\ \dot{i}_G \\ \dot{i}_Q \end{bmatrix}$$

This was equation 4.74 in VMAF (in case the “dots” don’t show up, note the right-hand-side vector of (L-1) is a vector of derivatives).

The torque equation was also (see notes “TorqueEquation”):

$$\dot{\omega} = \frac{T_m}{\tau_j} + \begin{bmatrix} \frac{-L_d}{3\tau_j} i_q & \frac{-kM_F}{3\tau_j} i_q & \frac{-kM_D}{3\tau_j} i_q & \frac{L_q}{3\tau_j} i_d & \frac{kM_G}{3\tau_j} i_d & \frac{kM_Q}{3\tau_j} i_d & \frac{-D}{\tau_j} \end{bmatrix} \begin{bmatrix} i_d \\ i_F \\ i_D \\ \hline i_q \\ i_G \\ i_Q \\ \omega \end{bmatrix} \tag{L-2}$$

This is equation (4.101) in VMAF.

And finally, we had (equation (4.102) in VMAF):

$$\dot{\delta} = \omega - 1 \quad (\text{L-3})$$

Note that the state variables are  $i_d$ ,  $i_F$ ,  $i_D$ ,  $i_q$ ,  $i_G$ ,  $i_Q$ ,  $\omega$ , and  $\delta$ , i.e.,

$$\underline{x} = \begin{bmatrix} i_d \\ i_F \\ i_D \\ i_q \\ i_G \\ i_Q \\ \omega \\ \delta \end{bmatrix} \quad (\text{L-4})$$

So we have 8 state variables and 8 equations.

We want to develop a model that will allow us to investigate the stability properties of the system represented by these equations using linear system theory, because the theory of linear systems is so powerfully well-developed, as we have seen in our previous work on eigenvalues, eigenvectors, mode shape, and participation factors.

However, to employ linear system theory, the state variables must only appear in our equations in the form of linear terms; that is, state variables may only appear as a term like  $a_i x_i$  where  $a_i$  is a constant.

Our equations have some state variables appearing as product terms of the form  $x_i x_j$ . For example, note in eq. 4.74 (L-1) the appearance of  $\omega$  in the first matrix, which means that it will get multiplied by a current state variable to result in a term like  $\omega L_q i_q$ . Also, note that almost every term in the torque equation is the product of two current state variables.

So our current-state-space model is non-linear. The process of *linearization* is where we convert the equations so that they contain only linear terms. To do this, we will focus on each equation and convert each nonlinear term to a linear term.

Nothing is free. In order to linearize, we will need to select a “point” in the state-space. Then the price we will pay for the simplification is that the resulting model will only be valid for a region “close to” that selected point.

Our general strategy is based on the recognition that each of our equations in 4.74 look like

$$v = f( \underline{x}, \dot{\underline{x}} ) \quad (\text{L-5})$$

We will select our point of linearization as  $\underline{x}_0$  and then we will develop a model based on the idea that

- Only changes in the state variables may occur (e.g., no changes in parameters);
- The changes in the state variables must be small.

We will denote the changes as:

$$\underline{x} = \underline{x}_0 + \underline{\Delta x} \quad (\text{L-6})$$

Differentiating, we also have that:

$$\dot{\underline{x}} = \dot{\underline{x}}_0 + \underline{\Delta \dot{x}} \quad (\text{L-7a})$$

The term  $\dot{\underline{x}}_0$  in eq. (L-7a) should not be understood as an attempt to take a set of derivatives on a set of constants (which would be zero), but rather a set of derivatives on  $\underline{x}$  that are subsequently evaluated at  $\underline{x}_0$ . This understanding is emphasized by rewriting (L-7a) as:

$$\dot{\underline{x}} = \left. \dot{\underline{x}} \right|_{\underline{x}=\underline{x}_0} + \underline{\Delta \dot{x}} \quad (\text{L-7b})$$

The effect of changes in the state variables given by eqs. (L-6) and (L-7a) on eq. (L-5) can be expressed by substituting eqs. (L-6) and (L-7a) into eq. (L-5) and adjusting the right-hand-side of eq. (L-5) accordingly. That is:

$$v_0 + \Delta v = f( \underline{x}_0 + \underline{\Delta x}, \dot{\underline{x}}_0 + \underline{\Delta \dot{x}} ) \quad (\text{L-8})$$

Therefore, every time we see a state variable, we will replace it with  $\underline{x}_0 + \underline{\Delta x}$ , and every time we see a state-variable derivative, we will replace it with  $\dot{\underline{x}}_0 + \underline{\Delta \dot{x}}$ .

Once this is done, however, the terms corresponding to products of state variables, i.e.,  $x_1 x_2$ , will appear like

$$(x_{10} + \Delta x_1)(x_{20} + \Delta x_2) \quad (\text{L-9})$$

(Fortunately, we have no products of terms containing derivatives of state variables).

Eq. (L-9) is evaluated as:

$$(x_{10} + \Delta x_1)(x_{20} + \Delta x_2) = x_{10}x_{20} + x_{10}\Delta x_2 + x_{20}\Delta x_1 + \Delta x_1\Delta x_2 \quad (\text{L-10})$$

Here, for small deviations,  $\Delta x_1\Delta x_2$  is negligible and therefore

$$(x_{10} + \Delta x_1)(x_{20} + \Delta x_2) \approx x_{10}x_{20} + x_{10}\Delta x_2 + x_{20}\Delta x_1 \quad (\text{L-11})$$

Noting that

- the left-hand-side of eq. (L-11) gives the product term under the changed condition and
- the first term on the right-hand side gives the product term for the unchanged condition,

we see that the change in the product term is given by:

$$(x_{10} + \Delta x_1)(x_{20} + \Delta x_2) - x_{10}x_{20} = x_{10}\Delta x_2 + x_{20}\Delta x_1 \quad (\text{L-12})$$

We will repeatedly employ this strategy of substituting and then recognizing on the right-hand-side the “unchanged” expression.

Before we go through this procedure, it is worthwhile to note that the process of linearization comes about more formally through the Taylor series expansion (TSE) about a selected point.

Performing a TSE about the point  $\underline{x}_0$ , we have:

$$f(\underline{x}_0 + \Delta \underline{x}) = f(\underline{x}_0) + f'(\underline{x}_0)\Delta \underline{x} + \frac{1}{2!} f''(\underline{x}_0)\Delta \underline{x}^2 + \dots \quad (\text{L-13})$$

Neglecting the higher-order terms with more than one derivative (justified by the idea that the higher order powers of  $\Delta \underline{x}$  are very small), we have that

$$f(\underline{x}_0 + \Delta \underline{x}) \approx f(\underline{x}_0) + f'(\underline{x}_0)\Delta \underline{x}. \quad (\text{L-14})$$

**For example**, consider  $f(x_1, x_2) = x_1 x_2$ , which is just a function with a single product term (which was what (L-12) was derived to address). The TSE of  $f(x_1, x_2) = x_1 x_2$  about the point  $\underline{x}_0 = [x_{10}, x_{20}]^T$  is:

$$f(\underline{x}_0 + \Delta \underline{x}) \approx x_{10}x_{20} + \begin{bmatrix} x_{20} & x_{10} \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} = x_{10}x_{20} + x_{20}\Delta x_1 + x_{10}\Delta x_2$$

where we see that

$$\begin{aligned} & f(\underline{x}_0 + \Delta \underline{x}) - f(\underline{x}_0) \\ &= (x_{10}x_{20} + x_{20}\Delta x_1 + x_{10}\Delta x_2) - (x_{10}x_{20}) \\ &= x_{20}\Delta x_1 + x_{10}\Delta x_2 \end{aligned}$$

which is in agreement with (L-12).

The eight state equations must be linearized one at a time. We linearize one of them, for purposes of illustration. We take the first voltage equation, the one for  $v_d$ , lifted out of eq. 4.74 (or L-1 above), repeated below for convenience (done in 3 lines via 8.26 of VMAF):

$$v_d = -ri_d - \omega L_q i_q - \omega kM_G i_G - \omega kM_Q i_Q - L_d \dot{i}_d - kM_F \dot{i}_F - kM_D \dot{i}_D \quad (\text{L-15})$$

Substituting (note VMAF nomenclature is  $v_{d\Delta}$ ; we use  $\Delta v_d$  here).

$$\begin{aligned} v_d &\leftarrow v_{d0} + \Delta v_d \\ i_d &\leftarrow i_{d0} + \Delta i_d, \quad \dot{i}_d \leftarrow \dot{i}_{d0} + \Delta \dot{i}_d \\ i_q &\leftarrow i_{q0} + \Delta i_q \\ i_G &\leftarrow i_{G0} + \Delta i_G \\ i_Q &\leftarrow i_{Q0} + \Delta i_Q \\ \dot{i}_F &\leftarrow \dot{i}_{F0} + \Delta \dot{i}_F \\ \dot{i}_D &\leftarrow \dot{i}_{D0} + \Delta \dot{i}_D \end{aligned}$$

$$\omega \leftarrow \omega_0 + \Delta \omega$$

we have:

$$\begin{aligned} & v_{d0} + \Delta v_d = \\ & -r(i_{d0} + \Delta i_d) \\ & -(\omega_0 + \Delta \omega)L_q(i_{q0} + \Delta i_q) \leftarrow \\ & -(\omega_0 + \Delta \omega)kM_G(i_{G0} + \Delta i_G) \leftarrow \\ & -(\omega_0 + \Delta \omega)kM_Q(i_{Q0} + \Delta i_Q) \leftarrow \\ & -L_d(\dot{i}_{d0} + \Delta \dot{i}_d) - kM_F(\dot{i}_{F0} + \Delta \dot{i}_F) - kM_D(\dot{i}_{D0} + \Delta \dot{i}_D) \end{aligned} \quad (\text{L-16})$$

Eq. (8.26) of VMAF has error on 2<sup>nd</sup> line in that the “=” sign should not be there.

where the “product terms” are indicated by the arrow “←”.

Recall eq. (L-11) which is the linearized expression for products, repeated below for convenience:

$$(x_{10} + \Delta x_1)(x_{20} + \Delta x_2) = x_{10}x_{20} + x_{10}\Delta x_2 + x_{20}\Delta x_2$$

We apply this equation to each of the product terms in eq. (L-16).

$$\begin{aligned} v_{d0} + \Delta v_d = & \\ -r(i_{d0} + \Delta i_d) & \\ -L_q(\omega_0 i_{q0} + \omega_0 \Delta i_q + \Delta \omega i_{q0}) \leftarrow & \\ -kM_G(\omega_0 i_{G0} + \omega_0 \Delta i_G + i_{G0} \Delta \omega) \leftarrow & \quad (L-17) \\ -kM_Q(\omega_0 i_{Q0} + \omega_0 \Delta i_Q + i_{Q0} \Delta \omega) \leftarrow & \\ -L_d(\dot{i}_{d0} + \Delta \dot{i}_d) - kM_F(\dot{i}_{F0} + \Delta \dot{i}_F) - kM_D(\dot{i}_{D0} + \Delta \dot{i}_D) & \end{aligned}$$

Now separate, on the right-hand-side, initial condition terms (with zero subscript) from “deviation” terms (with  $\Delta$  in front):

$$\begin{aligned} v_{d0} + \Delta v_d = & \left[ -ri_{d0} - L_q \omega_0 i_{q0} - kM_Q \omega_0 i_{Q0} - kM_G \omega_0 i_{G0} - L_d \dot{i}_{d0} - kM_F \dot{i}_{F0} - kM_D \dot{i}_{D0} \right] \\ & + \left\{ -r\Delta i_d - L_q(\omega_0 \Delta i_q + \Delta \omega i_{q0}) - kM_Q(\omega_0 \Delta i_Q + i_{Q0} \Delta \omega) \right. \\ & \left. - kM_G(\omega_0 \Delta i_G + i_{G0} \Delta \omega) - L_d \Delta \dot{i}_d - kM_F \Delta \dot{i}_F - kM_D \Delta \dot{i}_D \right\} \end{aligned} \quad (L-18)$$

Recognizing that the term in square brackets is just  $v_{d0}$ , we subtract  $v_{d0}$  from both sides, leaving only what is inside the curly brackets:

$$\begin{aligned} \Delta v_d = & -r\Delta i_d - L_q(\omega_0 \Delta i_q + \Delta \omega i_{q0}) - kM_Q(\omega_0 \Delta i_Q + i_{Q0} \Delta \omega) \\ & - kM_G(\omega_0 \Delta i_G + i_{G0} \Delta \omega) - L_d \Delta \dot{i}_d - kM_F \Delta \dot{i}_F - kM_D \Delta \dot{i}_D \end{aligned} \quad (L-19)$$

Now combining terms in  $\Delta \omega$ , we obtain:

$$\begin{aligned} \Delta v_d & \\ = & -r\Delta i_d - L_q \omega_0 \Delta i_q - kM_Q \omega_0 \Delta i_Q \\ & - kM_G \omega_0 \Delta i_G - L_d \Delta \dot{i}_d - kM_F \Delta \dot{i}_F - kM_D \Delta \dot{i}_D \\ & - (L_q i_{q0} + kM_Q i_{Q0} + kM_G i_{G0}) \Delta \omega \end{aligned} \quad (L-20)$$

The last term in (L-20) is interesting. Let me show you why.

We have previously derived that

$$\begin{bmatrix} \lambda_0 \\ \lambda_d \\ \lambda_q \\ \lambda_F \\ \lambda_G \\ \lambda_D \\ \lambda_Q \end{bmatrix} = \begin{bmatrix} L_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & L_d & 0 & kM_F & 0 & kM_D & 0 \\ 0 & 0 & L_q & 0 & kM_G & 0 & kM_Q \\ 0 & kM_F & 0 & L_F & 0 & M_R & 0 \\ 0 & 0 & kM_G & 0 & L_G & 0 & M_Y \\ 0 & kM_D & 0 & M_R & 0 & L_D & 0 \\ 0 & 0 & kM_Q & 0 & M_Y & 0 & L_Q \end{bmatrix} \begin{bmatrix} i_0 \\ i_d \\ i_q \\ i_F \\ i_G \\ i_D \\ i_Q \end{bmatrix} \quad (4.20)$$

This is eq. 4.20 in VMAF. (We developed this equation in the notes on “mach\_eqts.” It was also used in the notes on per-unitization and elsewhere.)

From the above, we see that

$$\begin{aligned} \lambda_d &= L_d i_d + kM_F i_F + kM_D i_D \\ \lambda_q &= L_q i_q + kM_Q i_Q + kM_G i_G \end{aligned}$$

Noting the equation for  $\lambda_q$ , we see that the coefficient on  $\Delta\omega$  in eq. (L-20) is  $\lambda_{q0}$ . Therefore, eq. (L-20) becomes:

$$\begin{aligned} \Delta v_d &= -r\Delta i_d - L_q\omega_0\Delta i_q - kM_Q\omega_0\Delta i_Q \\ &\quad - kM_G\omega_0\Delta i_G - L_d\Delta \dot{i}_d - kM_F\Delta \dot{i}_F - kM_D\Delta \dot{i}_D - \lambda_{q0}\Delta\omega \end{aligned} \quad (L-21)$$

And this is our linearized equation for the  $v_d$  voltage equation.

Similar analysis applies for the other 7 state equations. Once this is done, we may form a matrix relation like that of eq. 8.36, pg. 286 in the text, and the matrices will be  $8 \times 8$ , as shown in (L-22).

$$\begin{aligned}
\underbrace{\begin{bmatrix} \Delta v_d \\ -\Delta v_F \\ 0 \\ \Delta v_q \\ \Delta v_G \\ 0 \\ \Delta T_m \\ 0 \end{bmatrix}}_{\mathbf{v}} &= - \underbrace{\begin{bmatrix} r & 0 & 0 & L_q \omega_0 & kM_G \omega_0 & kM_Q \omega_0 & \lambda_{q0} & 0 \\ 0 & r_F & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r_D & 0 & 0 & 0 & 0 & 0 \\ -L_q \omega_0 & -kM_F \omega_0 & -kM_D \omega_0 & r & 0 & 0 & -\lambda_{d0} & 0 \\ 0 & 0 & 0 & 0 & r_G & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & r_Q & 0 & 0 \\ \lambda_{q0} - L_d i_{q0} & -kM_F i_{q0} & -kM_D i_{q0} & -\lambda_{d0} + L_q i_{d0} & -kM_G i_{d0} & kM_Q i_{d0} & -D & 0 \\ \underbrace{0}_{\mathcal{I}} & \underbrace{0}_{\mathcal{I}} & \underbrace{0}_{\mathcal{I}} & \underbrace{0}_{\mathcal{I}} & \underbrace{0}_{\mathcal{I}} & \underbrace{0}_{\mathcal{I}} & -1 & 0 \end{bmatrix}}_{\mathbf{K}} \underbrace{\begin{bmatrix} \Delta i_d \\ \Delta i_F \\ \Delta i_D \\ \Delta i_q \\ \Delta i_G \\ \Delta i_Q \\ \Delta \omega \\ \Delta \delta \end{bmatrix}}_{\mathbf{x}} \\
- \underbrace{\begin{bmatrix} L_d & kM_F & kM_D & 0 & 0 & 0 & 0 & 0 \\ kM_F & L_F & M_R & 0 & 0 & 0 & 0 & 0 \\ kM_D & M_R & L_D & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & L_q & kM_G & kM_Q & 0 & 0 \\ 0 & 0 & 0 & kM_G & L_G & M_Y & 0 & 0 \\ 0 & 0 & 0 & kM_Q & M_Y & L_Q & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\tau_j & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} \Delta \dot{i}_d \\ \Delta \dot{i}_F \\ \Delta \dot{i}_D \\ \Delta \dot{i}_q \\ \Delta \dot{i}_G \\ \Delta \dot{i}_Q \\ \Delta \dot{\omega} \\ \Delta \dot{\delta} \end{bmatrix}}_{\dot{\mathbf{x}}}
\end{aligned}$$

(L-22)

The resulting matrix relation (which includes the 6 voltage equations and the 2 inertial equations) can be written more compactly as:

$$\mathbf{v} = -\mathbf{K}\mathbf{x} - \mathbf{M}\dot{\mathbf{x}} \quad (\text{eq. 8.37})$$

where the “ $\Delta$ ” on the variables  $\mathbf{v}$ ,  $\mathbf{x}$ , and  $\dot{\mathbf{x}}$  is implied, and nomenclature  $\mathbf{v}$ ,  $\mathbf{K}$ ,  $\mathbf{M}$ ,  $\mathbf{x}$ , and  $\dot{\mathbf{x}}$  are defined in (L-22).

Then we may solve for  $\dot{\mathbf{x}}$  according to:

$$\dot{\mathbf{x}} = -\mathbf{M}^{-1}\mathbf{K}\mathbf{x} - \mathbf{M}^{-1}\mathbf{v} \text{ pu} \quad (\text{eq. 8.38})$$

which is in the form of

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad (\text{eq. 8.39})$$

where

$$\mathbf{A} = -\mathbf{M}^{-1}\mathbf{K}$$

$$\mathbf{B} = -\mathbf{M}^{-1}$$

The VMAF text, p. 287, also shows that the matrix  $\mathbf{M}$  is related to the matrix we called  $\mathbf{L}$  in chapter 4, according to

$$\mathbf{M} = \begin{bmatrix} \mathbf{L} & 0 & 0 \\ 0 & -\tau_j & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

On p. 4 of our notes called “LoadEquations,” and in VMAF, Section 4.13 (p. 124), we developed the matrix  $\mathbf{L}$  as

$$\mathbf{L} = \begin{bmatrix} L_d & kM_F & kM_D & 0 & 0 & 0 \\ kM_F & L_F & M_R & 0 & 0 & 0 \\ kM_D & M_R & L_D & 0 & 0 & 0 \\ 0 & 0 & 0 & L_q & kM_G & kM_Q \\ 0 & 0 & 0 & kM_G & L_G & M_Y \\ 0 & 0 & 0 & kM_Q & M_Y & L_Q \end{bmatrix}$$

We easily see in (L-22) that  $\mathbf{L}$  is the upper left  $6 \times 6$  submatrix of  $\mathbf{M}$ .

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**Final Exam Question #4:** Derive the 7<sup>th</sup> equation in (L-22), i.e., linearize equation (L-2) (equation (4.101) in VMAF). Although this is done in a very cursory way at the top of p. 286 of VMAF, I expect you to show work which follows a procedure similar to what is done above for the  $v_d$  equation.

Note: Final Exam questions #1-#3 are found on p. 23 of the notes called “pssdamp.” Again, all final exam questions posed are to be worked untimed, with open book/notes, but without interaction with anyone else including classmates.

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### Eigenanalysis of linearized one-machine to infinite bus system:

Section 8.4 provides a linearization for a one-machine infinite bus problem, similar to what we did in the notes called “LoadEquations” corresponding to Section 4.13 of VMAF. Here, the generator is connected to an infinite bus having voltage of  $V_\infty$  through an impedance of  $R_e + jL_e$  ( $\omega=1$ ). The changes to (4.103) of VMAF necessary to model the load are again what are necessary to (L-22):

- Where you see  $r$ , replace it with  $r + R_e$  and denote it with  $\hat{R}$
- Where you see  $L_d$ , replace it with  $L_d + L_e$  and denote it with  $\hat{L}_d$
- Where you see  $L_q$ , replace it with  $L_q + L_e$  and denote it with  $\hat{L}_q$
- Where you see  $\lambda_{d0}$ , replace it with  $\hat{\lambda}_{d0} = \lambda_{d0} + L_e i_{d0}$
- Where you see  $\lambda_q$ , replace it with  $\hat{\lambda}_{q0} = \lambda_{q0} + L_e i_{q0}$
- Insert  $-K\cos(\delta_0 - \alpha)$  as element (1,8) (the  $v_d$  equation) of matrix  $\mathbf{K}$
- Insert  $-K\sin(\delta_0 - \alpha)$  as element (3,8) (the  $v_q$  equation) of matrix  $\mathbf{K}$

Note that:

- speed deviation equation contains un-hatted values for  $L_d$ ,  $L_q$ ,  $\lambda_{d0}$ , and  $\lambda_{q0}$ ;
- the last two bullets represent the Park’s transformation of a set of balanced voltages at the infinite bus, where  $K = \sqrt{3} V_\infty$  (and is **not** the same  $K$  as used in the saturation notes).
- $\alpha$  is the angle of the infinite bus phasor voltage  $V_\infty$ .

These changes to (L-22) result in

$$\begin{bmatrix} 0 \\ -v_F \\ 0 \\ \dots \\ 0 \\ 0 \\ 0 \\ T_m \\ 0 \end{bmatrix} = - \underbrace{\begin{bmatrix} \hat{R} & 0 & 0 & \omega_0 \hat{L}_q & \omega_0 kM_G & \omega_0 kM_Q & \hat{\lambda}_{q0} & -K \cos(\delta_0 - \alpha) \\ 0 & r_F & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r_D & 0 & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ -\omega_0 \hat{L}_d & -\omega_0 kM_F & -\omega_0 kM_D & \hat{R} & 0 & 0 & -\hat{\lambda}_{d0} & -K \sin(\delta_0 - \alpha) \\ 0 & 0 & 0 & 0 & r_G & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & r_Q & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{\lambda_{q0} - L_d i_{q0}}{3} & \frac{-L_{AD} i_{q0}}{3} & \frac{-L_{AD} i_{q0}}{3} & \frac{-\lambda_{d0} + L_q i_{d0}}{3} & \frac{L_{AQ} i_{d0}}{3} & \frac{L_{AQ} i_{d0}}{3} & -D & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}}_{\mathbf{K}} \underbrace{\begin{bmatrix} \hat{L}_d & kM_F & kM_D & 0 & 0 & 0 & 0 & 0 \\ kM_F & L_F & M_X & 0 & 0 & 0 & 0 & 0 \\ kM_D & M_X & L_D & 0 & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \hat{L}_q & kM_G & kM_Q & 0 & 0 \\ 0 & 0 & 0 & kM_G & L_G & M_Y & 0 & 0 \\ 0 & 0 & 0 & kM_Q & M_Y & L_Q & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & -\tau_j & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{M}} \begin{bmatrix} \dot{i}_d \\ \dot{i}_d \\ \dot{i}_d \\ \dots \\ \dot{i}_d \\ \dot{i}_d \\ \dot{i}_d \\ \dots \\ \dot{\omega} \\ \dot{\delta} \end{bmatrix}$$

Observe that in the above per-unit formulation, we have replaced the respective mutual inductances in the torque equation with  $L_{AD}$  and  $L_{AQ}$ .

Example 8.2 then computes the  $\mathbf{M}$  and  $\mathbf{K}$  matrices using the loading conditions of Example 5.3, with  $D=0$ . The  $\mathbf{M}$  matrix (which is independent of loading) is shown to be

$$\mathbf{M} = \begin{bmatrix} 2.100 & 1.550 & 1.550 & & & & & & \\ 1.550 & 1.651 & 1.550 & & \mathbf{0} & & & & \mathbf{0} \\ 1.550 & 1.550 & 1.605 & & & & & & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ & \mathbf{0} & & 2.040 & 1.490 & 1.490 & & & \\ & & & 1.490 & 1.76 & 1.490 & & & \mathbf{0} \\ & & & 1.490 & 1.490 & 1.526 & & & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ & \mathbf{0} & & & \mathbf{0} & & -1786.9 & 0 \\ & & & & & & 0 & 1 \end{bmatrix}$$

Necessary computations to obtain the  $\mathbf{K}$  matrix are shown on p. 291 of VMAF, from which the  $\mathbf{K}$  matrix is expressed as

$$\mathbf{K} = \begin{bmatrix} 0.0211 & 0 & 0 & 2.040 & 1.490 & 1.490 & 1.4304 & -1.025 \\ 0 & 0.0007 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.0131 & 0 & 0 & 0 & 0 & 0 \\ \hline -2.100 & -1.550 & -1.550 & 0.0211 & 0 & 0 & -1.0396 & -1.397 \\ 0 & 0 & 0 & 0 & 0.00584 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.0540 & 0 & 0 \\ \hline -0.014 & -0.362 & -0.362 & -1.428 & -0.790 & -0.790 & -D & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

The  $\mathbf{A}$  matrix, given by  $\mathbf{A} = -\mathbf{M}^{-1}\mathbf{K}$ , is then expressed as

$$\mathbf{A} = \begin{bmatrix} -36.080 & 0.412 & 14.177 & -3488.32 & -2547.84 & -2547.84 & -2445.93 & 1752.71 \\ 12.435 & -4.684 & 77.202 & 1202.24 & 878.103 & 878.103 & 842.98 & -604.06 \\ 22.835 & 4.125 & -96.410 & 2207.74 & 1612.517 & 1612.517 & 1548.02 & -1109.28 \\ \hline 3613.82 & 2667.34 & 2667.34 & -36.310 & 1.158 & 80.282 & 1789.02 & 2404.05 \\ -416.28 & -307.26 & -307.26 & 4.183 & -19.27 & 163.54 & -206.08 & -276.93 \\ -3122.11 & -2304.41 & -2304.41 & 31.369 & 17.68 & -273.46 & -1545.59 & -2076.95 \\ \hline -0.0078 & -0.2026 & -0.2026 & -0.7991 & -0.4421 & -0.4421 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0 & 0.0 & 1000 & 0.0 \end{bmatrix} 10^{-3}$$

Example 8.3 then computes the eigenvalues from the  $\mathbf{A}$  matrix (this can easily be done using Matlab) to obtain

$$\begin{aligned} \lambda_1 &= -0.03594 + j0.9986 & \lambda_5 &= -0.00132 + j0.03135 \\ \lambda_2 &= -0.03594 - j0.9986 & \lambda_6 &= -0.00132 - j0.03135 \\ \lambda_3 &= -0.09928 & \lambda_7 &= -0.000643 \\ \lambda_4 &= -0.28464 & \lambda_8 &= -0.007113 \end{aligned}$$

A look at the imaginary parts indicates that corresponding modes have very low frequencies, e.g., eigenvalues 1 and 2 correspond to a 0.16Hz mode, and eigenvalues 5 and 6 correspond to a 0.005Hz mode. However, we are forgetting that time has been normalized, and so these eigenvalues are expressed in  $\text{rad}/(\text{sec}/t_{\text{base}}) = (\text{rad}/\text{sec}) * t_{\text{base}}$ . To express the eigenvalues in units of  $\text{rad}/\text{sec}$ , we need to divide by  $t_{\text{base}}$ , which is equivalent to multiplying by 377.

Multiplying these frequencies by 377, we obtain 60 Hz and 1.88Hz. Some comments:

Not sure that “balance” is the right word here. Perhaps it should be “represent.”

- VMAF on p. 292 explain the 60Hz mode as being “the 60-Hz component injected into the rotor circuits to **balance** the MMF caused by the stator DC currents.” I add that we observe this mode as a result of the fact that we used the full IEEE 2.2 model where we still represent stator transients, i.e., we have NOT made the assumption in this model that stator transients are negligible.
- The 1.88 Hz mode is a little higher than the modes we were studying in the eigenanalysis of our project system, which were 0.28-0.30 Hz and 0.41-0.42 Hz. However, we looked only at these modes because they were the ones giving us trouble (they are *interarea* modes). If we would have inspected the entire list of modes that SSAT computed for us, we would have found many modes in the 1-2.5Hz range. For example, here are a few I picked out:

161	-2.3566	7.5618	1.2095	29.75	Base.SSA	Base Scenario	No fault
					Dominant State :	18 : NAVAJO 226.0 :	0 : : 1 : GENROU :
							: Angle
119	-2.4101	18.2805	2.1197	17.86	Base.SSA	Base Scenario	No fault
					Dominant State :	77 : JOHN DAY19.8 :	0 : : 1 : ESDC2A : TE :
							1
144	-2.4529	10.3648	1.6496	23.03	Base.SSA	Base Scenario	No fault
					Dominant State :	47 : OWENS G 11.5 :	0 : : 1 : GENROU :
							: Speed
206	-2.4778	2.9667	0.4722	64.10	Base.SSA	Base Scenario	No fault
					Dominant State :	140 : LITEHIPE20.0 :	0 : : 1 : ESDC2A : TE :
							1

The vertical box identifies the frequency, in Hz, of four modes, were we observe frequencies at 1.2 Hz (dominated by Navajo angle state), 2.1 Hz (dominated by an exciter state), 1.65 Hz (dominated by Owens speed state). The 2.1 and 1.65 Hz modes are referred to as “local modes.” If we looked at their participation factors, we would find significant participation from states associated with only those units.

**Point:** Interarea modes typically have frequencies 0.1-1.0Hz. Local modes typically have frequencies above 1.0Hz.

**Last comments on linearized analysis:**

- Section 8.5 of VMAF shows how to linearize the flux-linkage state-space model; it is satisfying that on p. 298 it indicates that eigenanalysis of the resulting **A**-matrix provides the same eigenvalues as we obtained using the current state-space model.

- Section 8.6, called “State Matrix for Multimachine System,” provides a very good articulation of how to build the  $\mathbf{A}$ -matrix for a multimachine system. This section is analogous to the treatment for time-domain simulation given in Chapter 7 where we saw we must express all dynamic models on a common reference frame. Section 8.6 also shows how to account for nonlinear loads such as a ZIP or exponential models (again, analogous to Chapter 7), an issue of particular interest since the nonlinear equations must be linearized. It is interesting to note at the beginning of this section that it says,  
“The basis for this procedure is the work done in reference [4] in the development of small-signal stability analysis software, which is now available as part of Powertech Labs’ DSATools package.”

Reference [4] is

[4] Kundur, P., Rogers G.J., Wong D.Y., Arabi, S., and Wang, L. Small Signal Stability Analysis Program: Version 3.1, Vol. 1 Final Report, EPRI TR-101850-V1R1, May 1994.

which is an EPRI report that came out a few years after the 1990 paper on the website.

Kundur, Rogers, Wong, Wang, and Lauby, “A comprehensive computer program package for small signal stability analysis of power systems,” IEEE Transactions on Power Systems, Vol. 5, No. 4, Nov., 1990.

Although there are several major commercial-grade time-domain simulation packages used in the industry (PSLF, PSS\E, DSAtools, and PowerWorld), there is only one commercial-grade eigenanalysis package, which is SSAT.