The Swing Equation: Power Form, Per-Unit, Error

1.0 Power Form of Swing Equation

Recall from the previous notes eq. (25)

$$\frac{2J}{p}\ddot{\delta}(t) = T_a = \frac{P_a}{\omega_m} \tag{1}$$

where p is the number of pole pairs, J is the moment of inertia (kg-m^2), and T_a is the accelerating torque, P_a is the accelerating power, and ω_m is the mechanical angular velocity. Replacing $\ddot{\delta}(t) = \dot{\omega}(t)$, and using the power expression on the right-hand-side, we obtain

$$\frac{2J}{p}\dot{\omega}(t) = \frac{P_a}{\omega_m} \tag{2}$$

Normalizing both sides of (2) by $T_B=S_{B3}/\omega_R$, we obtain

$$\frac{2J\omega_R}{pS_{B3}}\dot{\omega}(t) = \frac{P_a\omega_R}{S_{B3}\omega_m} \tag{3}$$

Now assume $\omega_m = \omega_R$, then (3) becomes

$$\frac{2J\omega_R}{pS_{B3}}\dot{\omega}(t) = \frac{P_a}{S_{B3}} \equiv P_{au} \tag{4}$$

Now recall (28) from previous notes:

$$\frac{2J\omega_R}{pS_{B3}}\dot{\omega}(t)=\frac{T_a}{T_B}=T_{au} \tag{28}$$

Comparing (4) with (28), it is clear that the numerical value of T_{au} is the same as the numerical value of P_{au}, on the condition that $\omega_m=\omega_R$. Therefore (under this condition) when the swing equation is written in per-unit, the numerical value of the torque version is the same as the numerical value of the power version.

Proceeding as in Section 5 of the previous notes, we get:

$$\frac{2H}{\omega_{\mathrm{Re}}}\dot{\omega}(t)=P_{au} \tag{5}$$

However, if you do not allow that $\omega_m=\omega_R$ on the right-hand-side, then we get:

$$\frac{2H}{\omega_{\mathrm{Re}}}\dot{\omega}(t)=\frac{P_{au}}{\omega_m/\omega_R} \tag{6}$$

We will come back to (6) to analyze error in the power form of the swing equation. But before we do that, we need to define pu speed.

2.0 Swing equation in per-unit speed

You will often see the swing equation given in terms of per-unit speed. This section derives this equation, starting from the per-unit power equation (5):

$$\frac{2H}{\omega_{\mathrm{Re}}}\dot{\omega}(t) = P_{au} \tag{5}$$

Recall that ω (where we earlier agreed to drop the "e" subscript) is the electrical radial frequency or speed. We define the per-unit speed as

$$\omega_u = \frac{\omega}{\omega_{\mathrm{Re}}} \tag{7}$$

where $\omega_{Re}=377$ rad/sec in the US.

Writing (5) in a slightly different format,

$$2H\frac{\dot{\omega}(t)}{\omega_{\mathrm{Re}}} = P_{au} \tag{8}$$

or

$$2H\frac{\frac{d\omega(t)}{dt}}{\omega_{\mathrm{Re}}}=P_{au} \tag{9}$$

or

$$2H\frac{d(\omega(t)/\omega_{\mathrm{Re}})}{dt}=P_{au} \tag{10}$$

By (7), we recognize what is in the parentheses as ω_u, leading to:

$$2H\frac{d\omega_u(t)}{dt}=P_{au} \tag{11}$$

or

$$2H\dot{\omega}_u(t)=P_{au} \tag{12}$$

You will often see the swing equation written as (12) instead of as (5), and you need to know that omission of ω_{Re} on the denominator implies that the speed is in per-unit.

As a final note to per-unitization of speed, we can write (7) as

$$\omega_u = \frac{\omega}{\omega_{\mathrm{Re}}} = \frac{\frac{2}{p}\omega}{\frac{2}{p}\omega_{\mathrm{Re}}} = \frac{\omega_m}{\omega_R} \tag{13}$$

which shows us that per unit speed in mechanical rad/sec is the same as the per unit speed in electrical rad/sec.

3.0 Error in power form of swing equation

Let's return now to (6).

$$\frac{2H}{\omega_{\mathrm{Re}}}\dot{\omega}(t) = \frac{P_{au}}{\omega_m / \omega_R} \tag{6}$$

Now let's account for the fact that $\omega_m \neq \omega_R$ on the right-hand-side of (6), i.e.,

$$\omega_m = \omega_R + \Delta\omega_m \tag{14}$$

Dividing (14) through by ω_R, we obtain

$$\frac{\omega_m}{\omega_R} = \frac{\omega_R}{\omega_R} + \frac{\Delta\omega_m}{\omega_R} = 1 + \Delta\omega_u \tag{15}$$

Substituting (15) into (6) results in

$$\frac{2H}{\omega_{\mathrm{Re}}}\dot{\omega}(t) = \frac{P_{au}}{1+\Delta\omega_u} \tag{16}$$

Recognizing $\dot{\omega}(t)/\omega_{\mathrm{Re}}$ on the left-hand-side of (16) is $\dot{\omega}_u(t)$, we have

$$2H\dot{\omega}_u(t) = \frac{P_{au}}{1+\Delta\omega_u} \tag{17}$$

From the binomial expansion for small numbers, we have

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots \tag{18}$$

Using (18) to expand the right-hand-side of (17) results in

$$2H\dot{\omega}_u(t) = P_{au}\left(1 - \Delta\omega_u + (\Delta\omega_u)^2 - (\Delta\omega_u)^3 + \ldots\right) \tag{19}$$

If $\Delta\omega_u$ is small, then the higher-order terms of (19) may be neglected, resulting in

$$2H\dot{\omega}_u(t) = P_{au}\left(1 - \Delta\omega_u\right) \tag{20}$$

or

$$2H\dot{\omega}_u(t) = P_{au} - P_{au}\Delta\omega_u \tag{21}$$

But yet, we always solve just

$$2H\dot{\omega}_u(t) = P_{au} \quad (12)$$

Comparison of (12) to (21) shows that in the power form of the swing equation, there is some error on the right-hand-side of (12), approximated by $P_{au}\Delta\omega_u$.

This error can be reduced by solving

$$2H\dot{\omega}_u(t) = P_{au} - D\Delta\omega_u \quad (22)$$

where $D=P_{a0}$ is set equal to the initial accelerating power.

An even better approach is, however, to solve the torque equation rather than the power equation when getting numerical solutions.

$$2H\dot{\omega}_u(t) = T_{au} \quad (23)$$

This avoids the right-hand-side error altogether.

4.0 Speed vs. Speed Deviation

Equation (14) is

$$\omega_m = \omega_R + \Delta\omega_m \quad (14)$$

Or in electrical rad/sec, we have

$$\omega = \omega_{\mathrm{Re}} + \Delta\omega \quad (24)$$

Normalize by ω_{Re},

$$\omega_u = 1 + \Delta\omega_u \tag{25}$$

Now differentiate both sides of (25)

$$\frac{d\omega_u}{dt} = \frac{d\Delta\omega_u}{dt} \tag{26}$$

The implication of (26) is that (23) may be written in terms of speed or speed deviation since their derivatives are equal. That is, we can solve either of the below:

$$2H\dot{\omega}_u(t) = T_{au} \tag{23}$$

$$2H\Delta\dot{\omega}_u(t) = T_{au} \tag{27}$$

When you solve numerically, you must use the appropriate constant of integration in order to get what you want. If you use a constant of integration equal to 1, your answer will be in terms of per-unit speed. If you use a constant of integration equal to 0, your answer will be in terms of per-unit speed deviation. This is expressed below, where we use $\dot{x}(t) = \dot{\omega}_u = \Delta\dot{\omega}_u$:

$$\int \dot{x}(t)dt = \begin{matrix} x(t) + 1 = \omega_u & \text{(pu speed)} \\ x(t) = \Delta\omega_u & \text{(pu speed deviation)} \end{matrix}$$