Torque Equation

(See sections 4.9, 4.10)

Our goal is to combine the state-space voltage equations with the state-space torque equations.

To achieve this, we need to do the following three things to the torque equation:

1. Address the difference in power bases.
2. Address the difference in speed (time) bases.
3. Express the electromagnetic torque in terms of i_d and i_q quantities instead of a-b-c quantities.

Let's take them in that order.

1. <u>Power Base (see first part of Section 4.9):</u>

Consider the electrical torque in MKS units (ntn-meters); denote it as T_e.

The electrical torque that is computed from the voltage equations will be on a per-phase base, because, as we have seen, all quantities in the voltage equations are per-unitized on a per-phase base. Let's denote this torque as $T_{e\phi u}$ (VMAF call it $T_{e\phi}$ - see eqt. 4.80 in text). Therefore,

$$T_{e\phi u} = \frac{T_e}{(S_B / \omega_B)}$$

However, the swing equation is usually written on a 3-phase power base, e.g.,

In (4.79), ω_B is used in place of ω_{Re} →

$$\frac{2H}{\omega_{Re}} \dot{\omega} = T_{mu} - T_{eu} \quad \text{in pu} \qquad (4.79)$$

Here, we have that T_{eu} is in pu on a 3-phase power base, i.e.,

$$T_{eu} = \frac{T_e}{(3S_B / \omega_B)}$$

We selected per-phase power and line-neutral voltage bases for the voltage equations because it enabled us develop voltage equations in terms of per-phase circuits.

We will continue to write the swing equation (and use it) like (4.79), because the network equations are typically given on a three-phase base. In addition, this is the convention in the literature.

But because our voltage equations have been per-unitized on a per-phase base, we need to divide the torque obtained from the voltage equations by 3 before using it in the swing equation, i.e.,

$$T_{eu} = \frac{T_{e\phi u}}{3}$$

2. Speed (time) Base (see Section 4.9.1)

In the voltage equations, both speed and time were per-unitized, so we also need to do this in the swing equation.

$$\frac{2H}{\omega_{\mathrm{Re}}}\frac{d\omega}{dt} = T_{mu} - T_{eu}$$

Let's substitute for speed and time according to $\omega=\omega_u\omega_B$ and $t=t_u t_B$, resulting in

$$\frac{2H}{\omega_{\mathrm{Re}}}\frac{d(\omega_u\omega_B)}{d(t_u t_B)} = T_{mu} - T_{eu} \rightarrow \frac{2H}{\omega_{\mathrm{Re}}}\frac{d(\omega_u\omega_B)}{d(t_u/\omega_B)} = T_{mu} - T_{eu}$$

$$\rightarrow \frac{2H}{\omega_{\mathrm{Re}}}\omega_B{}^2\frac{d(\omega_u)}{d(t_u)} = T_{mu} - T_{eu}$$

With $\omega_{Re} = \omega_B$, we have that

$$2H\omega_B\frac{d\omega_u}{dt_u} = T_{mu} - T_{eu}$$

Now define $\tau_j=2H\omega_B$, and the swing equation becomes

$$\tau_j\frac{d\omega_u}{dt_u} = T_{mu} - T_{eu} \qquad (4.82)$$

Mechanical starting time: total time required to accelerate the unit from standstill to rated speed ω_R if rated torque (T_{au}=1.0) is applied as a step function at t=0.

Aside: Recall (eq. (46) of "Swing equation" notes, and pg. 450 A&F) that the *mechanical starting time* is $T_4=2H$, therefore we see $\tau_j=T_4\omega_B=T_4/t_B$.

3. The electromagnetic torque (see Section 4.10)

Our basic approach is to obtain an expression for the electric power in terms of the 0dq quantities and then use that to obtain an expression for the electric torque in terms of the 0dq quantities.

In what follows, assume that all quantities are in pu on a per-phase base. This treatment is similar to that done in "macheqts," p. 13 to show power invariance from an orthogonal Park's transformation.

The instantaneous 3-phase power is given in terms of a-b-c quantities as

$$p_{out} = v_a i_a + v_b i_b + v_c i_c = \underline{v}_{abc}^{\,T}\, \underline{i}_{abc}$$

We want it in terms of the 0dq quantities.

Recall that $\underline{v}_{odq} = \underline{P}\underline{v}_{abc}$ and $\underline{i}_{odq} = \underline{P}\underline{i}_{abc}$, implying that $\underline{v}_{abc} = \underline{P^{-1}v}_{0dq}$ and $\underline{i}_{abc} = \underline{P^{-1}i}_{0dq}$.

But we need $\underline{v}_{abc}^T$, which will be $\underline{v}_{abc}^T = [\,\underline{P^{-1}v}_{0dq}\,]^T$.

How do we deal with the transpose of a vector product? Consider:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 17 \\ 39 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix}^T = \begin{bmatrix} x & y \end{bmatrix} = \begin{bmatrix} 5 & 6 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 17 & 39 \end{bmatrix}$$

From the above illustration, we may infer that

$$\underline{v}_{abc}^T = [\,\underline{P^{-1}v}_{0dq}\,]^T = \underline{v}_{0dq}^T [\underline{P^{-1}}]^T$$

But here, we recall that $\underline{P}$ is orthogonal. Then $[\underline{P^{-1}}]^T = \underline{P}$.

So finally, we have that $\underline{v}_{abc}^T = \underline{v}_{0dq}^T \underline{P}$. Therefore, we can substitute into the instantaneous power expression to obtain:

$$p_{out} = \underline{v}_{abc}^T \underline{i}_{abc} = [\underline{v}_{0dq}^T \underline{P}][\underline{P}^{-1} \underline{i}_{0dq}] = \underline{v}_{0dq}^T \underline{i}_{0dq}$$

The above proves that our version of Park's transformation is power invariant, i.e., the instantaneous power is obtained from either the a-b-c quantities or the 0-d-q quantities using the same form of expression, according to:

$$p_{out} = v_a i_a + v_b i_b + v_c i_c = v_0 i_0 + v_d i_d + v_q i_q$$

Aside: Observe that power invariance depends on the orthogonality of $\underline{P}$. Without an orthogonal $\underline{P}$, then $[\underline{P}^{-1}]^T \neq \underline{P}$, and

$$p_{out} = \underline{v}_{abc}^T \underline{i}_{abc} = [\underline{v}_{0dq}^T (\underline{P}^{-1})^T][\underline{P}^{-1} \underline{i}_{0dq}] \neq \underline{v}_{0dq}^T \underline{i}_{0dq}$$

We will again consider only balanced conditions so that zero-sequence quantities are zero, and

$$p_{out} = v_d i_d + v_q i_q$$

Returning to the voltage equations we had before we folded in the speed voltage terms (see p. 30 of "perunitization" notes), we can extract the expressions for v_d and v_q as:

$$v_d = -ri_d - L_d \dot{i}_d - kM_F \dot{i}_F - kM_D \dot{i}_D - \omega\lambda_q$$

$$v_q = -ri_q - L_q \dot{i}_q - kM_Q \dot{i}_Q - kM_G \dot{i}_G + \omega\lambda_d$$

Now substitute this into the expression for p_{out} to obtain:

$$p_{out} = -ri_d^2 - L_d \dot{i}_d i_d - kM_F \dot{i}_F i_d - kM_D \dot{i}_D i_d - \omega\lambda_q i_d$$

$$-ri_q^2 - L_q \dot{i}_q i_q - kM_Q \dot{i}_Q i_q - kM_G \dot{i}_G i_q + \omega\lambda_d i_q$$

Gathering together

- The derivative terms
- The ω terms
- The resistive terms

we obtain:

$$p_{out} = -i_d \overbrace{(L_d \dot{i}_d + kM_F \dot{i}_F + kM_D \dot{i}_D)}^{\dot{\lambda}_d} - i_q \overbrace{(L_q \dot{i}_q - kM_Q \dot{i}_Q - kM_G \dot{i}_G)}^{\dot{\lambda}_q}$$
$$+\omega(\lambda_d i_q - \lambda_q i_d)$$
$$-r(i_d^{\,2} + i_q^{\,2})$$

Note the expressions in parenthesis of the first line are flux linkage derivatives according to the notes in “macheqts” (see eq. 4.20).

Making the substitution indicated by the brackets above the first line of the expression,

$$p_{out} = \underbrace{-[i_d \dot{\lambda}_d + i_q \dot{\lambda}_q]}_{TERM\,1} + \underbrace{\omega(\lambda_d i_q - \lambda_q i_d)}_{TERM\,2} - \underbrace{r(i_d^{\,2} + i_q^{\,2})}_{TERM\,3} \qquad \text{eq. 4.94'}$$

Note that this is identical to eq. 4.94 in VMAF except for the minus sign in front of the term 1. I believe that this is an error in VMAF. But it does not matter, because we will not use this term anyway.

Eq. 4.20

$$\begin{bmatrix} \lambda_0 \\ \lambda_d \\ \lambda_q \\ \lambda_F \\ \lambda_G \\ \lambda_D \\ \lambda_Q \end{bmatrix} = \begin{bmatrix} L_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & L_d & 0 & kM_F & 0 & kM_D & 0 \\ 0 & 0 & L_q & 0 & kM_G & 0 & kM_Q \\ 0 & kM_F & 0 & L_F & 0 & M_R & 0 \\ 0 & 0 & kM_G & 0 & L_G & 0 & M_Y \\ 0 & kM_D & 0 & M_R & 0 & L_D & 0 \\ 0 & 0 & kM_Q & 0 & M_Y & 0 & L_Q \end{bmatrix} \begin{bmatrix} i_0 \\ i_d \\ i_q \\ i_F \\ i_G \\ i_D \\ i_Q \end{bmatrix}$$

On p. 115 of VMAF, and Charles Concordia in his book on synchronous machines (see p. 28 of “Synchronous Machines: Theory and Performance,” 1951) indicate that the three terms may be understood to represent:

- Term 1: rate of change in the stator magnetic field energy (recognizing flux linkage derivatives as voltages, $i_d \dot{\lambda}_d$ is d-axis winding power and $i_q \dot{\lambda}_q$ is q-axis winding power, where “power” here is of course reactive).
- Term 2: Power crossing the air gap (the speed-voltage terms)
- Term 3: Stator ohmic losses due to the armature resistance

Therefore, terms 1 and 3 represent power that is entirely on the stator side. But we need the power transferred from the rotor to the stator, which corresponds to the electromagnetic torque. Therefore, we are only interested in term 2.

From any text on mechanics or electromechanics (see, for example, p. 104 of Fitzgerald, Kingsley, and Kusko), we know that a body experiencing a force f over a distance ∂x undergoes a change in energy according to

$$\partial W = f\partial x$$

Analogously, a body experiencing a torque T over an angle $\partial\theta$ undergoes a change in energy according to

$$\partial W = T\partial\theta$$

For magnetically coupled coils for which at least one of them may experience rotation, the exerted electromagnetic torque is related to the variation in field energy with angular motion according to

$$T_{fld} = \frac{\partial W_{fld}}{\partial \theta_m}$$

But we may write this in terms of time derivatives according to:

$$T_{fld} = \frac{\partial W_{fld}}{\partial t}\frac{\partial t}{\partial \theta_m} = \frac{\partial W_{fld} / \partial t}{\partial \theta_m / \partial t}$$

Note that the numerator is the power and the denominator is the speed, therefore:

$$T_{fld} = \frac{\partial W_{fld} / \partial t}{\partial \theta_m / \partial t} = \frac{P_{fld}}{\omega_m}$$

Note that $T_{fld} = \dfrac{P_{fld}}{\omega_m}$ is expressed in MKS units. In per-unit, we have:

$$T_{fldu} = \frac{P_{fld} / S_B}{\omega_m / \omega_{mB}} = \frac{P_{fldu}}{\omega_e / \omega_B} = \frac{P_{fldu}}{\omega_u}$$

Here, ω_u is the same as ω in our voltage equation, eq. 4.94' above. Therefore,

$$T_{fldu} = \frac{P_{fldu}}{\omega_u} = \frac{\omega(\lambda_d i_q - \lambda_q i_d)}{\omega} = \lambda_d i_q - \lambda_q i_d$$

This is $T_{e\phi u}$, as discussed on page 1 above, and is therefore given on a per-phase base. (In Appendix A of these notes, we derive the torque expression in a different way).

Now, from "macheqts" (see p. 29), eqt. 4.20, we have:

$$\lambda_d = L_d i_d + kM_F i_F + kM_D i_D$$

$$\lambda_q = L_q i_q + kM_G i_G + kM_Q i_Q$$

Substitution of the above flux linkage relations into our torque expression yields:

$$\begin{aligned} T_{e\varphi u} = \lambda_d i_q - \lambda_q i_d &= (L_d i_q) i_d + (kM_F i_q) i_F + (kM_D i_q) i_D \\ &\quad - (L_q i_d) i_q - (kM_G i_d) i_G - (kM_Q i_d) i_Q \end{aligned}$$

The above can be written as the product of 2 vectors, according to

$$T_{e\varphi u} = \begin{bmatrix} L_d i_q & kM_F i_q & kM_D i_q & -L_q i_d & -kM_G i_d & -kM_Q i_d \end{bmatrix} \begin{bmatrix} i_d \\ i_F \\ i_D \\ i_q \\ i_G \\ i_Q \end{bmatrix} \qquad (4.98)$$

Recall the swing equation (see p. 2 above):

$$\tau_j \frac{d\omega_u}{dt_u} = T_{mu} - T_{eu} \qquad (4.82)$$

where $\tau_j = 2H\omega_B$ and the torque is given on a three-phase base.

<u>Three issues</u>:

1. As discussed before, we must divide $T_{e\phi u}$ in (4.98) by 3 to account for power base difference before using it in the above.
2. We will bring in a damping term.
3. Drop the per-unit notation, and realize that per-unit is implied throughout.

In the below equation, we have used $T_{e\varphi}$ instead of $T_{e\varphi u}$ because, as said at the bottom of p. 7, we have "dropped the pu notation."

So the swing equation becomes:

$$\tau_j \frac{d\omega}{dt} = T_m - \frac{T_{e\varphi}}{3} - T_d$$

Here, the damping term is T_d. Typically, it is written as a linear function of speed with the constant of proportionality D; thus, $T_d = D\omega$, and we have:

$$\tau_j \frac{d\omega}{dt} = T_m - T_e = T_m - \frac{T_{e\phi}}{3} - D\omega$$

We want a state-space equation so as to combine with our state-space "current-form" of the voltage equations (given by eq. 4.75, 4.76), which are

$$\underline{v} = -(\underline{R} + \omega\underline{N})\underline{i} - \underline{L}\dot{\underline{i}} \qquad \text{(eq. 4.75)}$$

$$\dot{\underline{i}} = -\underline{L}^{-1}(\underline{R} + \omega\underline{N})\underline{i} - \underline{L}^{-1}\underline{v} \qquad \text{(eq. 4.76)}$$

where each term is defined on pg. 27 of the "per-unitization" notes. So let's divide both sides of the swing equation above by τ_j.

$$\dot{\omega} = \frac{T_m}{\tau_j} + \frac{1}{3\tau_j}[-T_{e\phi}] + \left[\frac{-D}{\tau_j}\right]\omega$$

Recalling eqt. 4.98 (on previous page) for $T_{e\phi}$

$$T_{e\varphi u} = \begin{bmatrix} L_d i_q & kM_F i_q & kM_D i_q & -L_q i_d & -kM_G i_d & -kM_Q i_d \end{bmatrix} \begin{bmatrix} i_d \\ i_F \\ i_D \\ i_q \\ i_G \\ i_Q \end{bmatrix} \qquad (4.98)$$

and substituting it in into this last expression for $\dot{\omega}$, we have

$$\dot{\omega} = \frac{T_m}{\tau_j} + \begin{bmatrix} \frac{-L_d i_q}{3\tau_j} & \frac{-kM_F i_q}{3\tau_j} & \frac{-kM_D i_q}{3\tau_j} & \frac{L_q i_d}{3\tau_j} & \frac{kM_G i_d}{3\tau_j} & \frac{kM_Q i_d}{3\tau_j} \end{bmatrix} \begin{bmatrix} i_d \\ i_F \\ i_D \\ i_q \\ i_G \\ i_Q \end{bmatrix} + \left[\frac{-D}{\tau_j}\right]\omega$$

(4.101)

Now let's bring ω into the state vector….

$$\dot{\omega}=\frac{T_m}{\tau_j}+\left[\begin{array}{ccc:ccc:c}\frac{-L_d}{3\tau_j}i_q & \frac{-kM_F}{3\tau_j}i_q & \frac{-kM_D}{3\tau_j}i_q & \frac{L_q}{3\tau_j}i_d & \frac{kM_G}{3\tau_j}i_d & \frac{kM_Q}{3\tau_j}i_d & \frac{-D}{\tau_j}\end{array}\right]\left[\begin{array}{c}i_d\\ i_F\\ i_D\\ \hdashline i_q\\ i_G\\ i_Q\\ \hdashline \omega\end{array}\right]$$

Finally, we recall that there are two states for each machine: speed and angle, yet in the above, we only have speed. But we must be careful here, and use per-unit.

Recall that, on p. 93, VMAF give eq. (4.6) as follows:

$$\theta=\omega_{\text{Re}}t+\delta+\frac{\pi}{2}\ \ (4.6)\ \Rightarrow\ \dot{\theta}=\omega=\omega_{\text{Re}}+\dot{\delta}$$

Per-unitize by dividing through by $\omega_B=\omega_{Re}$, we obtain that

$$\omega_u=1+\dot{\delta}_u\ \Rightarrow\ \dot{\delta}_u=\omega_u-1$$

Dropping the per-unit notation, we have

$$\dot{\delta}=\omega-1 \tag{4.102}$$

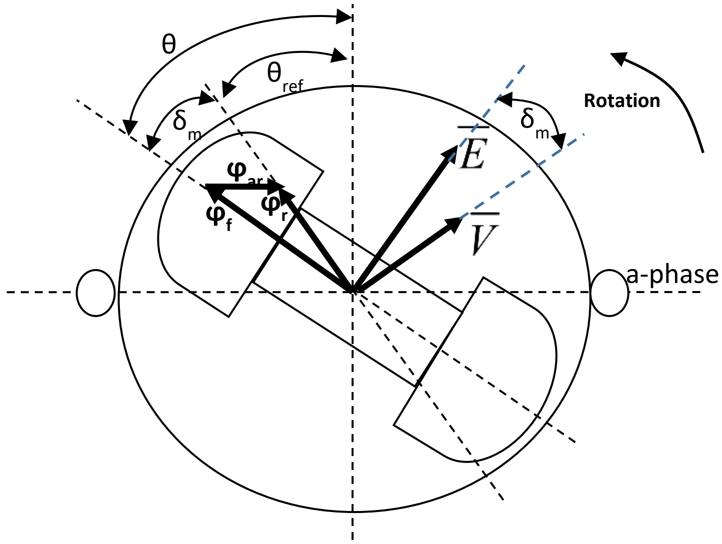
<u>**Aside for justification of (4.6)**</u>:

<u>**From p. 20, VMAF says**</u>:

"In relating the machine inertial performance to the network, it would be more useful to write (2.7) in terms of an electrical angle that can be conveniently related to the position of the rotor.

- Such an angle is the *torque angle* δ, which is the angle between the field MMF and the resultant MMF in the air gap, both rotating at synchronous speed.
- It is also the electrical angle between the generated EMF and the resultant stator voltage phasors."

In my notes (SwingEquation), I give the following picture for it, where here, I have also added the corresponding voltage vectors, $\overline{E}$ and $\overline{V}$, where, since voltage is a derivative of flux, then $\overline{E}$ lags φ_f by 90° and $\overline{V}$ lags φ_r by 90°, as shown in the (extended) figure below.

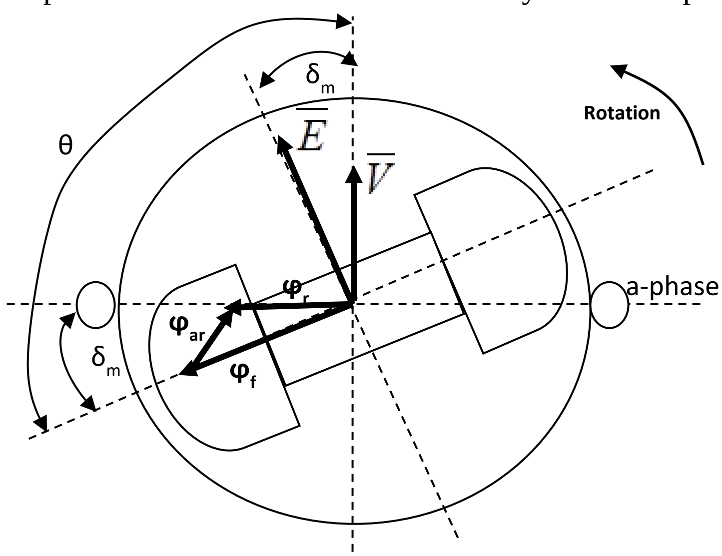


<u>But then on p. 93, VMAF says</u>:
"The main field-winding flux is along the direction of the *d* axis of the rotor. It produces an EMF that lags this flux by 90°. Therefore, the machine EMF *E* is primarily along the rotor *q* axis."

<u>COMMENT</u>: The figure above shows the EMF $\overline{E}$ that is "along the rotor *q* axis".

"Consider a machine having a constant terminal voltage *V*. For generator action the phasor $\overline{E}$ should be leading the phasor $\overline{V}$. The angle between $\overline{E}$ and $\overline{V}$ is the machine torque angle δ if the phasor $\overline{V}$ is in the direction of the reference phase (phase a)."

<u>COMMENT</u>: I have redrawn below the above figure so that it is consistent with this description. One observes that I have merely rotated all phasors an equal angle CCW.



$$\theta = \omega_{\text{Re}} t + \delta + \frac{\pi}{2}$$

"At $t=0$ the phasor $\bar{V}$ is located at the axis of phase *a*, i.e.,. at the reference axis in Figure 4.1. The *q* axis is located at an angle δ, and the *d* axis is located at $\theta=\delta+\pi/2$.
Comment: One observes that all of the above statements are accurate with respect to the above figure:

- $\bar{V}$ is located at the axis of phase *a*
- The *q* axis is located at an angle δ
- the *d* axis is located at $\theta=\delta+\pi/2$

"At $t>0$, the reference axis is located at an angle $\omega_R t$ with respect to the axis of phase *a*. The *d* axis of the rotor is therefore located at
$\theta=\omega_R t+\delta+\pi/2$ (4.6)"
Comment: As time increases, all phasors simply rotate CCW, adding $\omega_R t$ to θ(t=0).

Now we have three different sets of state equations, summarized as follows:

$$\dot{\underline{i}} = -\underline{L}^{-1}(\underline{R}+\omega\underline{N})\underline{i} - \underline{L}^{-1}\underline{v} \qquad \text{(eq. 4.75)}$$

$$\dot{\omega} = \frac{T_m}{\tau_j} + \begin{bmatrix} \frac{-L_d i_q}{3\tau_j} & \frac{-kM_F i_q}{3\tau_j} & \frac{-kM_D i_q}{3\tau_j} & \frac{L_q i_d}{3\tau_j} & \frac{kM_G i_d}{3\tau_j} & \frac{kM_Q i_d}{3\tau_j} & \frac{-D}{\tau_j} \end{bmatrix} \begin{bmatrix} i_d \\ i_F \\ i_D \\ i_q \\ i_G \\ i_Q \\ \omega \end{bmatrix}$$

$$\dot{\delta} = \omega - 1 \qquad (4.102)$$

And we can put all of this together into a single state equation that looks like the following:

$$\begin{bmatrix} \dot{i}_d \\ \dot{i}_F \\ \dot{i}_D \\ \dot{i}_q \\ \dot{i}_G \\ \dot{i}_Q \\ \dot{\omega} \\ \dot{\delta} \end{bmatrix} = \left[\begin{array}{cccccc:cc} \multicolumn{6}{c:}{-\mathbf{L}^{-1}(\mathbf{R}+\omega\mathbf{N})} & \multicolumn{2}{c}{\mathbf{0}} \\ \hdashline \frac{-L_d i_q}{3\tau_j} & \frac{-kM_F i_q}{3\tau_j} & \frac{-kM_D i_q}{3\tau_j} & \frac{L_q i_d}{3\tau_j} & \frac{kM_G i_d}{3\tau_j} & \frac{kM_Q i_d}{3\tau_j} & \frac{-D}{\tau_j} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \begin{bmatrix} i_d \\ i_F \\ i_D \\ i_q \\ i_G \\ i_Q \\ \omega \\ \delta \end{bmatrix} + \begin{bmatrix} -\mathbf{L}^{-1}\mathbf{v} \\ \hdashline \frac{T_m}{\tau_j} \\ -1 \end{bmatrix}$$

The above relation is called the "**current state-space model**." Observe that it is in the form of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t)$, i.e., we have found **f**. Some comments about **f**:

- The torque equation makes it a nonlinear function.
- The second term (the one with $-\mathbf{L}^{-1}\mathbf{v}$, T_m/τ_j, and -1) contain the system "inputs," which are the voltages in **v** (this would be v_d, v_q, and v_F) and the mechanical torque T_m. Here, we write "inputs" in quotations to account for the fact that the voltages v_d and v_q are actually determined by the network and so are considered inputs only in this model (which does not represent the network).
- The order of the model is 8.
- We develop a very crude network model next, by jumping to Sec. 4.13 (we have been in Sec. 4.10) called "Load equations."

Appendix A: Alternative Derivation of Torque for Synchronous Machine

Note: This appendix is an unfinished work. But I leave it to identify the source of another way to derive the torque equation.

The electromagnetic torque of the DFIG may be evaluated according to

$$T_{em} = \frac{\partial W_c}{\partial \Theta_m} \tag{A1}$$

where W_c is the co-energy of the coupling fields associated with the various windings; Θ_m is the angle in mechanical degrees between the main rotor axis and fixed reference.

We are not considering saturation here, assuming the flux-current relations are linear, in which case the co-energy W_c of the coupling field equals its energy, W_f, so that:

$$T_{em} = \frac{\partial W_f}{\partial \Theta_m} \tag{A2}$$

We use electric rad/sec by substituting $\Theta_m=\theta_m/p$ where p is the number of pole pairs.

$$T_{em} = p\frac{\partial W_f}{\partial \theta_m} \tag{A3}$$

For a linear electromagnetic system with J electrical inputs (windings), the total field energy is given by:

$$W_f = \frac{1}{2}\sum_{p=1}^{J}\sum_{q=1}^{J} L_{pq} i_p i_q \tag{A4}$$

where L_{pq} is the winding's self inductance when p=q and when p≠q, it is the mutual inductance between the two windings[1]. Thus, the stored energy is the sum of

- The self inductances (less leakage[2]) of each winding times one-half the square of its current and
- All mutual inductances, each times the currents in the two windings coupled by the mutual inductance

[1] Derivation is given in pp 22-24 of Krause, Wasynchzuk, and Sudhoff, "Analysis of electric machinery," 1995.

[2] * See pg. 178 of Krause, 1995.

Observe that the energy stored in the leakage inductances is not a part of the energy stored in the coupling field[2].

Now let's apply (A4) to the synchronous machine. We must account for the stored energy associated with the stator windings alone, the stator-rotor windings, and the rotor windings alone. To accomplish this, let's first describe the needed matrices.

$$\begin{bmatrix} \underline{\lambda}_{abc} \\ \underline{\lambda}_{FDQG} \end{bmatrix} = [\underline{L}] \begin{bmatrix} \underline{i}_{abc} \\ \underline{i}_{FDQG} \end{bmatrix} = \begin{bmatrix} \underline{L}_{aa} & \underline{L}_{aR} \\ \underline{L}_{Ra} & \underline{L}_{RR} \end{bmatrix} \begin{bmatrix} \underline{i}_{abc} \\ \underline{i}_{FDQG} \end{bmatrix}$$

where

$$\begin{bmatrix} \lambda_a \\ \lambda_b \\ \lambda_c \\ \lambda_F \\ \lambda_D \\ \lambda_Q \\ \lambda_G \end{bmatrix} = \begin{bmatrix} L_S + L_m\cos 2\theta & -[M_S + L_m\cos 2(\theta+30°)] & -[M_S + L_m\cos 2(\theta+150°)] & M_F\cos\theta & M_D\cos\theta & M_Q\sin\theta & M_G\sin\theta \\ -[M_S + L_m\cos 2(\theta+30°)] & L_S + L_m\cos 2(\theta-120°) & -[M_S + L_m\cos 2(\theta-90°)] & M_F\cos(\theta-120°) & M_D\cos(\theta-120°) & M_Q\sin(\theta-120°) & M_G\sin(\theta-120°) \\ -[M_S + L_m\cos 2(\theta+150°) & -[M_S + L_m\cos 2(\theta-90°)] & L_S + L_m\cos 2(\theta-240°) & M_F\cos(\theta-240°) & M_D\cos(\theta-240°) & M_Q\sin(\theta-240°) & M_G\sin(\theta-240°) \\ M_F\cos\theta & M_F\cos(\theta-120°) & M_F\cos(\theta-240°) & L_F & M_R & 0 & 0 \\ M_D\cos\theta & M_D\cos(\theta-120°) & M_D\cos(\theta-240°) & M_R & L_D & 0 & 0 \\ M_Q\sin\theta & M_Q\sin(\theta-120°) & M_Q\sin(\theta-240°) & 0 & 0 & L_Q & M_Y \\ M_G\sin\theta & M_G\sin(\theta-120°) & M_G\sin(\theta-240°) & 0 & 0 & M_Y & L_G \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_c \\ i_F \\ i_D \\ i_Q \\ i_G \end{bmatrix}$$

$$\underline{L}_{aa} = \begin{bmatrix} L_S + L_m\cos 2\theta & -[M_S + L_m\cos 2(\theta+30°)] & -[M_S + L_m\cos 2(\theta+150°)] \\ -[M_S + L_m\cos 2(\theta+30°)] & L_S + L_m\cos 2(\theta-120°) & -[M_S + L_m\cos 2(\theta-90°)] \\ -[M_S + L_m\cos 2(\theta+150°) & -[M_S + L_m\cos 2(\theta-90°)] & L_S + L_m\cos 2(\theta-240°) \end{bmatrix}$$

$$\underline{L}_{aR} = \begin{bmatrix} M_F\cos\theta & M_D\cos\theta & M_Q\sin\theta & M_G\sin\theta \\ M_F\cos(\theta-120°) & M_D\cos(\theta-120°) & M_Q\sin(\theta-120°) & M_G\sin(\theta-120°) \\ M_F\cos(\theta-240°) & M_D\cos(\theta-240°) & M_Q\sin(\theta-240°) & M_G\sin(\theta-240°) \end{bmatrix}$$

$$\underline{L}_{Ra} = \begin{bmatrix} M_F\cos\theta & M_F\cos(\theta-120°) & M_F\cos(\theta-240°) \\ M_D\cos\theta & M_D\cos(\theta-120°) & M_D\cos(\theta-240°) \\ M_Q\sin\theta & M_Q\sin(\theta-120°) & M_Q\sin(\theta-240°) \\ M_G\sin\theta & M_G\sin(\theta-120°) & M_G\sin(\theta-240°) \end{bmatrix}$$

$$\underline{L}_{RR} = \begin{bmatrix} L_F & M_R & 0 & 0 \\ M_R & L_D & 0 & 0 \\ 0 & 0 & L_Q & M_Y \\ 0 & 0 & M_Y & L_G \end{bmatrix}$$

Recalling that the leakage inductance does not contribute to the energy stored in the coupling field, we must subtract off the leakage inductance from the diagonals of $\underline{L}_{aa}$ and $\underline{L}_{RR}$ ($\underline{L}_{aR}$ and L_{Ra} contain only mutual inductances and so have no leakage inductance in them). And so the modified matrices are

$$\underline{L}_{aa}-\underline{L}_{s\ell}=\begin{bmatrix} L_S+L_m\cos 2\theta & -[M_S+L_m\cos 2(\theta+30°)] & -[M_S+L_m\cos 2(\theta+150°)] \\ -[M_S+L_m\cos 2(\theta+30°)] & L_S+L_m\cos 2(\theta-120°) & -[M_S+L_m\cos 2(\theta-90°)] \\ -[M_S+L_m\cos 2(\theta+150°) & -[M_S+L_m\cos 2(\theta-90°)] & L_S+L_m\cos 2(\theta-240°) \end{bmatrix} - L_{s\ell}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$=\underline{L}_{aa}-L_{s\ell}\underline{U}_3=\begin{bmatrix} L_S-L_{s\ell}+L_m\cos 2\theta & -[M_S+L_m\cos 2(\theta+30°)] & -[M_S+L_m\cos 2(\theta+150°)] \\ -[M_S+L_m\cos 2(\theta+30°)] & L_S-L_{s\ell}+L_m\cos 2(\theta-120°) & -[M_S+L_m\cos 2(\theta-90°)] \\ -[M_S+L_m\cos 2(\theta+150°) & -[M_S+L_m\cos 2(\theta-90°)] & L_S-L_{s\ell}+L_m\cos 2(\theta-240°) \end{bmatrix}$$

$$\underline{L}_{RR}-\underline{L}_{FDQG\ell}=\begin{bmatrix} L_F-L_{F\ell} & M_R & 0 & 0 \\ M_R & L_D-L_{D\ell} & 0 & 0 \\ 0 & 0 & L_Q-L_{Q\ell} & M_Y \\ 0 & 0 & M_Y & L_G-L_{G\ell} \end{bmatrix} \triangleq \underline{L}'_{RR}$$

Comment: I think the diagonal elements of $\underline{L}'_{RR}$ are identical. Need to check this. Though ultimately, I think it does not matter, because $\underline{L}'_{RR}$ is independent of angle.

This results in

$$W_f=\frac{1}{2}\sum_{p=1}^{J}\sum_{q=1}^{J}L_{pq}i_p i_q=\frac{1}{2}\mathbf{i}_{abc}^T\left(\mathbf{L}_{aa}-\mathbf{L}_{s\ell}\mathbf{U}_3\right)\mathbf{i}_{abc}-\mathbf{i}_{abc}^T\mathbf{L}_{aR}\mathbf{i}_{FDQG}+\frac{1}{2}\frac{3}{2}\mathbf{i}_{FDQG}^T\mathbf{L}'_{RR}\mathbf{i}_{FDQG} \quad \text{(A5)}$$

Equation (A5) is adapted from Krause, p. 217, equation (5.3-1). I have three comments about it:

1. There is no 1/2 in the middle term because that term actually incorporates both the $\underline{L}_{aR}$ contribution as well as the $\underline{L}_{Ra}$ contribution.
2. Why does the 3/2 show up in the last term? Ultimately, this does not matter because this term is independent of angle.
3. Why is the middle term negated? This will ultimately matter because the middle term is indeed dependent on angle. Krause says on pg. 218 of his 1995 edition, "…the second entry of Table 1.3-1 may be used with the factor P/2 included to account for a P-pole machine (Sec 4.3) and a negative sign to make T_e positive for generator action."

Accounting for the above comments, the torque is given by

$$T_e = \frac{\partial W_f}{\partial \theta} = \frac{p}{2}\frac{\partial}{\partial \theta}\left\{-\frac{1}{2}\mathbf{i}_{abc}^T\left(\mathbf{L}_{aa} - \mathbf{L}_{s\ell}\mathbf{U}_3\right)\mathbf{i}_{abc} + \mathbf{i}_{abc}^T\mathbf{L}_{aR}\mathbf{i}_{FDQG} - \frac{1}{2}\frac{3}{2}\mathbf{i}_{FDQG}^T\mathbf{L}_{RR}'\mathbf{i}_{FDQG}\right\} \tag{A-6}$$

And because the third term is independent of angle, we obtain:

$$T_e = \frac{p}{2}\left\{-\frac{1}{2}\frac{\partial}{\partial \theta}\left[\mathbf{i}_{abc}^T\left(\mathbf{L}_{aa} - \mathbf{L}_{s\ell}\mathbf{U}_3\right)\mathbf{i}_{abc}\right] + \frac{\partial}{\partial \theta}\left[\mathbf{i}_{abc}^T\mathbf{L}_{aR}\mathbf{i}_{FDQG}\right]\right\} \tag{A-7}$$

Now we apply the transformations

$$\mathbf{i}_{abc} = \mathbf{P}^{-1}\mathbf{i}_{0dq} \Rightarrow \mathbf{i}_{abc}^T = \left(\mathbf{P}^{-1}\mathbf{i}_{0dq}\right)^T = \mathbf{i}_{0dq}^T\mathbf{P} \tag{A-8}$$

This results in

$$T_e = \frac{p}{2}\left\{\underbrace{-\frac{1}{2}\frac{\partial}{\partial \theta}\left[\mathbf{i}_{0dq}^T\mathbf{P}\left(\mathbf{L}_{aa} - \mathbf{L}_{s\ell}\mathbf{U}_3\right)\mathbf{P}^{-1}\mathbf{i}_{0dq}\right]}_{TERM1} + \underbrace{\frac{\partial}{\partial \theta}\left[\mathbf{i}_{0dq}^T\mathbf{P}\mathbf{L}_{aR}\mathbf{i}_{FDQG}\right]}_{TERM2}\right\} \tag{A-8}$$

We will work on one term at a time.

TERM 1:

$$T_e = -\frac{1}{2}\frac{\partial}{\partial \theta}\left[\mathbf{i}_{0dq}^T\mathbf{P}\left(\mathbf{L}_{aa} - \mathbf{L}_{s\ell}\mathbf{U}_3\right)\mathbf{P}^{-1}\mathbf{i}_{0dq}\right]$$

$$= -\frac{1}{2}\frac{\partial}{\partial \theta}\left[\begin{bmatrix} i_0 & i_d & i_q \end{bmatrix}\sqrt{\frac{2}{3}}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ \cos\theta & \cos\left(\theta - 2\pi/3\right) & \cos\left(\theta + 2\pi/3\right) \\ \sin\theta & \sin\left(\theta - 2\pi/3\right) & \sin\left(\theta + 2\pi/3\right) \end{bmatrix}\left(\mathbf{L}_{aa} - \mathbf{L}_{s\ell}\mathbf{U}_3\right)\sqrt{\frac{2}{3}}\begin{bmatrix} 1/\sqrt{2} & \cos\theta & \sin\theta \\ 1/\sqrt{2} & \cos\left(\theta - 2\pi/3\right) & \sin\left(\theta - 2\pi/3\right) \\ 1/\sqrt{2} & \cos\left(\theta + 2\pi/3\right) & \sin\left(\theta + 2\pi/3\right) \end{bmatrix}\begin{bmatrix} i_0 \\ i_d \\ i_q \end{bmatrix}\right]$$

$$= -\frac{1}{2}\frac{2}{3}\frac{\partial}{\partial \theta}\left[\begin{bmatrix} i_0 & i_d & i_q \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ \cos\theta & \cos\left(\theta - 2\pi/3\right) & \cos\left(\theta + 2\pi/3\right) \\ \sin\theta & \sin\left(\theta - 2\pi/3\right) & \sin\left(\theta + 2\pi/3\right) \end{bmatrix}\right.$$

$$\left.\times\begin{bmatrix} L_S - L_{s\ell} + L_m\cos 2\theta & -[M_S + L_m\cos 2(\theta + 30°)] & -[M_S + L_m\cos 2(\theta + 150°)] \\ -[M_S + L_m\cos 2(\theta + 30°)] & L_S - L_{s\ell} + L_m\cos 2(\theta - 120°) & -[M_S + L_m\cos 2(\theta - 90°)] \\ -[M_S + L_m\cos 2(\theta + 150°) & -[M_S + L_m\cos 2(\theta - 90°)] & L_S - L_{s\ell} + L_m\cos 2(\theta - 240°) \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & \cos\theta & \sin\theta \\ 1/\sqrt{2} & \cos\left(\theta - 2\pi/3\right) & \sin\left(\theta - 2\pi/3\right) \\ 1/\sqrt{2} & \cos\left(\theta + 2\pi/3\right) & \sin\left(\theta + 2\pi/3\right) \end{bmatrix}\begin{bmatrix} i_0 \\ i_d \\ i_q \end{bmatrix}\right]$$

I will assume that L_s-L_{sl}=L_m. Therefore

$$T_e = -\frac{1}{2}\frac{2}{3}\frac{\partial}{\partial\theta}\left[\begin{bmatrix} i_0 & i_d & i_q \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ \cos\theta & \cos\left(\theta - 2\pi/3\right) & \cos\left(\theta + 2\pi/3\right) \\ \sin\theta & \sin\left(\theta - 2\pi/3\right) & \sin\left(\theta + 2\pi/3\right) \end{bmatrix}\right.$$

$$\left.\times\begin{bmatrix} L_m(1+\cos 2\theta) & -[M_S + L_m\cos 2(\theta+30°)] & -[M_S + L_m\cos 2(\theta+150°)] \\ -[M_S + L_m\cos 2(\theta+30°)] & L_m(1+\cos 2(\theta-120°)) & -[M_S + L_m\cos 2(\theta-90°)] \\ -[M_S + L_m\cos 2(\theta+150°) & -[M_S + L_m\cos 2(\theta-90°)] & L_m(1+\cos 2(\theta-240°)) \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & \cos\theta & \sin\theta \\ 1/\sqrt{2} & \cos\left(\theta - 2\pi/3\right) & \sin\left(\theta - 2\pi/3\right) \\ 1/\sqrt{2} & \cos\left(\theta + 2\pi/3\right) & \sin\left(\theta + 2\pi/3\right) \end{bmatrix}\begin{bmatrix} i_0 \\ i_d \\ i_q \end{bmatrix}\right]$$

$$T_e = -\frac{1}{2}\frac{2}{3}\frac{\partial}{\partial\theta}\left[\begin{bmatrix} i_0 & i_d & i_q \end{bmatrix}\begin{bmatrix} \begin{array}{c} \frac{1}{\sqrt{2}}\left[L_m(1+\cos 2\theta)\right. \\ -[M_S + L_m\cos 2(\theta+30°)] \\ \left.-[M_S + L_m\cos 2(\theta+150°)\right] \end{array} & \begin{array}{c} \frac{1}{\sqrt{2}}\left[-[M_S + L_m\cos 2(\theta+30°)]\right. \\ +L_m(1+\cos 2(\theta-120°)) \\ \left.-[M_S + L_m\cos 2(\theta+90°)\right] \end{array} & \begin{array}{c} \frac{1}{\sqrt{2}}\left[-[M_S + L_m\cos 2(\theta+150°)]\right. \\ -[M_S + L_m\cos 2(\theta-90°)] \\ \left.+L_m(1+\cos 2(\theta-240°))\right] \end{array} \\ \begin{array}{c} \left[L_m\cos\theta(1+\cos 2\theta)\right. \\ -\cos\left(\theta-120\right)[M_S + L_m\cos 2(\theta+30°)] \\ -\cos\left(\theta+120\right)[M_S + L_m\cos 2(\theta+150°)] \end{array} & \begin{array}{c} \left[-c\,os\,\theta[M_S + L_m\cos 2(\theta+30°)]\right. \\ +L_m c\,os\left(\theta-120\right)[(1+\cos 2(\theta-120°))] \\ -\cos\left(\theta+120\right)[M_S + L_m\cos 2(\theta-90°)] \end{array} & \begin{array}{c} \left[-c\,os\,\theta[M_S + L_m\cos 2(\theta-150°)]\right. \\ -c\,os\left(\theta-120\right)[M_S + L_m\cos 2(\theta-90°)] \\ +L_m(c\,os\left(\theta+120\right)(1+\cos 2(\theta-240°)) \end{array} \\ \begin{array}{c} \left[L_m\sin\theta(1+\cos 2\theta)\right. \\ -\sin\left(\theta-120\right)[M_S + L_m\cos 2(\theta+30°)] \\ -\sin\left(\theta+120\right)[M_S + L_m\cos 2(\theta+150°)] \end{array} & \begin{array}{c} \left[-c\,os\,\theta[M_S + L_m\cos 2(\theta+30°)]\right. \\ +L_m c\,os\left(\theta-120\right)[(1+\cos 2(\theta-120°))] \\ \left.-\cos\left(\theta+120\right)[M_S + L_m\cos 2(\theta-90°)\right] \end{array} & \begin{array}{c} \left[-\sin[M_S + L_m\cos 2(\theta-150°)]\right. \\ -\sin\left(\theta-120\right)[M_S + L_m\cos 2(\theta-90°)] \\ \left.+{}_m\sin\left(\theta+120\right)(1+\cos 2(\theta-240°))\right] \end{array} \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & \cos\theta & \sin\theta \\ 1/\sqrt{2} & \cos\left(\theta - 2\pi/3\right) & \sin\left(\theta - 2\pi/3\right) \\ 1/\sqrt{2} & \cos\left(\theta + 2\pi/3\right) & \sin\left(\theta + 2\pi/3\right) \end{bmatrix}\begin{bmatrix} i_0 \\ i_d \\ i_q \end{bmatrix}\right]$$

…

<u>TERM 2</u>:

$$\frac{\partial}{\partial\theta}\left[\mathbf{i}_{0dq}^{T}\mathbf{P}\mathbf{L}_{aR}\mathbf{i}_{FDQG}\right]\ldots$$