Time Constants (Section 4.14.1)

In a linear static circuit with no capacitance, i.e., an R-L circuit, the transient currents decay with time according to

\[ i(t) = i_0 e^{-t/T} \]  \hspace{1cm} (1)

where \( i_0 \) is the initial current and \( T \) is the time constant.

For an R-L circuit, we may show that

\[ i(t) = \frac{1}{R} e^{-t/(L/R)} \]  \hspace{1cm} (2)

where we see that \( T=L/R \). How do we think of \( T \)?

Let \( t=T \) and then we get that

\[ i(t) = i_0 e^{-T/T} = i_0 e^{-1} = 0.368 i_0 \]  \hspace{1cm} (3)

Thus we see that the time constant is:

1. The time in which the current decreases to 36.8% of its initial value OR
2. The time in which the current decrease equals 63.2% of its initial value.
3. The time in which the current would decrease to zero if it continued to decrease at its initial rate of decrease.

Figure 1 illustrates these three ways of thinking about \( T \).
So the time constant is a good measure of the speed of the dynamics. Low T \( \rightarrow \) fast dynamics.

For a salient pole machine, we have a time constant for each rotor circuit given as the ratio of some inductance to the circuit resistance.

We can get the time constants under one of two conditions:
1. Stator is open-circuited
2. Stator is short-circuited.

The procedure used in the text for developing these equations is as follows (see pp 124-125):
1. Write the voltage equation for the appropriate circuit. This equation will include a flux derivative. Use eq. (4.36') below:

\[
\begin{bmatrix}
  v_d \\
  v_q \\
  -v_F \\
  v_D = 0 \\
  v_Q = 0 \\
  v_G = 0
\end{bmatrix} =
\begin{bmatrix}
  r & 0 & 0 & 0 & 0 \\
  0 & r & 0 & 0 & 0 \\
  0 & 0 & r_F & 0 & 0 \\
  0 & 0 & 0 & r_D & 0 \\
  0 & 0 & 0 & 0 & r_G \\
  0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
  i_d \\
  i_q \\
  i_F \\
  i_D \\
  i_Q \\
  i_G
\end{bmatrix} +
\begin{bmatrix}
  -\omega \dot{\lambda}_q \\
  \omega \dot{\lambda}_d \\
  0 \\
  0 \\
  0 \\
  0
\end{bmatrix} -
\begin{bmatrix}
  \dot{\lambda}_q \\
  \dot{\lambda}_d \\
  \dot{\lambda}_F \\
  \dot{\lambda}_D \\
  \dot{\lambda}_Q \\
  \dot{\lambda}_G
\end{bmatrix}
\]

(4.36')

We assume a step change is applied to the field winding (with the stator winding open or short-circuited, it is the only way we can provide an external forcing function). We want to characterize the time constant of the D-winding. Therefore, we pull out of (4.36') the \( v_F \) and \( v_D \) equations:

\[
v_F = r_F i_F + \dot{\lambda}_F \quad (4.181a)
\]

\[
v_D = 0 = r_D i_D + \dot{\lambda}_D \quad (4.181b)
\]

2. Use eq. (4.20’) (see “macheqts”) to replace fluxes with currents.
For example, we see that

$$\lambda_F = \sqrt{\frac{3}{2}} M_F i_d + L_F i_F + M_R i_D$$  

(4.182a)

$$\lambda_D = \sqrt{\frac{3}{2}} M_D i_d + M_R i_F + L_D i_D$$  

(4.182b)

Notice that from (4.182b), for a step change applied to the field voltage, CFLT indicates that $\lambda_D(0^+)=0$, which implies that

$$0 = M_R i_F + L_D i_D \Rightarrow i_F = \frac{-M_R}{M_R} i_D$$  

(4.183)

4. Differentiate (4.182a) and (4.182b) and then substitute into (4.181a) and (4.181b), respectively. This results in

$$v_F = r_F i_F + L_F \dot{i}_F + M_R \ddot{i}_D$$  

(4.184a)

$$0 = r_D i_d + M_R \dot{i}_F + L_D \dot{i_D}$$  

(4.184b)

Divide (4.184a) by $L_F$ and (4.184b) by $M_R$ to get

$$\frac{v_F}{L_F} = \frac{r_F}{L_F} i_F + \dot{i}_F + \frac{M_R}{L_F} \ddot{i}_D$$  

(4.184c)

$$0 = \frac{r_D}{M_R} i_d + \dot{i}_F + \frac{L_D}{M_R} \dot{i_D}$$  

(4.184d)

Subtract (4.184c) from (4.184d) to get
\[
\frac{r_D}{M_R} i_D - \frac{r_F}{L_F} i_F + \left( \frac{L_D - M_R}{M_R} \right) \frac{i_D}{L_F} = -\frac{v_F}{L_F}
\]

Now replace \(i_F\) with (4.183) to get
\[
\left( \frac{r_D}{M_R} + \frac{r_F}{L_F} \frac{L_D}{M_R} \right) i_D + \left( \frac{L_D - M_R}{M_R} \right) \frac{i_D}{L_F} = -\frac{v_F}{L_F}
\]

Now divide through by the coefficient of the derivative term:
\[
\frac{\left( r_D + \frac{r_F L_D}{L_F} \right)}{\left( L_D - M_R^2 / L_F \right)} i_D + \frac{i_D}{L_F} = -\frac{M_R v_F}{L_F}
\]

Multiply top and bottom of the first term on the left-hand-side by \(M_R\), and do the same to the right-hand-side, to get
\[
\frac{\left( r_D + \frac{r_F L_D}{L_F} \right)}{\left( L_D - M_R^2 / L_F \right)} i_D + \frac{i_D}{L_F} = -\frac{M_R v_F}{L_F}
\]

5. Use the following information (in per-unit):
- Damper circuits are very fast, which makes \(r_D\) and \(r_Q\) large.
- Field circuits are very slow, which makes \(r_F\) and \(r_G\) small.

Reference to Example 4.1 (p. 98) indicates, in per-unit:
\[
\begin{align*}
    r_F &= 0.000742 \\
    r_D &= 0.0131 \\
    r_Q &= 0.054
\end{align*}
\]

In our example, A&F (pg. 125) make the statement that “usually in pu \(r_D >> r_F\) while \(L_D\) and \(L_F\) are of similar magnitude.” This means \(r_D >> r_F L_D / L_F\), and so the above becomes
\[
\frac{r_D}{\left( L_D - M_R^2 / L_F \right)} i_D + \frac{i_D}{L_F} = -\frac{M_R v_F}{L_F}
\]

Rearranging, we obtain
\[
i_D + \frac{r_D}{\left( L_D - M_R^2 / L_F \right)} i_D = -\frac{M_R}{L_F} \frac{v_F}{\left( L_D - M_R^2 / L_F \right)} \quad (4.186a)
\]

Now define
\[ K_1 = \frac{r_D}{L_D - M_R^2 / L_F} \quad K_2 = -v_F \frac{M_R / L_F}{L_D - M_R^2 / L_F} \]

Then (4.186a) becomes
\[ \dot{i}_D + K_i i_D = K_2 \quad (4.186b) \]

Using LaPlace transforms, we get
\[ sI_D(s) + K_I I_D(s) = K_2 / s \]
\[ I_D(s)(s + K_1) = K_2 / s \quad (4.186c) \]
\[ I_D(s) = \frac{K_2}{s(s + K_1)} \]

Taking partial fraction expansion, we have:
\[ I_D(s) = \frac{K_2}{s(s + K_1)} = \frac{K_2 / K_1}{s} - \frac{K_2 / K_1}{s + K_1} = \frac{K_2}{K_1} \left( \frac{1}{s} - \frac{1}{s + K_1} \right) \]

The inverse LaPlace transform is then
\[ i_D(t) = \frac{K_2}{K_1} \left( 1 - e^{-K_1 t} \right) u(t) \quad (4.186d) \]

This shows that if we were to apply a current unit-step function to the circuit, it would rise to the level of that unit-step function (which is 1) in accordance with a time constant of \( K_1 \). For example, the function \( y=(1-e^{-1t})u(t) \) is as indicated in the below.

Replacing \( K_1 \) and \( K_2 \), we obtain
\[ i_D(t) = -v_F \frac{M_R / L_F}{(L_D - M_R^2 / L_F)} \left( L_D - M_R^2 / L_F \right) \left( 1 - e^{-r_D \left( L_D - M_R^2 / L_F \right) t} \right) u(t) \]

\[ = -v_F \frac{M_R / L_F}{r_D} \left( 1 - e^{-r_D \left( L_D - M_R^2 / L_F \right) t} \right) u(t) \]  

(4)

Recall

\[ i(t) = \frac{1}{R} e^{-t/(L/R)} \]  

(2)

where \( T=L/R \), and so we see that \( 1/K_1 \) is the time constant. We define this time constant as the open circuit subtransient time constant, i.e.

\[ \tau_{d0}'' = \frac{1}{K_1} = \frac{L_D - M_R^2 / L_F}{r_D} \]

It’s name comes from the fact that

- it is computed when the stator windings are open circuit,
- it characterizes the behavior of the D-winding and is therefore a subtransient response.

Application of similar procedures results in the expressions that Kundur calls the “classical expressions” given as follows:

**Without G-winding (salient pole machine):**

**OC/DA/T/TC:** \[ \tau_{d0}' = \frac{L_F}{r_F} \]  
(D-axis field)

**OC/DA/ST/TC:** \[ \tau_{d0}' = \frac{L_D - (L_{AD})^2 / L_F}{r_D} \]  
(D-axis damper)

**OC/QA/ST/TC:** \[ \tau_{q0}'' = \frac{L_Q}{r_Q} \]  
(Q-axis damper)
With G-Winding (round rotor machine):

\[ \tau'_{d0} = \frac{L_F}{r_F} \quad \text{(D-axis field)} \]

\[ \tau''_{d0} = \frac{L_D - (L_{AD})^2 / L_F}{r_D} \quad \text{(D-axis damper)} \]

\[ \tau''_{q0} = \frac{L_Q}{r_Q} \quad \text{(Q-axis damper)} \]

\[ \tau'_{q0} = \frac{L_Q - (L_{AQ})^2 / L_G}{r_G} \quad \text{(Q-axis field)} \]

In the above:
OC : Open-circuit
DA : direct-axis
QA : quadrature axis
T : transient
ST : subtransient
TC : time constant

The short circuit time constants are as follows:

Without G-winding (salient pole machine):

\[ \tau'_{d} = \tau'_{d0} \frac{L'_d}{L_d} \quad \text{(D-axis field)} \]

\[ \tau''_{d} = \tau''_{d0} \frac{L''_d}{L'_d} \quad \text{(D-axis damper)} \]
SC/QA/ST/TC: \[ \tau''_q = \tau''_q 0 \frac{L''_q}{L_q} \] (Q-axis damper)

With G-Winding (round-rotor machine):

SC/DA/T/TC: \[ \tau'_d = \tau'_d 0 \frac{L'_d}{L_d} \] (D-axis field)

SC/DA/ST/TC: \[ \tau''_d = \tau''_d 0 \frac{L''_d}{L'_d} \] (D-axis damper)

SC/QA/ST/TC: \[ \tau''_q = \tau''_q 0 \frac{L''_q}{L_q} \] (Q-axis damper)

SC/QA/ST/TC: \[ \tau'_q = \tau'_q 0 \frac{L'_q}{L_q} \] (Q-axis damper)

Another time constant used to characterize synchronous machines is the stator time constant, given by

\[ \tau_a = \frac{(L'_d + L'_q)}{2} \]

Note that the text uses \( L_q \) in the above equation instead of \( L'_q \) (since \( L_q = L'_q \) when the G-winding is not represented).

Table 4.3, pg. 126 in your text, provides a comparison of typical numerical range for time constants. Kundur also provides such a table, Table 4.2, pg. 150. Note transient \( T >> \) subtransient \( T \).

Another way to get the time constants is to use the equivalent circuits.

Then derive the inductances in terms of the LaPlace variable “s” according to
I will not go through the development here, but you can find it on pp. 140-143 of Kundur’s text.

The denominator of the above expressions is the characteristic equation for the circuit. The roots of this equation are the inverse of the time constants.

This approach makes no approximations, and therefore Kundur refers to the resulting expressions for the parameters as the “accurate expressions.”

The relationship between our nomenclature and that used by Kundur is as follows:

**Kundur ➔ A&F**

- $L_{ad} \Rightarrow L_{AD}$
- $L_{fd} \Rightarrow l_F$
- $R_{FD} \Rightarrow r_F$
- $R_{1d} \Rightarrow r_D$
- $L_{1d} \Rightarrow l_D$
- $L_{l} \Rightarrow l_d$

You can review some of the data in appendix C of your text to see if it conforms to our conclusions about “fast” vs. “slow” circuits.

And you can check Kundur, page 153, for some comparative data for both salient pole and round-rotor machines, which I have copied out below.
Similar data is in Chapter 4 of Anderson & Fouad, p. 126: Note Table 4.3 comes from Kimbark, see next page of these notes.
Table 4.3. Typical Synchronous Machine Time Constants in Seconds

<table>
<thead>
<tr>
<th>Time constant</th>
<th>Turbogenerators</th>
<th>Waterwheel generators</th>
<th>Synchronous condensers</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Low</td>
<td>Avg.</td>
<td>High</td>
</tr>
<tr>
<td>$\tau_0$</td>
<td>2.8</td>
<td>5.6</td>
<td>9.2</td>
</tr>
<tr>
<td>$\tau_d$</td>
<td>0.4</td>
<td>1.1</td>
<td>1.8</td>
</tr>
<tr>
<td>$\tau''_d$</td>
<td>0.02</td>
<td>0.035</td>
<td>0.05</td>
</tr>
<tr>
<td>$\tau_q$</td>
<td>0.04</td>
<td>0.16</td>
<td>0.35</td>
</tr>
</tbody>
</table>


Table 4.4. Typical Turbogenerator and Synchronous Condenser Characteristics

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Generators</th>
<th>Synchronous condensers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Range</td>
<td>Recommended average</td>
<td>Range</td>
</tr>
<tr>
<td>Nominal rating</td>
<td>300-1000 MW</td>
<td>...</td>
</tr>
<tr>
<td>Power factor</td>
<td>0.80-0.95</td>
<td>0.90</td>
</tr>
<tr>
<td>Direct axis synchronous reactance $x_d$</td>
<td>140-180</td>
<td>160</td>
</tr>
<tr>
<td>Transient reactance $x'_d$</td>
<td>23-35</td>
<td>25</td>
</tr>
<tr>
<td>Subtransient reactance $x''_d$</td>
<td>15-23</td>
<td>20</td>
</tr>
<tr>
<td>Quadrature axis synchronous reactance $x_q$</td>
<td>150-160</td>
<td>155</td>
</tr>
<tr>
<td>Negative-sequence reactance $x_2$</td>
<td>18-20</td>
<td>19</td>
</tr>
<tr>
<td>Zero-sequence reactance $x_0$</td>
<td>12-14</td>
<td>13</td>
</tr>
<tr>
<td>Short circuit ratio</td>
<td>0.05-0.72</td>
<td>0.64</td>
</tr>
<tr>
<td>Inertia constant $H$ (kW·s)</td>
<td>3600 r/min</td>
<td>3.0-5.0</td>
</tr>
<tr>
<td>(kV·A)</td>
<td>1800 r/min</td>
<td>5.0-8.0</td>
</tr>
</tbody>
</table>

Source: From the 1964 National Power Survey made by the U.S. Federal Power Commission. USGPO.
Note: All reactances in percent on rated voltage and kVA base. kW losses for typical synchronous condensers in the range of sizes shown, excluding losses associated with step-up transformers, are in the order of 1.2-1.5% on rated kVA base. No attempt has been made to show kW losses associated with generators, since generating plants are generally rated on a net power output basis and losses vary widely dependent on the generator plant design.

Table 4.5. Typical Hydrogenerator Characteristics

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Small units</th>
<th>Large units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Range</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>Nominal rating (MVA)</td>
<td>0.40</td>
<td>40-200</td>
</tr>
<tr>
<td>Power factor</td>
<td>0.80-0.95*</td>
<td>0.80-0.95*</td>
</tr>
<tr>
<td>Speed (r/min)</td>
<td>70-350</td>
<td>70-200</td>
</tr>
<tr>
<td>Inertia constant $H$ (kW·s)</td>
<td>1.5-4.0</td>
<td>3.0-5.5</td>
</tr>
<tr>
<td>(kV·A)</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>Direct axis synchronous reactance $x_d$</td>
<td>90-110</td>
<td>80-100</td>
</tr>
<tr>
<td>Transient reactance $x'_d$</td>
<td>25-45</td>
<td>20-40</td>
</tr>
<tr>
<td>Subtransient reactance $x''_d$</td>
<td>20-35</td>
<td>15-30</td>
</tr>
<tr>
<td>Quadrature axis synchronous reactance $x_q$</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>Negative-sequence reactance $x_2$</td>
<td>20-45</td>
<td>20-35</td>
</tr>
<tr>
<td>Zero-sequence reactance $x_0$</td>
<td>10-35</td>
<td>10-25</td>
</tr>
<tr>
<td>Short circuit ratio</td>
<td>1.0-2.0</td>
<td>1.0-2.0</td>
</tr>
</tbody>
</table>

Source: From the 1964 National Power Survey made by the U.S. Federal Power Commission. USGPO.
Note: All reactances in percent on rated voltage and kVA base. No attempt has been made to show kW losses associated with generators, since generating plants are generally rated on a net power output basis and losses vary widely dependent on the generator plant design.
*These power factors cover conditions for generators installed either close to or remote from load centers.

And From Kimbark, p. 40 (note time constant data at bottom of table is used in A&F’s table, given on previous page).
Typical Constants of Three-Phase Synchronous Machines
(Adapted from Refs. 15, 34, and 41)

<table>
<thead>
<tr>
<th></th>
<th>Turbo-generators (solid rotor)</th>
<th>Water-Wheel Generators (with dampers)†</th>
<th>Synchronous Condensers</th>
<th>Synchronous Motors (general purpose)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Low</td>
<td>Avg.</td>
<td>High</td>
<td>Low</td>
</tr>
<tr>
<td>Reactances in per unit</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_d$</td>
<td>0.95</td>
<td>1.10</td>
<td>1.45</td>
<td>0.60</td>
</tr>
<tr>
<td>$x_q$</td>
<td>0.92</td>
<td>1.08</td>
<td>1.42</td>
<td>0.40</td>
</tr>
<tr>
<td>$x_d'$</td>
<td>0.12</td>
<td>0.23</td>
<td>0.28</td>
<td>0.20</td>
</tr>
<tr>
<td>$x_q'$</td>
<td>0.12</td>
<td>0.23</td>
<td>0.28</td>
<td>0.40</td>
</tr>
<tr>
<td>$x_d''$</td>
<td>0.07</td>
<td>0.12</td>
<td>0.17</td>
<td>0.13</td>
</tr>
<tr>
<td>$x_q''$</td>
<td>0.10</td>
<td>0.15</td>
<td>0.20</td>
<td>0.23</td>
</tr>
<tr>
<td>$x_p$</td>
<td>0.07</td>
<td>0.14</td>
<td>0.21</td>
<td>0.17</td>
</tr>
<tr>
<td>$x_z$</td>
<td>0.07</td>
<td>0.12</td>
<td>0.17</td>
<td>0.13</td>
</tr>
<tr>
<td>$x_0^*$</td>
<td>0.01</td>
<td>0.10</td>
<td>0.21</td>
<td>0.02</td>
</tr>
<tr>
<td>Resistances in per unit</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$r_d$(d-c.)</td>
<td>0.0015</td>
<td>0.005</td>
<td>0.003</td>
<td>0.020</td>
</tr>
<tr>
<td>$r$(a-c.)</td>
<td>0.003</td>
<td>0.008</td>
<td>0.003</td>
<td>0.015</td>
</tr>
<tr>
<td>$r_a$</td>
<td>0.025</td>
<td>0.045</td>
<td>0.012</td>
<td>0.20</td>
</tr>
<tr>
<td>Time constants in seconds</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$T_{ds}'$</td>
<td>2.8</td>
<td>5.6</td>
<td>9.2</td>
<td>1.5</td>
</tr>
<tr>
<td>$T_d'$</td>
<td>0.4</td>
<td>1.1</td>
<td>1.8</td>
<td>0.5</td>
</tr>
<tr>
<td>$T_d'' = T_{q''}$</td>
<td>0.02</td>
<td>0.035</td>
<td>0.05</td>
<td>0.01</td>
</tr>
<tr>
<td>$T_a$</td>
<td>0.04</td>
<td>0.16</td>
<td>0.35</td>
<td>0.03</td>
</tr>
</tbody>
</table>

* $x_0$ varies from about 0.15 to 0.60 of $x_d''$, depending upon winding pitch.
† For water-wheel generators without damper windings,

$$x_d'' = 0.85 x_d', \quad x_q'' = x_q' = x_q, \quad x_2 = (x_d' + x_q)/2,$$

and $x_0$ is as listed.

† For curves showing the normal value of $x_d'$ of water-wheel-driven generators as a function of kilovolt-ampere rating and speed, see Ref. 54.