State-Space Model for a Multi-Machine System

These notes parallel section 3.4 in the text. We are dealing with classically modeled machines (IEEE Type 0.0), constant impedance loads, and a network reduced to its internal machine terminals.

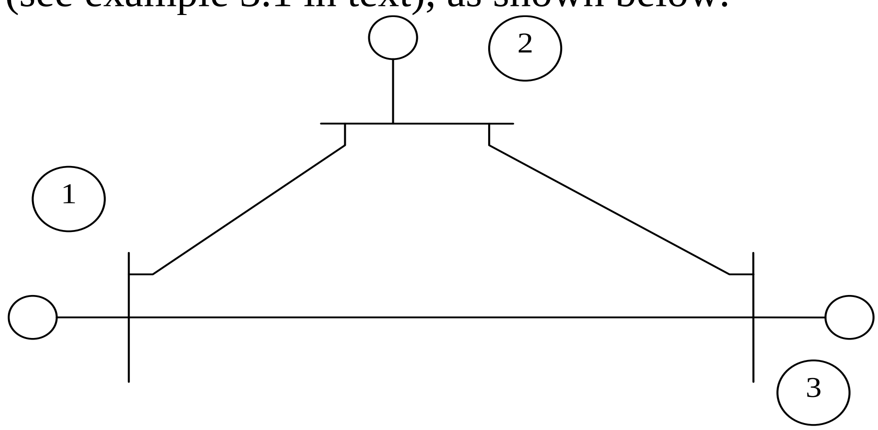
We have found that the linearized swing equation is given by:

$$\frac{2H_i}{\omega_{\mathrm{Re}}}\frac{d^2\Delta\delta_i}{dt^2}+\sum_{\substack{i=1\\ i\neq j}}^{n}P_{Sij}\Delta\delta_{ij}=0, \quad \text{i=1,…,n} \qquad \text{(eq. 3.26)}$$

where

$$P_{Sij}=\left.\frac{\partial P_{ij}}{\partial\delta_{ij}}\right|_{\delta_{ij0}}=E_iE_j\left\{B_{ij}cos\delta_{ij0}\text{ - }G_{ij}sin\delta_{ij0}\right\} \qquad \text{(Eqt. 3.24)}$$

and it is important to observe that we have assumed no damping. Let's consider writing the linearized swing equations for a test system (see example 3.1 in text), as shown below.



The three swing equations are:

$$\frac{2H_1}{\omega_{\mathrm{Re}}}\frac{d^2\Delta\delta_1}{dt^2}+P_{S12}\Delta\delta_{12}+P_{S13}\Delta\delta_{13}=0$$

$$\frac{2H_2}{\omega_{\mathrm{Re}}}\frac{d^2\Delta\delta_2}{dt^2}+P_{S21}\Delta\delta_{21}+P_{S23}\Delta\delta_{23}=0$$

$$\frac{2H_3}{\omega_{\mathrm{Re}}}\frac{d^2\Delta\delta_3}{dt^2}+P_{S31}\Delta\delta_{31}+P_{S32}\Delta\delta_{32}=0$$

<u>One important fact</u>: The stability of a power system depends on relative rotor angles $\Delta\delta_{ij}$ NOT absolute rotor angles $\Delta\delta_i$. This is because *synchronism* is a relative phenomenon. That is, it makes no sense to say "generator 1 is in synchronism." Rather, we must say with what it is in synchronism, i.e., "Generator 1 is in synchronism with generators 2 and 3," or "Generator 1 is in synchronism with the rest of the system."

So we need to define our states in terms of <u>relative rotor angles</u>. In the above equations, the states (derivatives) are in terms of absolute rotor angle. We can deal with this in the following way.

First, multiply through each equation by $\omega_{Re}/2H_i$, resulting in:

$$\frac{d^2\Delta\delta_1}{dt^2}+\frac{\omega_{\mathrm{Re}}}{2H_1}\left[P_{S12}\Delta\delta_{12}+P_{S13}\Delta\delta_{13}\right]=0$$

$$\frac{d^2\Delta\delta_2}{dt^2}+\frac{\omega_{\mathrm{Re}}}{2H_2}\left[P_{S21}\Delta\delta_{21}+P_{S23}\Delta\delta_{23}\right]=0$$

$$\frac{d^2\Delta\delta_3}{dt^2}+\frac{\omega_{\mathrm{Re}}}{2H_3}\left[P_{S31}\Delta\delta_{31}+P_{S32}\Delta\delta_{32}\right]=0$$

Now subtract the last equation from each of the other two. When we do this, the derivative terms on the left will be affected as follows (and this is the main motivation for making this subtraction):

$$\frac{d^2\Delta\delta_1}{dt^2}-\frac{d^2\Delta\delta_3}{dt^2}=\frac{d^2(\Delta\delta_1-\Delta\delta_3)}{dt^2}=\frac{d^2\Delta\delta_{13}}{dt^2}$$

which is $d\omega_{13}/dt$. However, this very convenient substitution of variable will not help if damping is modeled ($D_i\omega_i$) in the swing equation AND the damping is nonuniform (we will see below what this means) because then it is not possible to combine the corresponding speed variables. Let's look at this issue by re-writing the above equations with damping.

$$\frac{d^2\Delta\delta_1}{dt^2}+\frac{D_1}{2H_1}\frac{d\Delta\delta_1}{dt}+\frac{\omega_{\text{Re}}}{2H_1}\left[P_{S12}\Delta\delta_{12}+P_{S13}\Delta\delta_{13}\right]=0$$

$$\frac{d^2\Delta\delta_2}{dt^2}+\frac{D_2}{2H_2}\frac{d\Delta\delta_2}{dt}+\frac{\omega_{\text{Re}}}{2H_2}\left[P_{S21}\Delta\delta_{21}+P_{S23}\Delta\delta_{23}\right]=0$$

$$\frac{d^2\Delta\delta_3}{dt^2}+\frac{D_3}{2H_3}\frac{d\Delta\delta_3}{dt}+\frac{\omega_{\text{Re}}}{2H_3}\left[P_{S31}\Delta\delta_{31}+P_{S32}\Delta\delta_{32}\right]=0$$

Subtracting the last equation from the other two (in the case of the first equation) affects the 2nd derivative terms on the left according to:

$$\frac{d^2\Delta\delta_1}{dt^2}-\frac{d^2\Delta\delta_3}{dt^2}=\frac{d^2(\Delta\delta_1-\Delta\delta_3)}{dt^2}=\frac{d^2\Delta\delta_{13}}{dt^2}\text{ (as before)}$$

and the 1rst derivative terms on the left according to:

$$\frac{D_1}{2H_1}\frac{d\Delta\delta_1}{dt}-\frac{D_3}{2H_3}\frac{d\Delta\delta_3}{dt}$$ ➔ Cannot combine angles if D_i/H_i differ

We can combine angles if ratios D_i/H_i are the same for all i, which is the condition for *uniform damping*. In this case, the first derivative terms become (when we subtract last equation from first):

$$\frac{D_1}{2H_1}\frac{d\Delta\delta_1}{dt}-\frac{D_3}{2H_3}\frac{d\Delta\delta_3}{dt}=\frac{D_1}{2H_1}\left(\frac{d\Delta\delta_1}{dt}-\frac{d\Delta\delta_3}{dt}\right)=\frac{D_1}{2H_1}\frac{d\Delta\delta_{13}}{dt}=\frac{D_1}{2H_1}\Delta\omega_{13}$$

The implication is that we can ALWAYS reduce the number of states by 1 due to the ability to use relative angles. But an *additional* reduction of states by 1 due to the ability to use relative speeds only occurs in the cases of *no damping* or of *uniform damping*.

In general, D_i/H_i ratios will be different, and so modeling nonuniform damping is necessary. When this is the case, we are only able to get the state reduction for relative angles, but not for relative speeds. The resulting system appears as below.

$$\frac{d^2\Delta\delta_{13}}{dt^2}+\frac{D_1}{2H_1}\frac{d\Delta\delta_1}{dt}-\frac{D_3}{2H_3}\frac{d\Delta\delta_3}{dt}+\frac{\omega_{\text{Re}}}{2H_1}\left[P_{S12}\Delta\delta_{12}+P_{S13}\Delta\delta_{13}\right]-\frac{\omega_{\text{Re}}}{2H_3}\left[P_{S31}\Delta\delta_{31}+P_{S32}\Delta\delta_{32}\right]=0$$
$$\frac{d^2\Delta\delta_{23}}{dt^2}+\frac{D_2}{2H_2}\frac{d\Delta\delta_2}{dt}-\frac{D_3}{2H_3}\frac{d\Delta\delta_3}{dt}+\frac{\omega_{\text{Re}}}{2H_2}\left[P_{S21}\Delta\delta_{21}+P_{S23}\Delta\delta_{23}\right]-\frac{\omega_{\text{Re}}}{2H_3}\left[P_{S31}\Delta\delta_{31}+P_{S32}\Delta\delta_{32}\right]=0$$

And replacing the first derivative terms by speed deviations, we get:

$$\frac{d^2\Delta\delta_{13}}{dt^2}+\frac{D_1}{2H_1}\Delta\omega_1-\frac{D_3}{2H_3}\Delta\omega_3+\frac{\omega_{\text{Re}}}{2H_1}\left[P_{S12}\Delta\delta_{12}+P_{S13}\Delta\delta_{13}\right]-\frac{\omega_{\text{Re}}}{2H_3}\left[P_{S31}\Delta\delta_{31}+P_{S32}\Delta\delta_{32}\right]=0$$
$$\frac{d^2\Delta\delta_{23}}{dt^2}+\frac{D_2}{2H_2}\Delta\omega_2-\frac{D_3}{2H_3}\Delta\omega_3+\frac{\omega_{\text{Re}}}{2H_2}\left[P_{S21}\Delta\delta_{21}+P_{S23}\Delta\delta_{23}\right]-\frac{\omega_{\text{Re}}}{2H_3}\left[P_{S31}\Delta\delta_{31}+P_{S32}\Delta\delta_{32}\right]=0$$

In the particular special case of no damping, we obtain:

$$\frac{d^2\Delta\delta_{13}}{dt^2}+\frac{\omega_{\text{Re}}}{2H_1}\left[P_{S12}\Delta\delta_{12}+P_{S13}\Delta\delta_{13}\right]-\frac{\omega_{\text{Re}}}{2H_3}\left[P_{S31}\Delta\delta_{31}+P_{S32}\Delta\delta_{32}\right]=0$$
$$\frac{d^2\Delta\delta_{23}}{dt^2}+\frac{\omega_{\text{Re}}}{2H_2}\left[P_{S21}\Delta\delta_{21}+P_{S23}\Delta\delta_{23}\right]-\frac{\omega_{\text{Re}}}{2H_3}\left[P_{S31}\Delta\delta_{31}+P_{S32}\Delta\delta_{32}\right]=0$$

Recognizing that $\Delta\delta_{31}=-\Delta\delta_{13}$ and that $\Delta\delta_{32}=-\Delta\delta_{23}$, we may change the sign of the second term in each equation if we also make this change of variables. This results in:

$$\frac{d^2\Delta\delta_{13}}{dt^2}+\frac{\omega_{\text{Re}}}{2H_1}\left[P_{S12}\Delta\delta_{12}+P_{S13}\Delta\delta_{13}\right]+\frac{\omega_{\text{Re}}}{2H_3}\left[P_{S31}\Delta\delta_{13}+P_{S32}\Delta\delta_{23}\right]=0$$
$$\frac{d^2\Delta\delta_{23}}{dt^2}+\frac{\omega_{\text{Re}}}{2H_2}\left[P_{S21}\Delta\delta_{21}+P_{S23}\Delta\delta_{23}\right]+\frac{\omega_{\text{Re}}}{2H_3}\left[P_{S31}\Delta\delta_{13}+P_{S32}\Delta\delta_{23}\right]=0$$

eq. (#)

So we have derivatives on $\Delta\delta_{13}$ and $\Delta\delta_{23}$, and these are our states. But observe that there are two other variables, namely $\Delta\delta_{12}$, $\Delta\delta_{21}$.

This means we have 4 variables and only 2 equations.

Can we express $\Delta\delta_{12}$ and $\Delta\delta_{21}$ in terms of $\Delta\delta_{13}$ and $\Delta\delta_{23}$? Clearly, since $\Delta\delta_{12}=-\Delta\delta_{21}$, if we can do it for one, we can do it for the other.

This is done by noting first that

$$\Delta\delta_{12}+\Delta\delta_{23}+\Delta\delta_{31}=0 \qquad \text{(eq. *)}$$

We can prove this as follows:

$$\Delta\delta_{12} \quad + \quad \Delta\delta_{23} \quad + \quad \Delta\delta_{31}$$

$$=\Delta\delta_1-\Delta\delta_2+\Delta\delta_2-\Delta\delta_3+\Delta\delta_3-\Delta\delta_1=0$$

Therefore, from eq. (*), we can write that

$$\Delta\delta_{12}=-\Delta\delta_{23}-\Delta\delta_{31}$$

Reversing the subscript order of the last term on the right-hand-side, and changing signs, we get:

$$\Delta\delta_{12}=-\Delta\delta_{23}+\Delta\delta_{13} \qquad \text{(eq. **)}$$

Then, since $\Delta\delta_{12}=-\Delta\delta_{21}$, we get

$$\Delta\delta_{21}=\Delta\delta_{23}-\Delta\delta_{13} \qquad \text{(eq. ***)}$$

Substituting eq. (**) and (***) into eq (#), we obtain

$$\frac{d^2\Delta\delta_{13}}{dt^2}+\frac{\omega_{\text{Re}}}{2H_1}\left[P_{S12}\Delta\delta_{13}-P_{S12}\Delta\delta_{23}+P_{S13}\Delta\delta_{13}\right]+\frac{\omega_{\text{Re}}}{2H_3}\left[P_{S31}\Delta\delta_{13}+P_{S32}\Delta\delta_{23}\right]=0$$

$$\frac{d^2\Delta\delta_{23}}{dt^2}+\frac{\omega_{\text{Re}}}{2H_2}\left[P_{S21}\Delta\delta_{23}-P_{S21}\Delta\delta_{13}+P_{S23}\Delta\delta_{23}\right]+\frac{\omega_{\text{Re}}}{2H_3}\left[P_{S31}\Delta\delta_{13}+P_{S32}\Delta\delta_{23}\right]=0$$

Gathering terms in each variable, we get two differential equations:

$$\frac{d^2\Delta\delta_{13}}{dt^2}+\underbrace{\left[\frac{\omega_{\text{Re}}}{2H_1}P_{S12}+\frac{\omega_{\text{Re}}}{2H_1}P_{S13}+\frac{\omega_{\text{Re}}}{2H_3}P_{S31}\right]}_{\alpha_{11}}\Delta\delta_{13}+\underbrace{\left[\frac{\omega_{\text{Re}}}{2H_3}P_{S32}-\frac{\omega_{\text{Re}}}{2H_1}P_{S12}\right]}_{\alpha_{12}}\Delta\delta_{23}=0$$

$$\frac{d^2\Delta\delta_{23}}{dt^2}+\underbrace{\left[\frac{\omega_{\text{Re}}}{2H_3}P_{S31}-\frac{\omega_{\text{Re}}}{2H_2}P_{S21}\right]}_{\alpha_{21}}\Delta\delta_{13}+\underbrace{\left[\frac{\omega_{\text{Re}}}{2H_2}P_{S21}+\frac{\omega_{\text{Re}}}{2H_2}P_{S23}+\frac{\omega_{\text{Re}}}{2H_3}P_{S32}\right]}_{\alpha_{22}}\Delta\delta_{23}=0$$

Denote the coefficients of the above differential equations as α_{11}, α_{12}, α_{21}, and α_{22}, where (assuming the last equation, for bus n, is the one that gets subtracted off in the above steps):

$$\alpha_{ii}=\left\{\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{\omega_{\text{Re}}}{2H_i}P_{Sij}\right\}+\frac{\omega_{\text{Re}}}{2H_n}P_{Sni} \qquad \alpha_{ij}=\frac{\omega_{\text{Re}}}{2H_n}P_{Snj}-\frac{\omega_{\text{Re}}}{2H_i}P_{Sij}$$

Note these "alpha" expressions are the negation of the A_{ij}'s given in VMAF, pg. 68, (or in 2nd edition, in text's addendum, pg. 650), because the alphas are defined on the left-hand-side of the equation, whereas the A_{ij}'s are defined on the right-hand-side of the equation.

(3.29) can be further modified as

$$\frac{d^2\delta_{in\Delta}}{dt^2} + \sum_{j=1}^{n-1} \alpha_{ij}\delta_{jn\Delta} = 0 \quad i = 1, 2, \ldots, n-1 \tag{3.31}$$

where the coefficients α_{ij} depend on the machine inertias and synchronizing power coefficients.

Equation (3.31) represents a set of $n-1$ linear second-order differential equations or a set of $2(n-1)$ first-order differential equations. We will use the latter formulation to examine the free response of this system.

Let $x_1, x_2, \ldots, x_{n-1}$ be the angles $\delta_{1n\Delta}, \delta_{2n\Delta}, \ldots, \delta_{(n-1)n\Delta}$, respectively, and let $x_n, \ldots, x_{2n-2}$ be the time derivatives of these angles. The system equations are of the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdots \\ \dot{x}_{n-1} \\ \dot{x}_n \\ \dot{x}_{n+1} \\ \cdots \\ \dot{x}_{2n-2} \end{bmatrix} = \left[\begin{array}{cccc|cccc} & & & & 1 & 0 & \cdots & 0 \\ & & & & 0 & 1 & \cdots & 0 \\ & & 0 & & \cdots & \cdots & \cdots & \cdots \\ & & & & 0 & 0 & \cdots & 1 \\ \hline A_{11} & A_{12} & \cdots & A_{1,n-1} & & & & \\ A_{21} & A_{22} & \cdots & A_{2,n-1} & & & & \\ \cdots & \cdots & \cdots & \cdots & & & 0 & \\ A_{n-1,1} & A_{n-1,2} & \cdots & A_{n-1,n-1} & & & & \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_{n-1} \\ x_n \\ x_{n+1} \\ \cdots \\ x_{2n-2} \end{bmatrix} \tag{3.32}$$

where

$$A_{ii} = -\sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{\omega_R}{2H_i} P_{sij} - \frac{\omega_R}{2H_n} P_{sni}$$

$$A_{ij} = \frac{\omega_R}{2H_i} P_{sij} - \frac{\omega_R}{2H_n} P_{snj}$$

where n is the number of machines and a machine n is the reference.

Using the alphas (given p. 5 of these notes), I rewrite the differential equations as

$$\frac{d^2\Delta\delta_{13}}{dt^2} + \alpha_{11}\Delta\delta_{13} + \alpha_{12}\Delta\delta_{23} = 0$$

$$\frac{d^2\Delta\delta_{23}}{dt^2} + \alpha_{21}\Delta\delta_{13} + \alpha_{22}\Delta\delta_{23} = 0$$

We can now convert these second order linear differential equations into first order linear differential equations, in order to develop a state-space form.

We do this by recognizing that

$$\Delta\omega_{13} = \frac{d\delta_{13}}{dt}, \qquad \Delta\omega_{23} = \frac{d\delta_{23}}{dt}$$

Then, the above two second order differential equations become four first order differential equations, as follows:

$$\Delta\dot{\omega}_{13} = -\alpha_{11}\Delta\delta_{13} - \alpha_{12}\Delta\delta_{23} = 0$$
$$\Delta\dot{\omega}_{23} = -\alpha_{21}\Delta\delta_{13} - \alpha_{22}\Delta\delta_{23} = 0$$
$$\Delta\dot{\delta}_{13} = \Delta\omega_{13}$$
$$\Delta\dot{\delta}_{23} = \Delta\omega_{23}$$

So let's define the state vector as

$$\underline{x} = \begin{bmatrix} \Delta\delta_{13} \\ \Delta\delta_{23} \\ \Delta\omega_{13} \\ \Delta\omega_{23} \end{bmatrix}$$

Note for our 3-machine system, we have only 4 states due to the state reduction for relative angles and relative speeds.

Then

$$\underline{\dot{x}} = A\underline{x}$$

More explicitly,

$$\begin{bmatrix} \Delta\dot{\delta}_{13} \\ \Delta\dot{\delta}_{23} \\ \Delta\dot{\omega}_{13} \\ \Delta\dot{\omega}_{23} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\alpha_{11} & -\alpha_{12} & 0 & 0 \\ -\alpha_{21} & -\alpha_{22} & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\delta_{13} \\ \Delta\delta_{23} \\ \Delta\omega_{13} \\ \Delta\omega_{23} \end{bmatrix}$$

VMAF on page 71 shows the computation of the alpha-coefficients for the 9-bus, 3-generator system of Fig. 2.19. It is shown that

α_{11}=104.096

α_{12}=59.524

α_{21}=33.841

α_{22}=153.460

Then, the state-space equation is:

$$\begin{bmatrix} \Delta\dot{\delta}_{13} \\ \Delta\dot{\delta}_{23} \\ \Delta\dot{\omega}_{13} \\ \Delta\dot{\omega}_{23} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -104.096 & -59.524 & 0 & 0 \\ -33.841 & -153.460 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\delta_{13} \\ \Delta\delta_{23} \\ \Delta\omega_{13} \\ \Delta\omega_{23} \end{bmatrix}$$

Question is, now, what to do with the above in order to obtain useful information about the small-disturbance behavior of our system. We will investigate this next….