

## Sensitivity of the loading margin to voltage collapse with respect to arbitrary parameters

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**Abstract:** Loading margin is a fundamental measure of proximity to voltage collapse. Linear and quadratic estimates to the variation of the loading margin with respect to any system parameter or control are derived. Tests with a 118 bus system indicate that the estimates accurately predict the quantitative effect on the loading margin of altering the system loading, reactive power support, wheeling, load model parameters, line susceptance, and generator dispatch. The accuracy of the estimates over a useful range and the ease of obtaining the linear estimate suggest that this method will be of practical value in avoiding voltage collapse.

**Keywords:** voltage collapse, index, bifurcation, loading margin, control, sensitivity

## 1. Introduction

Voltage collapse is an instability of heavily loaded electric power systems characterized by monotonically decreasing voltages and blackout [1,2]. Secure operation of a power system requires appropriate planning and control actions to avoid voltage collapse. This paper describes and illustrates the use of loading margin sensitivities for the avoidance of voltage collapse.

For a particular operating point, the amount of additional load in a specific pattern of load increase that would cause a voltage collapse is called the loading margin. We are interested in how the loading margin of a power system changes as system parameters or controls are altered. *This paper shows how to compute linear and quadratic estimates to the variation of the loading margin with respect to any power system parameter or control.* The effect on the loading margin of changing the following controls and parameters is estimated:

- Emergency load shedding
- Reactive power support, shunt capacitance
- Variation in the direction of load increase
- Interarea redispatch, wheeling
- Changes to load model and load composition
- Varying line susceptance, FACTS device
- Generator redispatch

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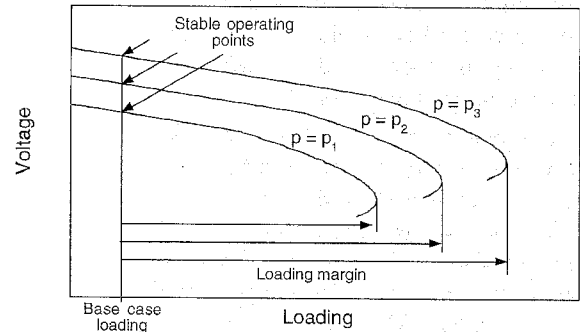


Figure 1A. Nose curves as parameter  $p$  varies

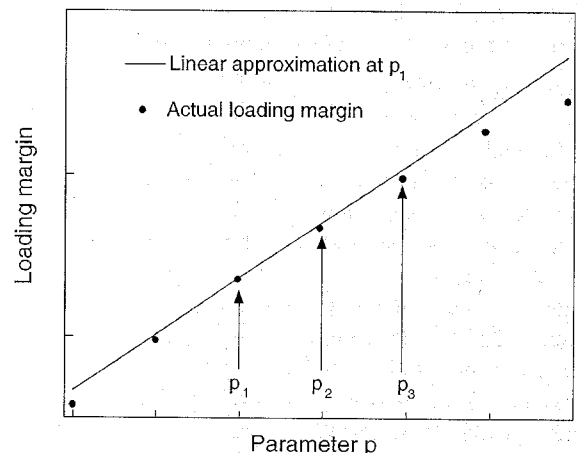


Figure 1B. Loading margin as parameter  $p$  varies

Loading margin sensitivities have a simple geometric meaning. Figure 1A shows nose curves of a large power system for three values of a power system parameter. The loading margin is the change in loading between the stable operating point and the nose of the curve corresponding to each parameter setting. (The nose corresponds to a bifurcation point of the power system when it is parameterized by loading.) As the parameter increases, the nose of the curve occurs at a higher loading and the loading margin increases. Figure 1B shows the loading margin as a function of the parameter value. Each nose curve in Figure 1A contributes one point to Figure 1B. The sensitivity of the loading margin with respect to the parameter at the nominal parameter value is given by the tangent linear approximation to the curve in Figure 1B. *The main idea of the paper is that after the loading margin has been computed for nominal parameters, the effect on the loading margin of altering the parameters can be predicted by using linear or quadratic estimates.* Exhaustively recomputing the nose for each parameter change is avoided.

Loading margin is an accurate measure of proximity to voltage collapse which takes full account of system limits and nonlinearities. (Every paper on other voltage collapse indices implicitly acknowledges the significance of loading margin by using it as the horizontal scale when the performance of the proposed index is graphed.) Moreover, loading margin estimates can be directly associated with costs, allowing for economic comparison of different strategies [4]. Methods to compute the nose and hence the loading margin are well developed [5,6,3,7,8,9]. This paper is different than these references because it assumes a loading margin computation and instead addresses the sensitivity of the loading margin.

Another approach to assessing proximity to voltage collapse uses fast time-domain simulation to predict whether the system will collapse (e.g. [10,11]). This approach has the advantage of better representing the potentially complex series of time dependent events which can influence voltage collapse. For example, the time dependence of generator reactive power limits can be represented. However, sensitivity information is difficult to obtain from time-domain simulations and requires a new simulation for each parameter variation considered. The loading margin and time-domain simulation approaches are complementary. Recent work combines aspects of both approaches [12].

There has been previous work on the sensitivity of various indices for voltage collapse. Tiranuchit and Thomas [13] computed the sensitivity of the minimum singular value of the system Jacobian, and Overbye and DeMarco [14] computed the sensitivity of an energy function index. The first order sensitivity of the loading margin was derived by Dobson and Lu [18]. This paper is an extension and application of [18].

## 2. Application to test system

The practical use of the sensitivity formulas derived in section 4 and appendix A is illustrated using a particular voltage collapse of the 118 bus IEEE standard test system [23] (see [23] for area and bus numbers and to reproduce the results). The system loading and loading margin is measured by the sum of all real load powers (an  $L^1$  norm). The stable operating point at which we test parameter variation has a total system loading of 5677 MW. Buses critical to the voltage collapse are in area two. The generator dispatch distributes the slack so that generators in each area provide additional real power roughly in proportion to their size. The loads increase proportionally from the base case loading and the voltage collapse occurs at a total load of 7443 MW and a loading margin of 1766 MW. Seven generators reach reactive power limits between the stable operating point and the voltage collapse. (Note that the reactive power limit for generator 4 is increased to avoid complications caused by an immediate instability that would have occurred just prior to the voltage collapse. An immediate instability [17] can be caused by a generator reaching a reactive power limit.)

The sensitivity formulas evaluated at the voltage collapse yield linear and quadratic estimates of the loading margin as a function of any parameter. The performance of these estimates is tested for seven different parameters representative of a range of control actions or system uncertainties. The solid lines and dotted curves in figures 2–8 are the respective linear and quadratic estimates for the loading margin variation as a function of the chosen parameter.

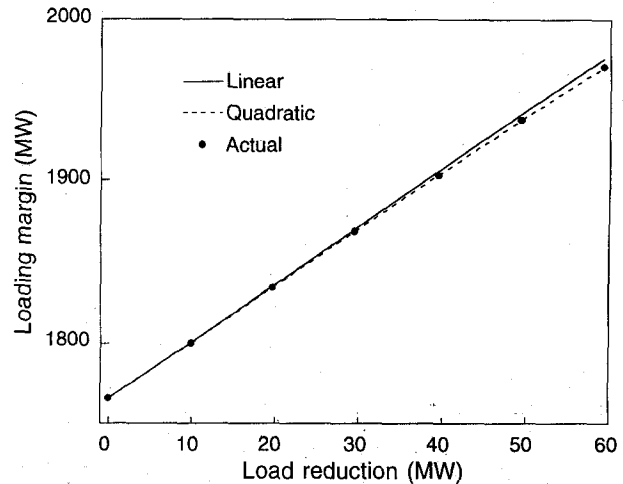


Figure 2. Effect of load shedding at bus 3

The dots in figures 2–8 represent the actual values of the loading margin as computed by combined continuation and direct methods [5]. The large dots represent the loading margin computed assuming that the reactive power limits which apply at the voltage collapse remain the same when the parameters are varied. (This assumption was used in deriving our sensitivity formulas.) The small dots represent the actual loading margin allowing different reactive power limits to apply at the voltage collapse. The small dots are computed by enforcing generator reactive power limits as the loading is increased from the stable operating point. In figures 2–6, the assumptions about limits make little difference and the large dots cover the small dots.

### Emergency load shedding:

At the stable operating point, bus 3 has a load of 60 MW and 15 MVARs and a voltage of 0.95 p.u. Fig. 2 shows the results for shedding up to 60 MW of base load at constant power factor. Each MW of load reduction increases the loading margin by 3.5 MW; and the relation remains almost linear over the entire range of load shed.

### Reactive Power Support:

The largest generator in area two is at bus 10, which is connected by a long transmission line to the high voltage side of the network. At the stable operating point, the generator at bus 10 is near its reactive power limit. Bus 9 represents the midpoint of the transmission line, and is a logical place to consider adding reactive power to alleviate the voltage collapse. Figure 3 shows that the linear estimate is accurate and quantifies the effectiveness of reactive power support at bus 9.

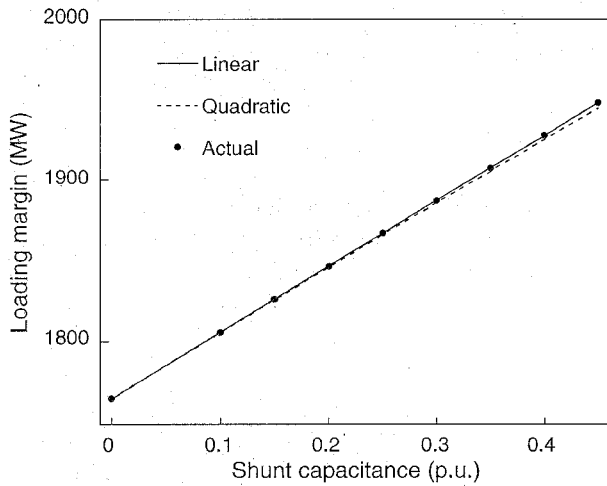


Figure 3. Effect of shunt capacitance at bus 9

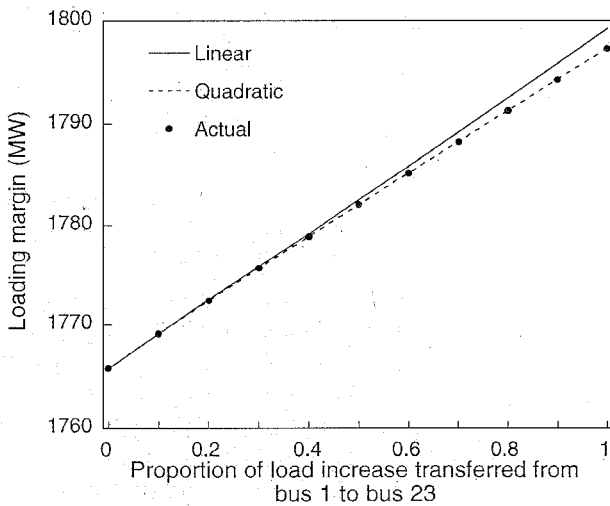


Figure 4. Effect of assumed loading direction

**Direction of load increase:**

Computing the loading margin requires a direction of load increase to be assumed. Variation in the direction of load increase can result from inaccuracies in forecasting. Thus it is useful to estimate the sensitivity of the loading margin to the direction of load increase. For this example, the direction of load increase is varied by transferring load increase from the critical bus 1 to a less critical bus in the same area, bus 23. For a particular loading factor, the total load remains the same but the proportion of load at bus 23 increases and the proportion at bus 1 decreases. Figure 4 shows that a linear estimate for the change in the loading margin performs well over the full range of variation.

**Area interchange:**

Recent trends in deregulation are expected to increase wheeling which can affect system security. The nominal interchange between the main area and area 2 is 103 MW.

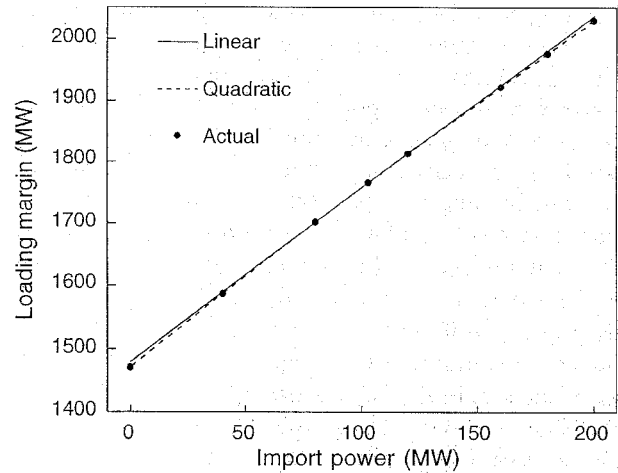


Figure 5. Effect of import into critical area

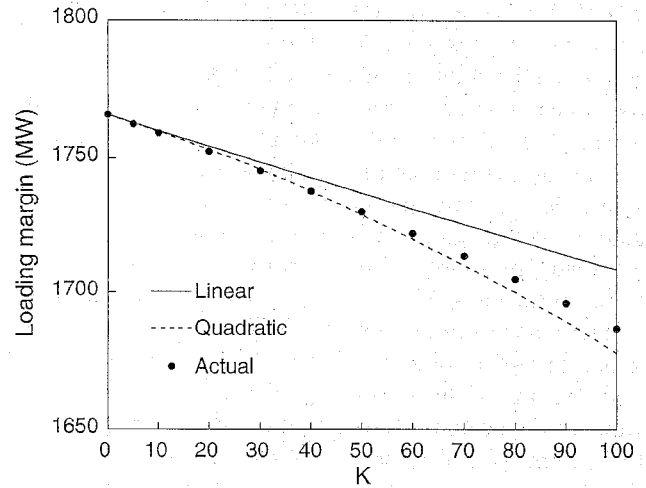


Figure 6. Effect of composition of load model

Figure 5 shows the effects on the loading margin of adjusting the flows between area 2 and the main area. Importing an additional 100 MW results in an increase in loading margin of over 200 MW, which is well predicted by the linear estimate.

**Load model:**

Load models are important in voltage collapse studies. The sensitivity of the loading margin with respect to parameters of a load model can be used to estimate the effect on the loading margin of using more detailed models. Figure 6 shows the effect on the loading margin of an additional reactive load  $Q$  at bus 3 linearly dependent upon the bus voltage  $V$  so that  $Q = KV$ .  $K$  can be interpreted as MVARs at a voltage of 1 p.u..

**Line susceptance:**

Variations in a line susceptance could represent the operation of a FACTS device or could reflect uncertainty in the network data. Figure 7 shows the effect of altering the susceptance of the line connecting bus 9 to bus 10.

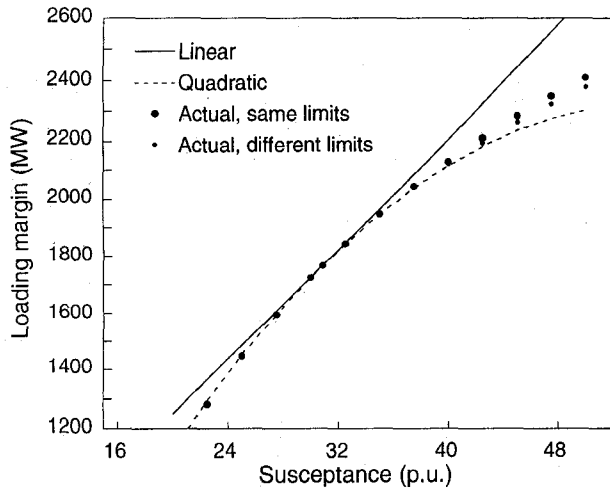


Figure 7. Effect of susceptance of line 9-10

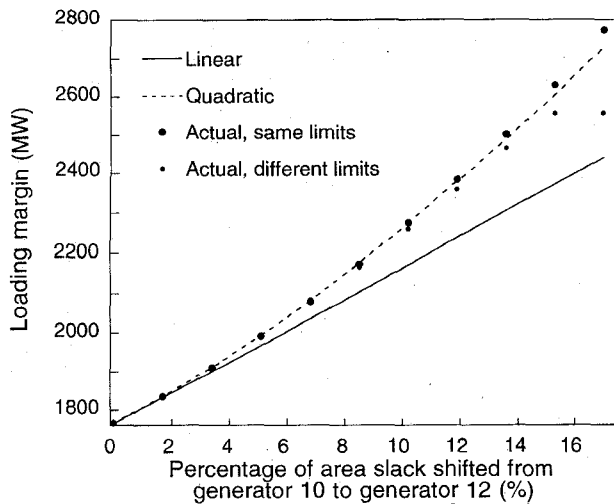


Figure 8. Effect of generator dispatch

For this example, the quadratic estimate is required to obtain accurate results over the full range of variation and the effects of changing limits are noticeable but not significant.

**Generator dispatch:** Generators 10 and 12 together assume 50% of the slack for area 2 with generator 10 alone picking up 42% for the nominal dispatch. Figure 8 shows the effect of shifting slack from generator 10 to generator 12. The two generators participate equally in the dispatch when 17% of the total area slack is moved from generator 10 to generator 12.

The quadratic estimate is accurate over a much larger range than the linear estimate. Moreover, the effects of limits can be significant. In this case, shifting more than 15% of the total area slack to generator 12 prevents generator 10 from reaching its reactive power limit; additional transfer past this point has little effect on the loading margin.

### 3. Theoretical background and assumptions

One influential theory of voltage collapse [15] models the power system as differential equations with slowly moving parameters and describes voltage collapse as the

dynamic consequence of a saddle node bifurcation. In a saddle node bifurcation, the stable operating equilibrium coalesces with an unstable equilibrium and disappears. The dynamic consequence of a generic saddle node bifurcation is a monotonic decline in system variables.

Although differential equations are the proper setting for understanding voltage collapse and are necessary for explaining why voltages dynamically decrease as a consequence of a saddle node bifurcation, it is possible and very advantageous to compute loading margins to voltage collapse and their sensitivities using static equations. Dobson [19] proves that there is no loss of accuracy in using static models in place of the underlying dynamic models when computing loading margins and their sensitivities.

The derivations and application of the sensitivity formulas require the choice of a nominal stable operating point at which parameters or controls are to be adjusted, and a projected pattern of load increase. The pattern of load increase determines the nominal bifurcation point (nose) and also defines the direction in which the loading margin is measured. *The bifurcation point should be computed by a method that takes into account system limits such as generator reactive power limits as they are encountered.* In general, the limits enforced at the bifurcation are different than those at the stable operating point. The derivation of the sensitivity formulas requires that the system equations remain the same as parameters are varied. In particular, the limits enforced at the bifurcation are assumed to stay the same as parameters are varied.

### 4. Informal derivation

This section informally derives the first order sensitivity of the loading margin  $L$  with respect to any parameter  $p$ . See the appendices for a rigorous derivation of this and the quadratic sensitivity formulas.

Suppose that the equilibria of the power system satisfy the equations

$$f(x, \lambda, p) = 0 \quad (1)$$

where  $x$  is the vector of state variables and  $\lambda$  is the vector of real and reactive load powers. Let  $\lambda_0$  be the real and reactive powers at the operating equilibrium. We specify a pattern of load increase with a unit vector  $\hat{k}$ . Then the load powers at the saddle node bifurcation causing voltage collapse are

$$\lambda = \lambda_0 + \hat{k}L \quad (2)$$

where  $L$  is the loading margin. The choice of norm is arbitrary,  $\hat{k}$  is a unit vector in whatever norm is used to measure the loading margin  $L$ . Since  $\hat{k}$  is a unit vector, it also follows that  $L = |\lambda - \lambda_0|$ .

At a saddle node bifurcation, the Jacobian matrix  $f_x$  is singular. For each  $(x, \lambda, p)$  corresponding to a bifurcation, there is a left eigenvector  $w(x, \lambda, p)$  (a row vector) corresponding to the zero eigenvalue of  $f_x$  such that

$$w(x, \lambda, p)f_x(x, \lambda, p) = 0. \quad (3)$$

The points  $(x, \lambda, p)$  satisfying (1) and (3) correspond to bifurcations and a curve of such points can be obtained by varying  $p$  about its nominal value  $p_*$ . Linearization of this curve about the bifurcation  $(x_*, \lambda_*, p_*)$  yields

$$f_x|_* \Delta x + f_\lambda|_* \Delta \lambda + f_p|_* \Delta p = 0 \quad (4)$$

where  $f_\lambda$  is the derivative of  $f$  with respect to the load powers  $\lambda$  and  $f_p$  is the derivative of  $f$  with respect to the parameter  $p$ . ' $|_*$ ' means 'evaluated at  $(x_*, \lambda_*, p_*)$ '. Premultiplication by  $w = w(x_*, \lambda_*, p_*)$  yields

$$w f_\lambda|_* \Delta \lambda + w f_p|_* \Delta p = 0 \quad (5)$$

since (3) implies that  $w f_x|_* = 0$ . Equation (5) can be interpreted as stating that  $(w f_\lambda|_*, w f_p|_*)$  is the normal vector at  $(\lambda_*, p_*)$  to the bifurcation set in a load power and parameter space [18].

Using the parameterization of  $\lambda$  by  $L$  from (2) yields  $\Delta \lambda = \hat{k} \Delta L$  and substitution in (5) gives

$$w f_\lambda \hat{k} \Delta L + w f_p \Delta p = 0 \quad (6)$$

and hence the sensitivity of the loading margin to the change in parameters is

$$L_p|_* = \frac{-w f_p|_*}{w f_\lambda|_* \hat{k}} \quad (7)$$

For the linear estimate we use (7) and

$$\Delta \lambda = L_p|_* \Delta p \quad (8)$$

The same formula holds for multiple parameters  $p$ , in which case  $w f_p|_*$  is a vector (see appendices). This is useful when approximating the combined effects of changes in several parameters or when comparing the effects of various parameters on the loading margin.

For the quadratic approximation we use (7), (A9) and

$$\Delta \lambda = L_p|_* \Delta p + \frac{1}{2} L_{pp}|_* (\Delta p)^2 \quad (9)$$

## 5. Discussion

The loading margin sensitivities only depend on quantities evaluated at the nominal bifurcation point. Evaluation of the linear sensitivity is particularly simple. Once the nominal bifurcation point is computed, the linear sensitivity (7) requires computation of the left eigenvector  $w$  and the derivative  $f_p|_*$  of the power system equations with respect to the parameter. In many cases  $f_p|_*$  has only one or two nonzero entries.  $w$  can be found by inverse power methods or as a byproduct of a direct method used to refine location of the bifurcation point [18]. Since  $w$  is the same regardless of the parameter chosen, it is very quick to compute the sensitivity to any additional parameters.

The quadratic estimate additionally requires solution of a sparse set of linear equations (A6,A8), the right eigenvector  $v$  and some second order derivatives. The second order derivatives include the matrix  $w f_{xx}|_*$ , where  $f_{xx}|_*$  is the Hessian tensor.  $w f_{xx}|_*$  can be obtained as a byproduct of a direct method that uses a Newton iteration. The other higher order derivatives are

more easily obtained and often evaluate to zero. When the quadratic term is small, it increases confidence in the accuracy of the linear estimate. When the quadratic term is not small, it serves as a more accurate estimate.

One source of inaccuracy is the neglect of higher order terms in the estimates. When the computed bifurcation is near a different bifurcation corresponding to voltage collapse of another area of the system, movement of the parameter can cause the voltage collapse to 'shift' from one area to the other. Since the set of critical parameters and loadings could have significant variations in curvature in this case, the linear and quadratic estimates would be useful only over a small parameter range.

Another source of inaccuracy is that the estimates assume a fixed set of equations whereas the form of the equations can change discretely whenever a parameter variation causes power system limits to change. The 118 bus system results are examples in which this does not significantly impair the usefulness of the estimates. However, this source of inaccuracy has the potential to be significant and requires awareness when using the estimates. Future work could address the effect of limits on loading margin sensitivities, perhaps by representing the effect of the limits using homotopy methods [21].

The loading margin and sensitivity computations require only static power system equations but accurately reflect the proximity to voltage collapse of the dynamic power system. In particular, explicit knowledge of load dynamics is not needed.

## 6. Conclusions

This paper computes linear and quadratic estimates to the variation of the loading margin with respect to any power system parameter or control. These estimates can be used to quickly assess the quantitative effectiveness of various control actions to maintain a sufficient loading margin to voltage collapse. That is, the estimates approximate the change in loading margin for a given change in each control. The estimates are also useful in determining the sensitivity of the loading margin to uncertainties in data. Estimates for any number of parameters or controls require computation of only one nose or bifurcation point.

The sensitivity formulas are rigorously derived in the appendix using bifurcation theory. The quadratic estimate is new and the derivation of the linear estimate improves on previous work in [18]. The derivation is independent of the norm chosen to measure the loading margin.

The practical use of the sensitivity computations is illustrated for a range of system parameters on a voltage collapse of the IEEE 118 bus system. The likely sources of inaccuracy discussed in section 5 include variations in the generator reactive power limits enforced at the nose. The results suggest that the linear estimate is good for many parameters and can sometimes be improved with the quadratic estimate. Direct comparison of different

control actions can be made in terms of their effect on the loading margin. The closeness of the estimates over a useful range of parameter variations and the ease of obtaining the linear estimate suggest that the sensitivity computations will be of practical value in avoiding voltage collapse.

## 7. Acknowledgments

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## Appendix A: Derivation of sensitivity formulas

Let  $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^l \rightarrow \mathbb{R}^n$  be a smooth function such that the solutions of

$$0 = f(x, \lambda, p) \quad (\text{A1})$$

are the equilibria of the power system near  $(x_*, \lambda_*, p_*)$ . We assume that  $f$  has a fold bifurcation at  $(x_*, \lambda_*, p_*)$  satisfying:

- F(a)  $f(x_*, \lambda_*, p_*) = 0$
- F(b)  $f_x|_*$  has rank  $n - 1$
- F(c)  $wf_\lambda|_* \hat{k} \neq 0$
- F(d)  $wf_{xx}|_*(v, v) \neq 0$ , where  $v$  and  $w$  are nonzero vectors satisfying  $f_x|_*v = 0$  and  $wf_x|_* = 0$ .

These are the generic conditions for a fold bifurcation [22]. (They differ slightly from the conditions for a saddle node bifurcation and the distinction between the two bifurcations is discussed in [19]. The fold bifurcation is more appropriate when working with static equations.)

Let  $\lambda_0$  be the base case loading and let the unit vector  $\hat{k} \in \mathbb{R}^m$  be a given direction in loading space. The loading is parameterized by  $\ell \in \mathbb{R}$ :

$$\lambda(\ell, \hat{k}, \lambda_0) = \lambda_0 + \ell \hat{k} \quad (\text{A2})$$

The loading may be measured with any norm, but different norms lead to different unit vectors  $\hat{k}$ . Let

$$g(x, \ell, p) = f(x, \lambda_0 + \ell \hat{k}, p) \quad (\text{A3})$$

Since  $g_x = f_x$ ,  $g_{xx} = f_{xx}$ , and  $g_\ell = f_\lambda \hat{k}$ ,  $g$  also has a fold bifurcation at  $(x_*, \ell_*, p_*)$  and the corresponding conditions F(a)-(d) are satisfied with  $g$  written for  $f$  except that F(c) becomes  $wg_\ell|_* \neq 0$ .

Appendix B proves that near  $(x_*, \ell_*, p_*)$  there is a smooth surface  $\Psi$  parameterized by  $p$  so that each point on  $\Psi$  corresponds to a fold bifurcation. Points on  $\Psi$  are of the form  $(X(p), L(p), p)$  where  $X(p)$  defines the variation of the bifurcating equilibrium with parameter  $p$  and  $L(p)$  defines the variation of the loading margin with parameter  $p$ . In the useful case of one dimensional  $p$ ,  $\Psi$  is a curve. Points of  $\Psi$  satisfy

$$g(X(p), L(p), p) = 0 \quad (\text{A4})$$

$$\mu(X(p), L(p), p) = 0 \quad (\text{A5})$$

(A4) states that  $(X(p), L(p), p)$  is an equilibrium and (A5) is the condition for bifurcation ( $\mu$  is defined in Appendix B).

Differentiation of (A4) with respect to  $p$  yields

$$g_x X_p + g_\ell L_p + g_p = 0 \quad (\text{A6})$$

Evaluation at  $(x_*, \ell_*, p_*)$  and premultiplication by  $w$  leads to  $wg_\ell|_* L_p|_* + wg_p|_* = 0$  and the desired first order result

$$L_p|_* = -\frac{wg_p|_*}{wg_\ell|_*} = -\frac{wf_p|_*}{wf_\lambda|_* \hat{k}} \quad (\text{A7})$$

The second order term  $L_{pp}|_*$  may be found as follows. Differentiation of (A5) (obtained by differentiating (B2)) and evaluation at  $(x_*, \ell_*, p_*)$  yields

$$wg_{xx}|_*(v, X_p|_*) + wg_{x\ell}|_* v L_p|_* + wg_{xp}|_* v = 0 \quad (\text{A8})$$

which, with (A6) evaluated at  $(x_*, \ell_*, p_*)$ , is a set of  $n + 1$  linear equations we may solve for  $X_p|_*$ . F(b) and F(d) imply that these  $n + 1$  equations have rank  $n$  and are uniquely solvable for  $X_p|_*$ .

Differentiation of (A6) gives

$$g_x X_{pp} + 2g_{x\ell} X_p L_p + g_{xx}(X_p, X_p) + 2g_{xp} X_p + g_{\ell\ell} L_p L_p + g_\ell L_{pp} + 2g_{\ell p} L_p + g_{pp} = 0$$

Evaluation at  $(x_*, \ell_*, p_*)$ , premultiplication by  $w$ , and solving for  $L_{pp}|_*$  gives

$$L_{pp}|_* = \frac{-1}{wg_\ell|_*} \left[ 2wg_{x\ell} X_p L_p + wg_{xx}(X_p, X_p) + 2wg_{xp} X_p + wg_{\ell\ell} L_p L_p + 2wg_{\ell p} L_p + wg_{pp} \right]_* \quad (\text{A9})$$

All terms on the right hand side are known and can easily be expressed in terms of  $f$ . If the loading  $\lambda$  appears only linearly in (A1) then (A9) simplifies to

$$L_{pp}|_* = \frac{-1}{wg_\ell|_*} \left[ wg_{xx}(X_p, X_p) + 2wg_{xp} X_p + wg_{pp} \right]_* \quad (\text{A10})$$

If, in addition, the parameters  $p$  also appear in (A1) as linear terms, then  $g_{xp} = g_{pp} = 0$  and the last two terms of the bracket in (A10) vanish.

## Appendix B: Construction of $\Psi$ .

The surface  $\Psi$  of bifurcation points is constructed as the zero section of a smooth function  $U$ . Write  $B^{ij}$  for the cofactor of the  $(i, j)$  element of a matrix  $B \in \mathbb{R}^{n \times n}$ . Since condition F(b) states that  $g_x|_*$  has rank  $n - 1$ , we can find  $i$  and  $j$  such that  $(g_x|_*)^{ij} \neq 0$ . Since the cofactors of a matrix are smooth functions of the entries of the matrix, there is a neighborhood  $S \subset \mathbb{R}^{n \times n}$  of  $g_x|_*$  such that  $B^{ij} \neq 0$  for  $B \in S$ . Define the smooth functions  $\tilde{w}(B) = (B^{1j}, B^{2j}, \dots, B^{nj})$  and  $\tilde{v}(B) = (B^{i1}, B^{i2}, \dots, B^{in})^T$ . Then

$$\tilde{w}(B)B = \det B e_j^T \quad \text{and} \quad B\tilde{v}(B) = \det B e_i \quad (\text{B1})$$

where  $e_i$  is a column vector of all zeros except that the  $i$ th position has value one.  $\tilde{w}(B)$  and  $\tilde{v}(B)$  are non-zero vectors for  $B \in S$ . Define  $w = \tilde{w}(g_x|_*)$  and  $v = \tilde{v}(g_x|_*)$ . It follows from (B1) and  $\det g_x|_* = 0$  that  $wg_x|_* = 0$  and  $g_x|_* v = 0$  so that  $w$  and  $v$  are nonzero vectors satisfying the conditions in the definition of the fold bifurcation.

Define the smooth map  $\beta : S \rightarrow \mathbb{R}$  by  $\beta(B) = \tilde{w}(B)B\tilde{v}(B)$ . It follows from (B1) that  $\beta(B) = B^{ij}\det B$ . Since  $g_x$  is smooth, there is a neighborhood  $N$  about  $(x_*, \ell_*, p_*)$  such that  $(x, \ell, p) \in N$  implies  $g_x|_{(x, \ell, p)} \in S$ . Define  $\mu : N \rightarrow \mathbb{R}$  by

$$\begin{aligned} \mu(x, \ell, p) &= \beta(g_x(x, \ell, p)) \\ &= \tilde{w}(g_x(x, \ell, p))g_x(x, \ell, p)\tilde{v}(g_x(x, \ell, p)) \quad (\text{B2}) \end{aligned}$$

and define  $U : N \rightarrow \mathbb{R}^n \times \mathbb{R}$  by  $U(x, \ell, p) = \begin{pmatrix} g(x, \ell, p) \\ \mu(x, \ell, p) \end{pmatrix}$ . Then  $\Psi$  is defined as the zero section of  $U$ :  $\Psi = U^{-1}(0)$ .

The matrix  $(U_x, U_\ell)|_* = \begin{pmatrix} g_x|_* & g_\ell|_* \\ wg_{xx}|_*v & wg_{x\ell}|_*v \end{pmatrix}$  is invertible if  $a = b = 0$  is the only solution to  $(U_x, U_\ell)|_* \begin{pmatrix} a \\ b \end{pmatrix} = 0$ , or, equivalently,

$$g_x|_*a + g_\ell|_*b = 0 \quad (\text{B3})$$

$$wg_{xx}|_*va + wg_{x\ell}|_*vb = 0 \quad (\text{B4})$$

Since F(c) implies that  $g_\ell|_*$  is not in the range of  $g_x|_*$ , to satisfy (B3),  $b = 0$  and then F(b) implies that  $a = \alpha v$  for some scalar  $\alpha$ . Then (B4) with  $b = 0$  yields  $\alpha wg_{xx}|_*(v, v) = 0$  and F(d) implies  $\alpha = 0$  and  $a = 0$ . Thus  $(U_x, U_\ell)|_*$  is invertible.

It then follows from the implicit function theorem that there is a neighborhood  $P$  of  $p_*$  and smooth functions  $X : P \rightarrow \mathbb{R}^n$  and  $L : P \rightarrow \mathbb{R}$  such that  $\{(X(p), L(p), p) \mid p \in P\} \subset \Psi$  and  $U(X(p), L(p), p) = 0$ , which can be rewritten as (A4) and (A5).

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## Discussion

**Claudio A. Cañizares** (University of Waterloo): This interesting and well written paper discusses the issue of how system parameters influence the maximum system loading, and proposes a simple method to locally predict the new location of the maximum loading points as these parameters change.

There are a couple of issues that this discussor would like to bring out, that are closely related to the discussions and results presented in the paper, and to which the authors' comments would be greatly appreciated.

1. As the authors mention and show in the paper, the proposed methods are successful in locally approximating the shape of the manifold of bifurcation points ( $\Psi$ ). However, how useful this approximation is depends on the actual shape of  $\Psi$ , which is very much contingent on the system characteristics and especially its limits, as indicated in the paper.

Figure A shows the  $\Psi$  curve for a 173 bus system [A], as the MVar rating of a SVC located at the system "critical" bus is changed. Notice the sharp change in the shape of  $\Psi$ , triggered by generator Q limits and SVC limits. In this case, the proposed methods would not be very useful around the knee of the curve, and probably a predictor-corrector approach that follows the  $\Psi$  manifold would be more adequate to determine the effect of the SVC rating on the maximum loading point. The proposed linear approximation (A7), however, could be readily used as a tangent predictor step of a continuation method in  $(\lambda, p)$  space to trace the  $\Psi$  manifold [B]. In the case of a multi-parameter systems, i.e., when  $p$  is a vector, (A7) would define a tangent hyperplane, which can then be used to define a direction of movement to trace  $\Psi$ .

2. It is important to highlight the fact that the proposed methods do not have to be dependent on a particular choice of  $\lambda$ , so that generic load models could be easily handled by the proposed techniques. These methods, however, assume a one dimensional parameterization  $\ell$  of  $\lambda$ , i.e., for all practical purposes  $m = 1$ . In some cases, such as the computation of the closest bifurcation points, one would be interested in allowing for multi-parameter loads ( $m > 1$ ). The predictor step of the GRG optimization

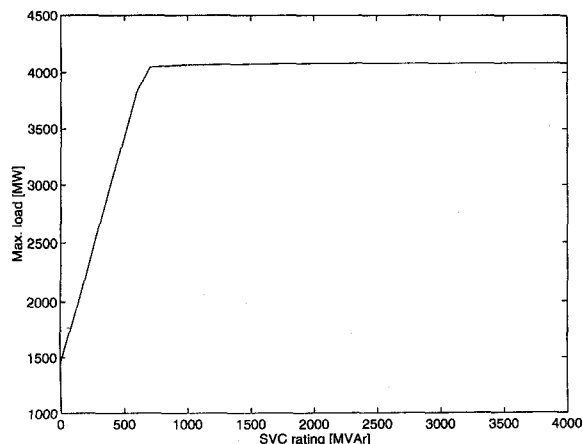


Fig. A.  $\Psi$  for SVC rating changes in a sample system.

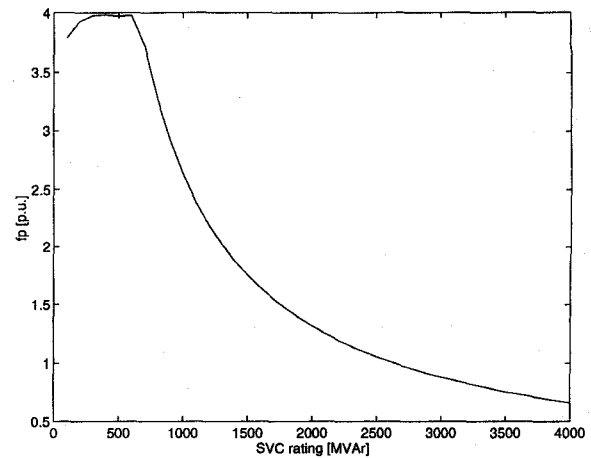


Fig. B. Optimum SVC rating in a sample system.

method can be modified to generically predict the effect of the parameters  $p$  on the manifold of closest bifurcation points [B], as this technique is designed to obtain tangent hyperplanes to a particular constraint manifold; however, this requires of some cumbersome computations. Thus, it would be very useful if the methods presented in the paper could be modified to do similar predictions.

3. Another interesting observation resulting from some of the figures in the paper, particularly Fig. 7, is that there appears to be a parameter value for which the loading margin is maximized. Determining the value of this parameter may be of interest, particularly when designing series and shunt compensation (e.g., FACTS design), as one may wish to "maximize" the distance to collapse using series and/or shunt devices [B]. The optimization, however, should not only be based on maximizing the loading margin, but how cost effective the devices are, e.g., how the rating of the device compares to changes in the loading margin [C]. This is depicted in Fig. B, where the factor  $f_p = \Delta MW_{margin}/MVAr_{SVC}$  is plotted against the SVC rating for the same 173 bus system used for Fig. A; observe that the maximum value of  $f_p$  does not correspond to the maximum loading margin. In these types of cases, determining the effect of the system parameters in an objective function other than the loading margin may be of more value.

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**T. Wu** (The University of Hong Kong), **R. Fischl**, **H. L. Chen** and **C.O. Nwankpa**, (Drexel University, Philadelphia, PA): The authors are to be congratulated for their work on predicting the loading margin due to changes in system parameters. This is particularly useful in estimating the effect of parameter uncertainties on the power transfer capability of the power network. We would like to seek the authors' clarification and comments on the following points:

1. The range of validity of the sensitivity results: It would be useful to have some quantitative measure as to the "size" of parameter variation around the base case operating the sensitivity approach is valid, since the nature of the problem is the estimation of an one-dimensional manifold (i.e., a curve that consists of the fold bifurcation points) relative to a specified point on it - a highly nonlinear problem. Specifically, we expect the range of validity to be different for changes in bus powers as compared to changes in network parameters such as line admittances and/or shunt capacitors. We have tried using the Taylor series expansion to predict the post-contingency voltage collapse point based on the pre-contingency voltage collapse point [Ref. A]. Curves similar to Fig. 7 were obtained, except that we used a larger range of susceptance values, i.e., from nominal value to zero susceptance which represents the line outage value. We found that even the quadratic estimates of the loading margins gave very poor prediction of the correct voltage collapse point. Therefore, we would be interested to know what the picture would look like if in Figure 7 the susceptance ran from 0 to 48 instead of 16 to 48. We are encouraged to see from Figs. 3 through 6, that the sensitivity method produced very accurate estimates in these cases. If the result in Fig. 5 represents a general phenomenon, where loading margin is linear with respect to import power level, it would be very helpful in the evaluation of inter-area power transfer capability. We would appreciate the authors comments on accuracy of the sensitivity method with respect to the potential applications.
2. Another observation regarding the results shown in the paper is that the sensitivity formula is used mainly for predicting the loading margin when the margin is increasing due to change of operation and/or adding equipment. Clearly, when equipment is outaged, the load margin will decrease and in effect this decrease may cause the specified operating point to be infeasible. Could the authors comment on this situation and if they have a method to predict the occurrence of this situation.
3. We would also like to know the type of load models used in the studies reported here. Since different load models produce different type of bifurcation points, it is important to clearly specify the load models.
4. It would be helpful if the authors give more details as to their statement that the accuracy of the prediction of the loading margin is poor when the desired bifurcation point is near a different bifurcation corresponding to voltage collapse of another area of the system, so that the movement of the parameter can cause the voltage collapse

to 'shift' from one area to the other. Could the authors give an example of this occurrence that they experienced. Do the authors have a criterion or method for predicting this phenomena?

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**Scott Greene, Ian Dobson, Fernando Alvarado:**  
We thank the discussers for their useful comments and suggestions.

**T. Wu, R. Fischl, H.L. Chen, C.O. Nwankpa:**

1. We agree that the main issue in applying the sensitivity formulas is the range over which the approximations remain accurate enough to be useful. We suspect that there are many useful applications for the sensitivity formulas and these will emerge as further experience is gained in particular applications. We discuss below the application to estimating the effect of line outages on the margin. We have extended the lower range of susceptance for Figure 7 as shown in Figure C1. The base case equilibrium disappears for a susceptance below 8 p.u. and this is predicted well by the quadratic estimate of the loading margin becoming negative.

In the case considered in our paper of a lossy line, reducing the susceptance to zero does not have exactly the same effect as removing the line (the line conductance remains fixed). A better way to represent a line outage is as follows: At the nominal bifurcation, the line may be removed and replaced by the real and reactive powers injected at the ends of the line into the network.

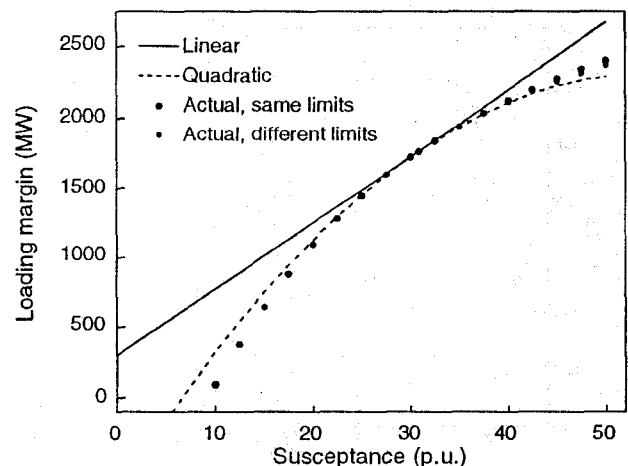


Figure C1. Effect of susceptance of line 9-10

These real and reactive powers are regarded as parameters and reducing these parameters to zero has the effect of removing the line. The sensitivity formulas can then be used to predict the effect of line outages. Some initial results using the paper's nominal voltage collapse of the 118 bus system are now described.

Four cases were chosen to sample several types of line outage. Losing the line between buses 1 and 2 disconnects two important load buses that are also independently well connected to the network. Removing the line connecting bus 9 and bus 10 disconnects the largest generator in area 2 from the network (variations in the susceptance of this line were considered above). Bus 33 connects some tie lines to the critical area through bus 15. Outaging the line connecting the generator at bus 26 to bus 30 weakens the link between generation and load in the critical area without isolating the generator. Table 1 shows the new loading margins resulting from the four outages, as well as the linear and quadratic estimates of the new loading margin.

The second column of Table 1 shows the loading margin in MW assuming that the same generator q-limits are enforced as at the nominal nose point. The third column shows the actual margin computed with the limits allowed to differ from those at the nominal nose point. The final two columns show the linear and quadratic estimates of the new loading margin computed from the sensitivity formulas. The negative margin estimates for the outage of line 9-10 are consistent with no solution for the equilibrium when the line is out. The outage of line 26-30 is a case in which the quadratic estimate is considerably better than the linear estimate and changes in the nose point limits have a noticeable effect. This initial testing seems to us promising and further work is needed.

We thank the discussers for bringing reference [A] to our attention; it is a useful reference for our paper. [A] uses a particular "loading margin" defined in import space and an optimization formulation to determine the nose point, and addresses the effect of line outages. Our paper does not address line outages but our general derivation of sensitivity formulas is valid for line outages.

The first order sensitivity formula (11) of [A] agrees with the first order sensitivity formula of our paper and reference [18] (the Lagrange multiplier  $-g_x H_x^{-1}$  of [A] is proportional to the left eigenvector  $w$  of our paper). However, the second order sensitivity formula (12) of [A] omits the first term of formula (A10) which describes the effect on the margin of the changes in the equilibrium

position due to changes in the parameter. This could explain the poor prediction in [A].

The detail of reducing our second order formula (A10) so that it can be compared with the formulas of [A] follows: The third term of (A10) vanishes in the case of line admittances and (A10) becomes

$$L_{pp}|_* = \frac{-1}{w g \ell|_*} \left[ w g_{xx}(X_p, X_p) + 2w g_{xp} X_p \right] \Big|_* \quad (C1)$$

or, in the notation of [A]:

$$\begin{aligned} \frac{d^2 r^*}{dy \lambda^2} = & -g_x H_x^{-1} F_{xx} (H_x^{-1} F_y, H_x^{-1} F_y) \\ & + 2g_x H_x^{-1} F_{xy} H_x^{-1} F_y \end{aligned} \quad (C2)$$

Equation (12) of [A] is

$$\frac{d^2 r^*}{dy \lambda^2} = g_x H_x^{-1} F_{xy} H_x^{-1} F_y \quad (12)$$

(the term of (12) is half the second term of (C2), but a factor of half is omitted from the Taylor series (10) of [A]).

2. The discussers ask about predicting the loss of an operating point when equipment is outaged. Infeasibility due to the loss of an operating point is indicated by the sensitivity formulas when they predict a negative loading margin and an example of this is discussed above.

3. The discussers ask about the type of load models used. The loads of the 118 bus system are modeled as constant power and the load increase assumes a constant power factor. An exception is the more detailed load model for bus 3 used for Figure 6, which included additional reactive load linearly dependent on the bus voltage. The intent of using this load model at bus 3 was to illustrate how the margin sensitivity to load model parameters or more detailed load models could be investigated.

The derivation of the sensitivity formulas is valid for any load model described by differential or static equations with parameters. Moreover, reference [19] proves the useful result that static load models suffice for the margin and sensitivity computations of this paper.

The discussers mention "different types of bifurcations". The paper addresses the margin to voltage collapse in which the operating point disappears in a saddle node or fold bifurcation; the paper does not address oscillatory instability via a Hopf bifurcation.

4. The discussers ask about the possibility of poor prediction accuracy when several bifurcations are nearby. This could arise when several areas of the power system are near voltage collapse. We answer this question by theory and an example.

Theory suggests how another saddle node bifurcation nearby could lead to poor prediction accuracy. Another bifurcation nearby implies there is another eigenvalue which is almost zero. It is known that close prox-

Table 1. Loading margin resulting from line outages  
(margin = 1766 MW for no outage)

line	same limits	different limits	linear estimate	quadratic estimate
1 - 2	1755	1755	1762	1760
9 - 10	no sol.	no sol.	-376	-1266
15 - 33	1770	1770	1766	1767
26 - 30	1317	1029	1502	1386

imity of the eigenvalues can cause the critical eigenvectors to be very sensitive to parameter changes. The sensitivity of the critical eigenvectors appears in formulas for the third and higher order terms in the Taylor series for  $L$  as a function of  $p$ . Large eigenvector sensitivities could cause large higher order Taylor series terms and poor prediction accuracy from the linear and quadratic estimates.

A small power system example follows: Consider a 3 bus power system consisting of an infinite bus connected by two identical transmission lines to two PQ load buses. The two PQ buses are joined by a weak transmission line of low admittance. This system is essentially two separate single line infinite bus systems which are weakly coupled. Then the bifurcation set appears as in Figure C2. Each of the flatter portions of the bifurcation set correspond to a voltage collapse of one of the PQ buses (we think of each PQ bus as a rather small area). One can conceive of varying the assumed direction of loading so that the bifurcation moves from the collapse of one PQ bus to the other PQ bus. The bifurcation would pass through the corner region in which the curvature of the bifurcation set changes significantly. We would expect the sensitivity formulas to lose accuracy when used to predict the effect of sizable changes near the corner region. Similar effects in large scale systems with many parameters are conceivable but have not yet been demonstrated to be a concern in practice. We suggest that the "corner" could be detected by another real eigenvalue being nearly zero at the bifurcation.

#### Claudio Cañizares:

1. We agree that our estimates would perform poorly near the knee of the curve of Figure A because of the way limits change and that more elaborate methods could be effective. Many of these methods use the sensitivity formulas repeatedly. For example, as Dr. Cañizares suggests, continuation methods can exploit the sensitivity formulas to help trace the curves in both the one and many parameter case. The lin-

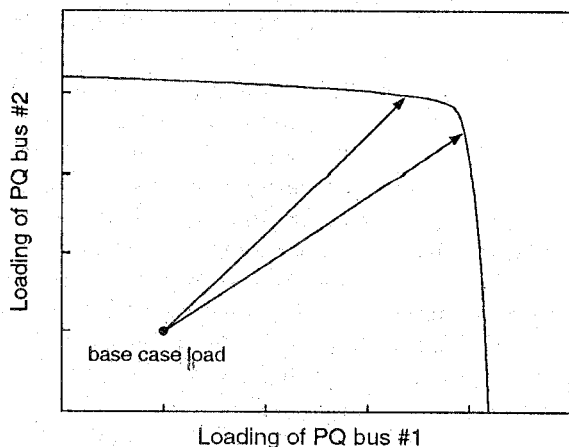


Figure C2. Bifurcation set of a 3 bus system

ear estimate would perform well over the nearly straight portions of the curve of Figure A.

2. Dr. Cañizares asks about sensitivity of the margin when we do not assume a particular direction of load increase  $\hat{k}$  but instead allow the direction of load increase  $\hat{k}$  to vary. One example was done in the paper in the case of the sensitivity of the margin to  $\hat{k}$  (see Figure 4). The example suggested by Dr. Cañizares in which  $\hat{k}$  varies with parameter  $p$  is when  $L$  is defined to be the margin to a locally closest bifurcation in loading space (minimum margin with respect to loading  $\lambda$ ). Then equation (A3) can be written as

$$g(x, \ell, p) = f(x, \lambda_0 + \ell \hat{k}(p), p) \quad (C3)$$

to emphasize the dependence of  $\hat{k}$  on parameter  $p$ . Equation (A7) becomes

$$L_p|_* = -\frac{wg_p|_* + wg_\ell|_* \hat{k}_p L}{wg_\ell|_*} = -\frac{wf_p|_* + wf_\lambda|_* \hat{k}_p L}{wf_\lambda|_* \hat{k}} \quad (C4)$$

At a locally closest bifurcation,

$$wf_\lambda|_* - |wf_\lambda|_* |\hat{k}^T = 0 \quad (C5)$$

where  $|wf_\lambda|_*$  is the norm of  $wf_\lambda|_*$ . Substituting for  $wf_\lambda|_*$  in the numerator of (C4) yields

$$L_p|_* = -\frac{wf_p|_* + |wf_\lambda|_* |\hat{k}^T \hat{k}_p L}{wf_\lambda|_* \hat{k}} \quad (C6)$$

In general, finding  $\hat{k}_p$  may require considerable calculation, but if the norm used is Euclidean, then  $\hat{k}^T \hat{k}_p = 0$  and (C5) reduces to the simple formula

$$L_p|_* = -\frac{wf_p|_*}{wf_\lambda|_* \hat{k}} \quad (C7)$$

which is the same as the first order sensitivity formula (7) of the paper. We agree with Dr. Cañizares that the first order sensitivity would be useful for a predictor step of an optimization method.

3. We agree that exploration of different margins such as those which measure cost would be useful. Our derivations are written generally to allow creativity in appropriately defining the margin when the formulas are applied. The system parameterization could also be chosen to include some parameters in dollars. The linear estimates could provide the marginal costs useful in economic analyses.

#### Correction of misprint:

The left hand sides of equations (8) and (9) of the paper should be  $\Delta L$ .

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