Preliminary Fundamentals

1.0 Introduction

In all of our previous work, we assumed a very simple model of the electromagnetic torque T_e (or power) that is required in the swing equation to obtain the accelerating torque.

This simple model was based on the assumption that there are no dynamics associated with the machine internal voltage. This is not true. We now want to construct a model that will account for these dynamics.

To do so, we first need to ensure that we have adequate background regarding preliminary fundamentals, which include some essential electromagnetic theory, and basics of synchronous machine construction & operation.

2.0 Some essential electromagnetic theory

2.1 Self inductance

Self inductance indicates the magnitude of the magnetic coupling between a circuit and itself. It is given, with units of henries, by

$$L_{11} = \frac{\lambda_{11}}{i_1} \tag{1}$$

We see that the self-inductance L_{11} is the ratio of

- the flux ϕ_{11} from coil 1 linking with coil 1, λ_{11}
- to the current in coil 1, i_1.

Since the flux linkage λ_{11} is the flux ϕ_{11} linking with coil 1, and since this flux "links" once per turn, and since the number of turns is N_1, then

$$\lambda_{11} = N_1 \phi_{11} \tag{2}$$

2.2 Faraday's law

Any change of flux linkages seen by a circuit induces a voltage in that circuit. The induced voltage is given by

$$e = \frac{d\lambda}{dt} = \frac{d(Li)}{dt} \tag{3}$$

where (3) accounts for the case of time variation in L, i, or both. If L does not vary with time, then

$$e = \frac{d\lambda}{dt} = L\frac{di}{dt} \tag{4}$$

2.3 Mutual inductance

For a pair of circuits, the mutual inductance L_{12} is

$$L_{12} = \frac{\lambda_{12}}{i_2} \tag{5}$$

We observe that L_{12} is the ratio of

- the flux from coil 2 linking with coil 1, λ_{12}
- to the current in coil 2, i_2.

More generally, for a group of circuits labeled 1, 2, …, we see that

$$\begin{aligned} \lambda_1 &= L_{11} i_1 + L_{12} i_2 + \ldots \\ \lambda_2 &= L_{21} i_1 + L_{22} i_2 + \ldots \\ &\vdots \end{aligned} \tag{6}$$

Here, L_{11}, L_{22}, … are self inductances, and L_{12}, L_{21},… are mutual inductances. From (6), we see a more general definition of self and mutual inductances, according to:

$$L_{ii} = \frac{\partial \lambda_i}{\partial i_i} \tag{8}$$

$$L_{ij} = \frac{\partial \lambda_i}{\partial i_j} \qquad (9)$$

In the case of self inductance, because λ_i is produced by i_i their directionalities will always be consistent such that current increases produce flux linkage increases. Therefore L_{ii} is always positive.

In the case of mutual inductance, whether current increases in one circuit produce flux linkage increases in the other circuit depends on the directionality of the currents and fluxes. The rule we will use is this:

L_{ij} is positive if positive currents in the two circuits produce self and mutual fluxes in the same direction.

2.4 Inductance and magnetic circuits

We define magnetomotive force (MMF), as the "force" that results from a current i flowing in N turns of a conductor. We will denote it with $\mathcal{F}$, expressed by:

$$MMF = \mathcal{F} = Ni \qquad (10)$$

If the conductor is wound around a magnetic circuit having reluctance $\mathcal{R}$, then the MMF will cause flux to flow in the magnetic circuit according to

$$\varphi = \frac{\mathcal{F}}{\mathcal{R}} = \frac{Ni}{\mathcal{R}} \tag{11}$$

If the cross-sectional area A and permeability μ of the magnetic circuit is constant throughout, then

$$\mathcal{R} = \frac{l}{\mu A} \tag{12a}$$

where *l* is the mean length of the magnetic circuit.

The permeance is given by

$$\mathcal{P} = \frac{1}{\mathcal{R}} \tag{12b}$$

Magnetic circuit relations described above are analogous to Ohm's law for standard circuits, in the following way:

$$\mathcal{F} \rightarrow V,\ \phi \rightarrow I,\ \mathcal{R} \rightarrow R,\ \mathcal{P} \rightarrow Y \tag{13}$$

So that

$$\phi = \frac{\mathcal{F}}{\mathcal{R}} \rightarrow I = \frac{V}{R} \tag{14}$$

We also show in the appendix (see eqs (A8), (A9a)) that

$$L_{21} = L_{12} = \frac{N_1 N_2}{\mathcal{R}} \qquad\qquad L_{11} = \frac{N_1^{\ 2}}{\mathcal{R}} \tag{15}$$

2.5 Constant flux linkage theorem

Consider any closed circuit having

- finite resistance
- flux linkage due to any cause whatsoever
- other emf's e not due to change in λ
- no series capacitance

Then

$$\sum ri + \frac{d\lambda}{dt} = \sum e \tag{16}$$

We know that flux linkages can change, and (16) tells us how: whenever the balance between the emfs and the resistance drops become non-zero, i.e.,

$$\frac{d\lambda}{dt} = \sum e - \sum ri \tag{17}$$

But, can they change instantly, i.e., can a certain flux linkage λ change from 4 to 5 weber-turns in 0 seconds?

To answer this question, consider integrating (16) with respect to time t from t=0 to t=Δt. We obtain

$$\underbrace{\sum r\int_0^{\Delta t} i\,dt}_{Term1} + \underbrace{\int_0^{\Delta t} \frac{d\lambda}{dt}dt}_{Term2} = \underbrace{\sum \int_0^{\Delta t} e\,dt}_{Term3} \qquad (18)$$

Notice that these terms are, for the interval 0→Δt,

- Term 1: The area under the curve of i(t) vs. t.
- Term 2: The area under the curve of dλ/dt vs. t, which is Δλ(Δt) (read "delta lambda of delta t").
- Term 3: The area under the curve of e(t) vs. t.

Now we know that we can get an instantaneous (step) change in current

➔short the circuit or open the circuit,

and we know that we can get an instantaneous (step) change in voltage

➔open/close a switch to insert a voltage source into the circuit.

And so i(t) and/or e(t) may change instantaneously in (18). But consider applying the limit as Δt→0 to (18). In this case, we have:

$$\underbrace{\lim_{\Delta t \to 0} \sum r \int_0^{\Delta t} i dt}_{Term1} + \lim_{\Delta t \to 0} \underbrace{\Delta\lambda(\Delta t)}_{Term2} = \underbrace{\lim_{\Delta t \to 0} \sum \int_0^{\Delta t} e dt}_{Term3} \qquad (19)$$

Even with a step change in i(t) or e(t) at t=0, their integrals will be zero in the limit. Therefore we have:

$$0 + \lim_{\Delta t \to 0} \underbrace{\Delta\lambda(\Delta t)}_{Term2} = 0 \qquad (20)$$

This implication of (20) is that the flux linkages cannot change instantaneously. This is the constant-flux-linkage theorem (CFLT).

CFLT: In any closed electric circuit, the flux linkages will remain constant immediately after any change in

- The current
- The voltage

- The position of other circuits to which the circuit is magnetically coupled.

There is an easier way to prove CFLT: Faraday's law states that if λ changes instantly, then there in an infinite induced voltage, but this is impossible, and so it must be the case that λ cannot change instantly.

The CFLT is particularly useful when L_{ii} or L_{ij} of a circuit changes quickly. It allows us to assume λ stays constant so that we can obtain currents after the change as a function of currents before the change.

3.0 Basics of synchronous machines

2.1 Basic construction issues

In this section, we present only the very basics of the physical attributes of a synchronous machine. We will go into more detail regarding windings and modeling later.

The synchronous generator converts mechanical energy from the turbine into electrical energy.

The turbine converts some kind of energy (steam, water, wind) into mechanical energy, as illustrated in Fig. 1 [i].

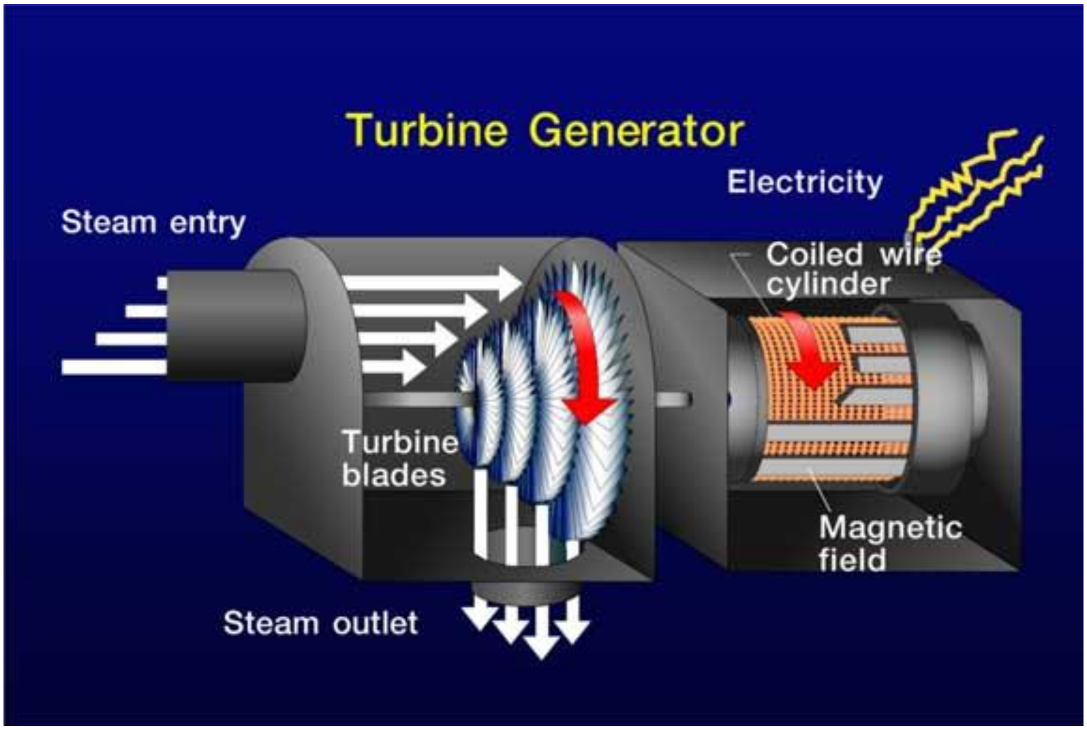

Fig. 1 [i]

The synchronous generator has two parts:

- Stator: carries 3 (3-phase) armature windings, AC, physically displaced from each other by 120 degrees
- Rotor: carries field windings, connected to an external DC source via slip rings and brushes or to a revolving DC source via a special *brushless* configuration.

Fig. 2 shows a simplified diagram illustrating the slip-ring connection to the field winding.

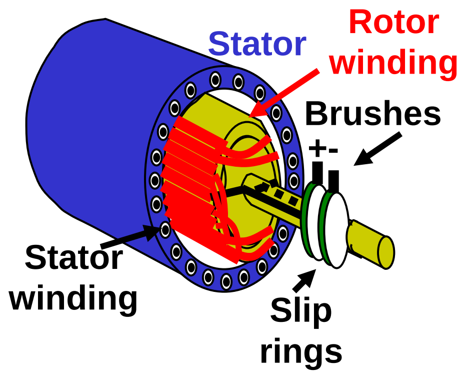

Fig. 2

Fig. 3 shows the rotor from a 200 MW steam generator. This is a smooth rotor.

Repairs to the overhand insulation of this 200MW generator rotor were carried out by experienced tradesmen working on shift to ensure the earliest possible return to service.

Fig. 3

Fig. 4 shows the rotor and stator of a hydro-generator, which uses a salient pole rotor.

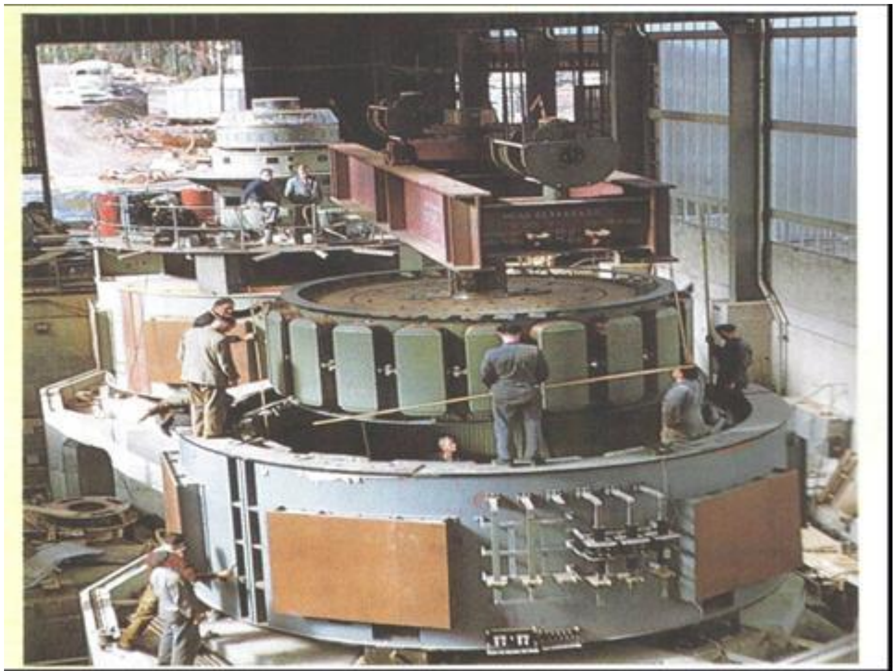

Fig. 4

Fig. 5 illustrates the synchronous generator construction for a salient pole machine, with 2 poles. Note that Fig. 5 only represents one "side" of each phase, so as to not crowd the picture too much. In other words, we should also draw the Phase A return conductor 180° away from the Phase A conductor shown in the picture. Likewise for Phases B and C.

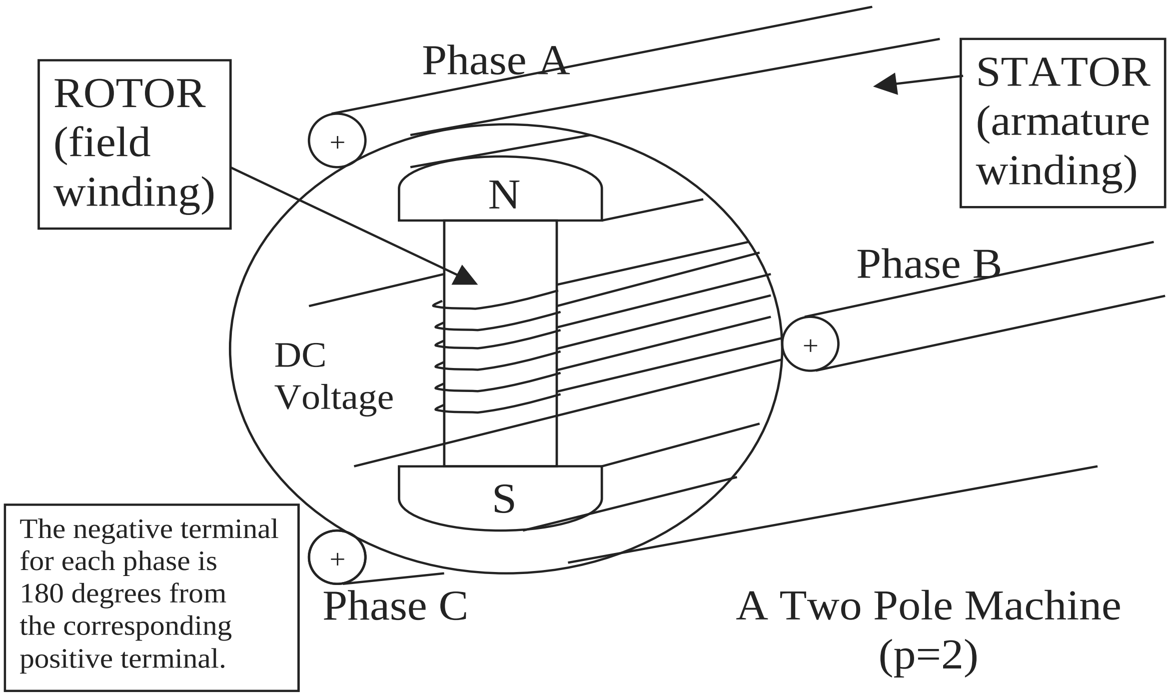

Fig. 5

Fig. 6 shows just the rotor and stator (but without stator winding) for a salient pole machine with 4 poles.

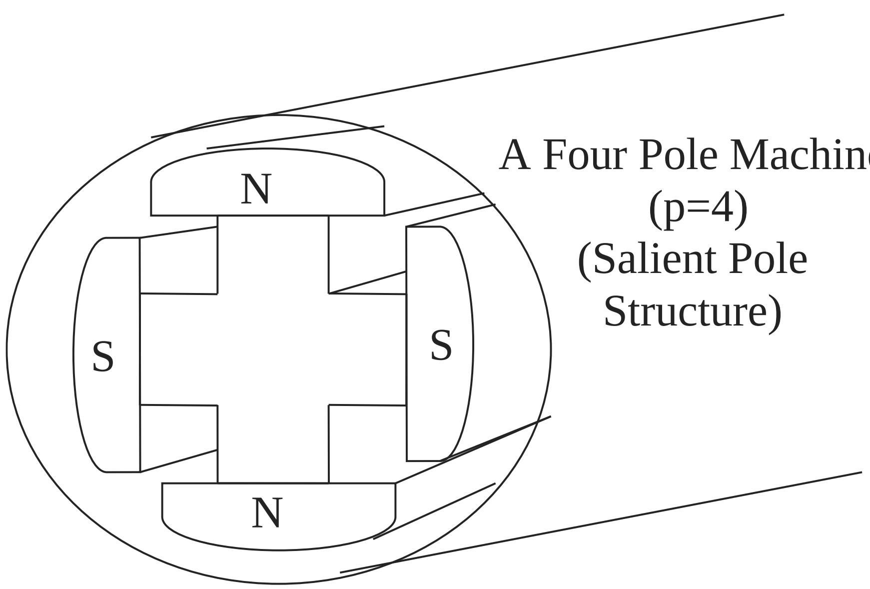

Fig. 6

The difference between smooth rotor construction and salient pole rotor construction is illustrated in Fig. 7. Note the air-gap in Fig. 7.

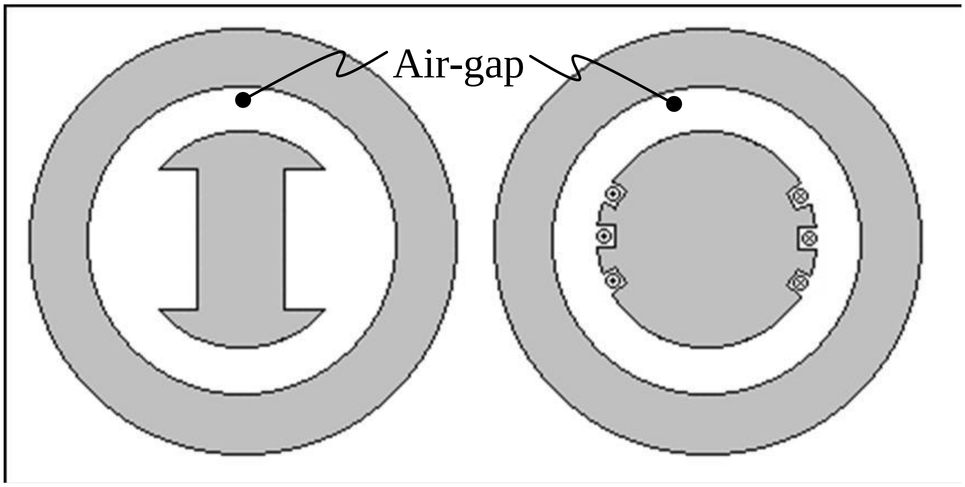

Fig. 7

We define synchronous speed as the speed for which the induced voltage in the armature (stator) windings is synchronized with (has same frequency as) the network voltage. Denote this as ω_{eR}.

In North America,

$$\omega_{eR}=2\pi(60)= 376.9911\approx 377 rad/sec$$

In Europe,

$$\omega_{eR}=2\pi(50)= 314.1593\approx 314 rad/sec$$

On an airplane,

$$\omega_{eR}=2\pi(400)= 2513.3\approx 2513 \text{rad/sec}$$

The mechanical speed of the rotor is related to the synchronous speed through:

$$\omega_m = \frac{2}{p}(\omega_e) \tag{21}$$

where both ω_m and ω_e are given in rad/sec. This may be easier to think of if we write

$$\omega_e = \frac{p}{2}(\omega_m) \tag{22}$$

Thus we see that, when p=2, we get one electric cycle for every one mechanical cycle. When p=4, we get two electrical cycles for every one mechanical cycle.

If we consider that ω_{eR} must be constant from one machine to another, then machines with more poles must rotate more slowly than machines with less.

It is common to express ω_{mR} in RPM, denoted by N; we may easily derive the conversion from analysis of units:

N_{mR}=(ω_m rad/sec)*(1 rev/2π rad)*(60sec/min)

= (30/π)ω_{mR}

Substitution of ω_{mR}=(2/p) ω_{eR}=(2/p)2πf=4πf/p

$$N_{mR}= (30/\pi)(4\pi f/p)=120f/p \tag{23}$$

Using (3), we can see variation of N_{mR} with p for f=60 Hz, in Table 1.

Table 1

No. of Poles (p)	Synchronous speed (N_{mR})
2	3600
4	1800
6	1200
8	900
10	720
12	600
14	514
16	450
18	400
20	360
24	300
32	225
40	180

Because steam-turbines are able to achieve high speeds, and because operation is more efficient at those speeds (due to use of higher pressure steam), most steam turbines are 2 pole, operating at 3600 RPM. At this rotational speed, the surface speed of a 3.5 ft diameter rotor is about 450 mile/hour. Salient poles incur very high mechanical stress and windage losses at this speed and therefore cannot be used. All steam-turbines use smooth rotor construction.

Because hydro-turbines cannot achieve high speeds, they must use a higher number of poles, e.g., 24 and 32 pole hydro-machines are common. But because salient pole construction is less expensive, all hydro-machines use salient pole construction.

Fig. 8 illustrates several different constructions for smooth and salient-pole rotors. The red arrows indicate the direction of the flux produced by the field windings.

- **Synchronous generator**

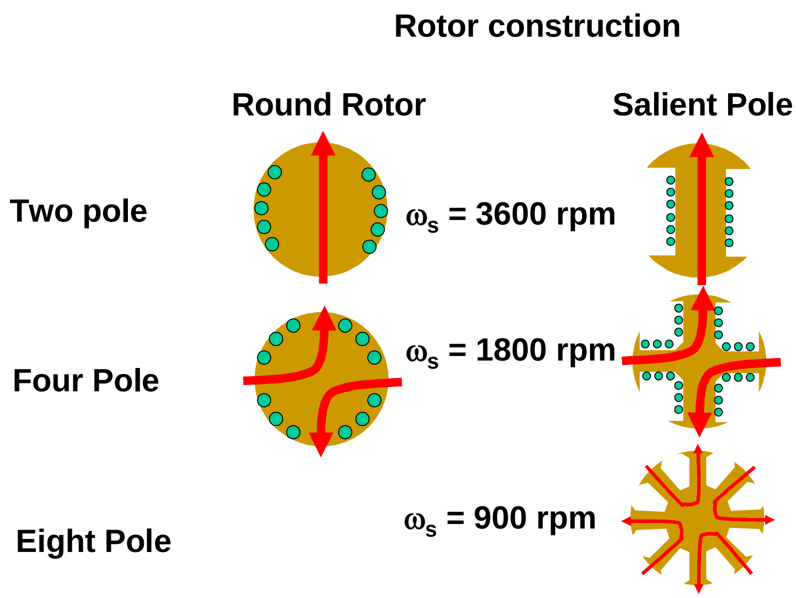

Fig. 8

The synchronous machine typically has two separate control systems – the speed governing system and the excitation system, as illustrated in Fig. 9 below. Our main interest in this course is synchronous machine modeling. However, we will still touch on a few issues related to the control systems.

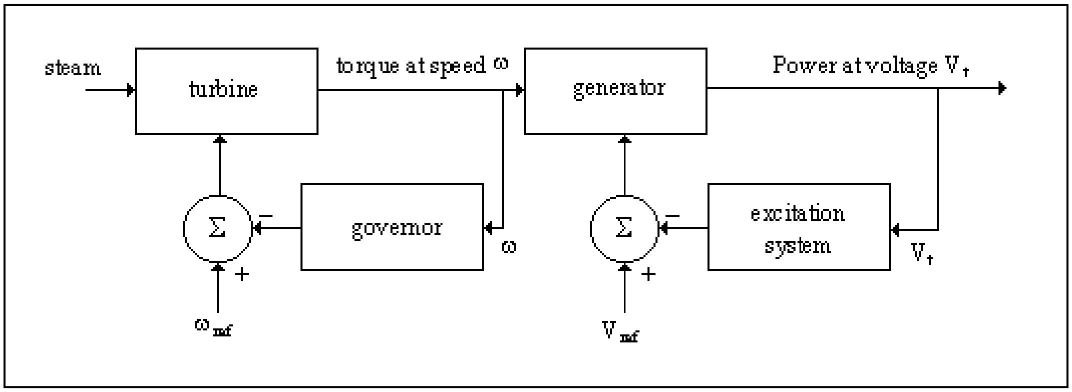

Fig. 9

2.2 Rotating magnetic field

The following outlines the conceptual steps associated with production of power in a synchronous generator.

1. DC is supplied to the field winding.
2. If the rotor is stationary, the field winding produces magnetic flux which is strongest radiating outwards from the center of the pole face and diminishes with distance along the air-gap away from the pole face center. Figure 10 illustrates. The left-hand-figure plots flux density as a function of angle from the main axis. The right-hand-figure shows the main axis and the lines of flux. The angle θ measures the point on the stator from the main axis, which is the a-phase axis. In this particular case, we have aligned the main axis with the direct-axis of the rotor.

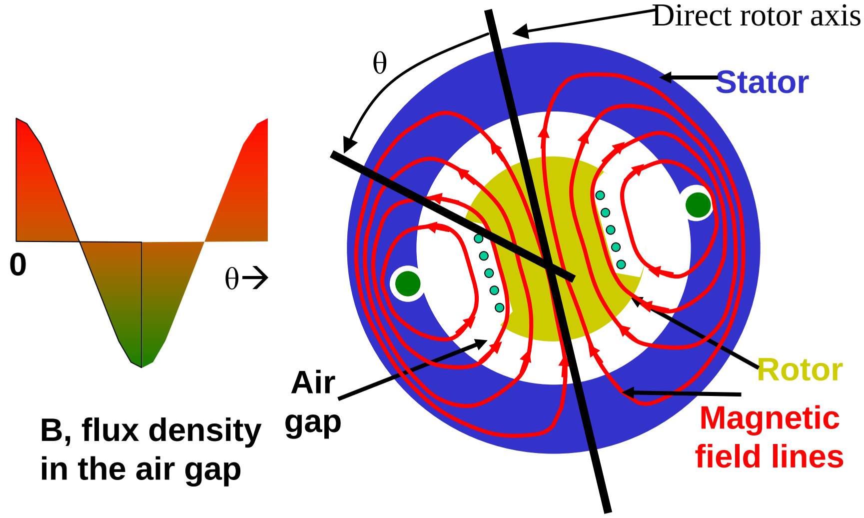

Fig. 10

3. The turbine rotates the rotor. This produces a **<u>rotating magnetic field</u>** (or a sinusoidal traveling wave) in the air gap, i.e., the plot on the left of Fig. 10 “moves” with time. Figure 11 illustrates, where we see that, *for fixed time (just one of the plots), there is sinusoidal variation of flux density with space.* Also, if we stand on a single point on the stator (e.g., θ=90°) and measure B as a function of time, we see that *for fixed space (the vertical dotted line at 90°, and the red eye on the pictures to the right), there is sinusoidal variation of flux density w/time*.

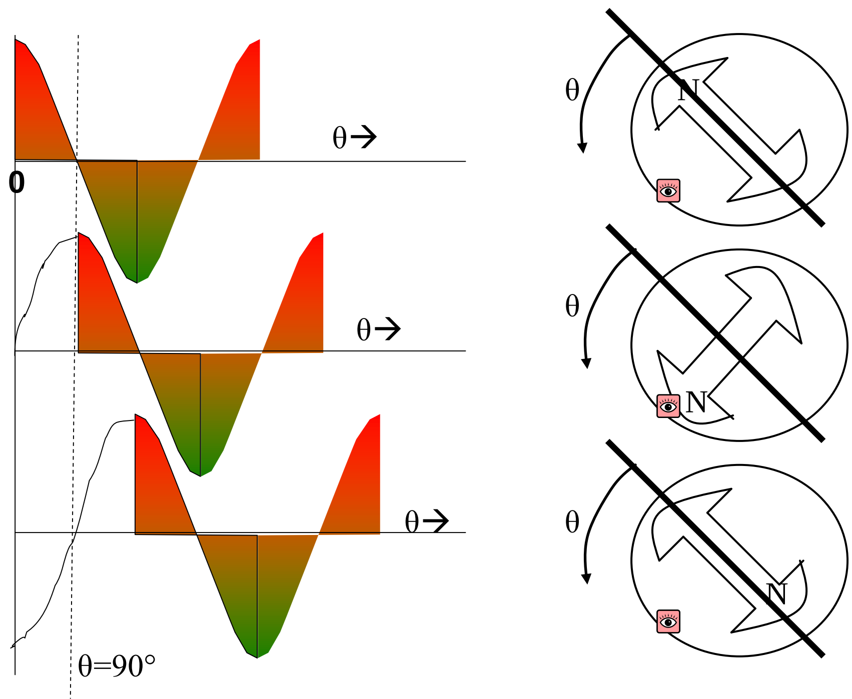

Fig. 11

4. Given that the stator windings, which run down the stator sides parallel to the length of the generator are fixed on the stator (like the eye of Fig. 11), those conductors will see a time varying flux. Thus, by Faraday's law, a voltage will be induced in those conductors.
   a. Because the phase windings are spatially displaced by 120°, then we will get voltages that are time-displaced by 120°.
   b. If the generator terminals are open-circuited, then the amplitude of the voltages are proportional to
      - Speed

- Magnetic field strength

And our story ends here if generator terminals are open-circuited.

5. If, however, the phase (armature) windings are connected across a load, then current will flow in each one of them. Each one of these currents will in turn produce a magnetic field. So there will be 4 magnetic fields in the air gap. One from the rotating DC field winding, and one each from the three stationary AC phase windings.
6. The three magnetic fields from the armature windings will each produce flux densities, and the composition of these three flux densities result in a single **rotating magnetic field** in the air gap. We develop this here…. Consider the three phase currents:

$$\begin{aligned} i_a &= I\cos\omega_e t \\ i_b &= I\cos(\omega_e t - 120°) \\ i_c &= I\cos(\omega_e t - 240°) \end{aligned} \tag{24}$$

Now, whenever you have a current carrying coil, it will produce a magnetomotive force (MMF) equal to *Ni*. And so each of the above three currents produce a time varying MMF around the stator. Each MMF will have a maximum in space, occurring on the axis of the phase, of $\mathcal{F}_{am}$, $\mathcal{F}_{bm}$, $\mathcal{F}_{cm}$, expressed as

$$\begin{aligned} \mathcal{F}_{am}(t) &= F_m \cos\omega_e t \\ \mathcal{F}_{bm}(t) &= F_m \cos(\omega_e t - 120^\circ) \\ \mathcal{F}_{cm}(t) &= F_m \cos(\omega_e t - 240^\circ) \end{aligned} \tag{25}$$

Recall that the angle θ is measured from the a-phase axis, and consider points in the airgap. At any time *t*, the spatial maximums expressed above occur on the axes of the corresponding phases and vary sinusoidally with θ around the air gap. We can combine the time variation with the spatial variation in the following way:

$$\begin{aligned} \mathcal{F}_a(\theta,t) &= \mathcal{F}_{am}(t)\cos\theta \\ \mathcal{F}_b(\theta,t) &= \mathcal{F}_{bm}(t)\cos(\theta - 120^\circ) \\ \mathcal{F}_c(\theta,t) &= \mathcal{F}_{cm}(t)\cos(\theta - 240^\circ) \end{aligned} \tag{26}$$

Note each individual phase MMF in (26)

- varies with θ around the air gap and
- has an amplitude that varies with time.

Substitution of (25) into (26) yields:

$$\mathcal{F}_a(\theta,t) = F_m \cos\omega_e t \cos\theta$$
$$\mathcal{F}_b(\theta,t) = F_m \cos(\omega_e t - 120°)\cos(\theta - 120°) \quad (27)$$
$$\mathcal{F}_c(\theta,t) = F_m \cos(\omega_e t - 240°)\cos(\theta - 240°)$$

Now do the following:

- Add the three MMFs in (27):

$$\begin{aligned}\mathcal{F}(\theta,t) &= \mathcal{F}_a(\theta,t) + \mathcal{F}_b(\theta,t) + \mathcal{F}_c(\theta,t) \\ &= F_m \cos\omega_e t \cos\theta \\ &+F_m \cos(\omega_e t - 120°)\cos(\theta - 120°) \\ &+F_m \cos(\omega_e t - 240°)\cos(\theta - 240°)\end{aligned} \quad (28)$$

- Use *cosαcosβ=0.5[cos(α-β)+cos(α+β)]* and then simplify, and you will obtain:

$$\mathcal{F}(\theta,t) = \frac{3}{2}F_m \cos(\omega_e t - \theta) \quad (29)$$

Equation (29) characterizes a rotating magnetic field, as illustrated in Fig. 11.

7. This rotating magnetic field from the armature will have the same speed as the rotating magnetic field from the rotor, i.e., these two rotating magnetic fields are *in synchronism*.
8. The two rotating magnetic fields, that from the rotor and the composite field from the armature, are “locked

in,” and as long as they rotate in synchronism, a torque (Torque=P/ω_m=Force×radius, where Force is tangential to the rotor surface), is developed. This torque is identical to that which would be developed if two magnetic bars were fixed on the same pivot [ii, pg. 171] as shown in Fig 3. In the case of synchronous generator operation, we can think of bar A (the rotor field) as pushing bar B (the armature field), as in Fig. 12a. In the case of synchronous motor operation, we can think of bar B (the armature field) as pulling bar A (the rotor field), as in Fig. 12b.

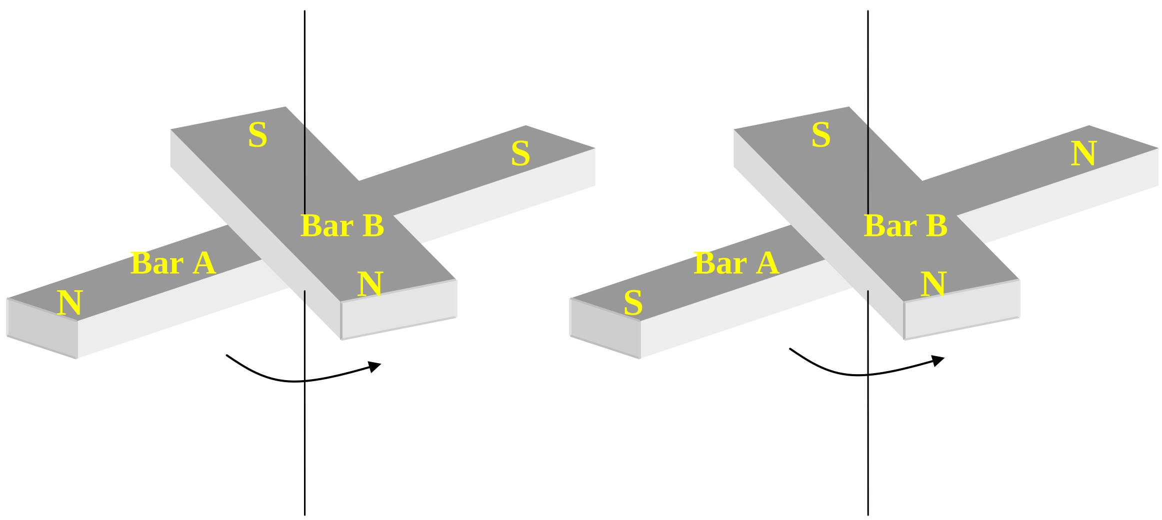

Fig 12a: Generator operation

Fig 12b: Motor operation

Fig. 12

Appendix: Mutual inductance

Let's consider another arrangement as shown in Fig. A1 below.

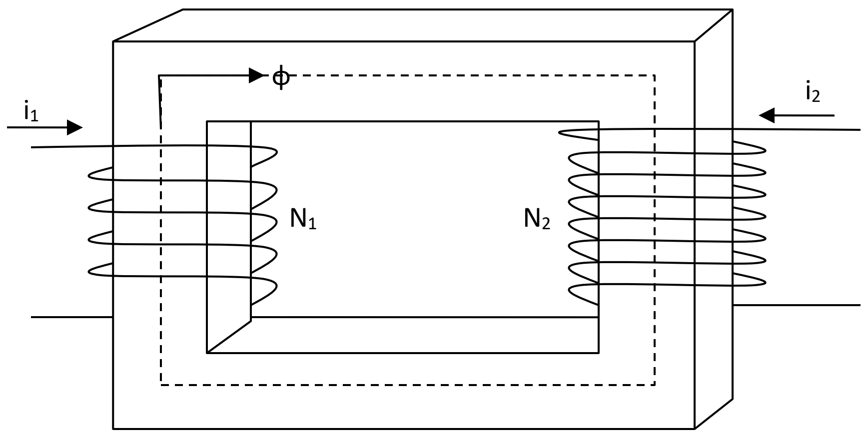

Fig. A1

We have for each coil:

$$L_{11} = \frac{\lambda_{11}}{i_1} \quad \text{(A1)}$$

$$L_{22} = \frac{\lambda_{22}}{i_2} \quad \text{(A2)}$$

We can also define L_{12} and L_{21}.

L_{12} is the ratio of

- the flux from coil 2 linking with coil 1, λ_{12}
- to the current in coil 2, i_2.

That is,

$$L_{12} = \frac{\lambda_{12}}{i_2} \quad \text{(A3)}$$

where the first subscript, 1 in this case, indicates "links with coil 1" and the second subscript, 2 in this case, indicates "flux from coil 2."

Here, we also have that

$$\lambda_{12} = N_1\phi_{12} \Rightarrow L_{12} = \frac{N_1\phi_{12}}{i_2} \tag{A4}$$

Likewise, we have that

$$L_{21} = \frac{\lambda_{21}}{i_1} \tag{A5a}$$

$$\lambda_{21} = N_2\phi_{21} \Rightarrow L_{21} = \frac{N_2\phi_{21}}{i_1} \tag{A5b}$$

Now let's assume that all flux produced by each coil links with the other coil. The implication of this is that there is no leakage flux, as illustrated in Fig. A2.

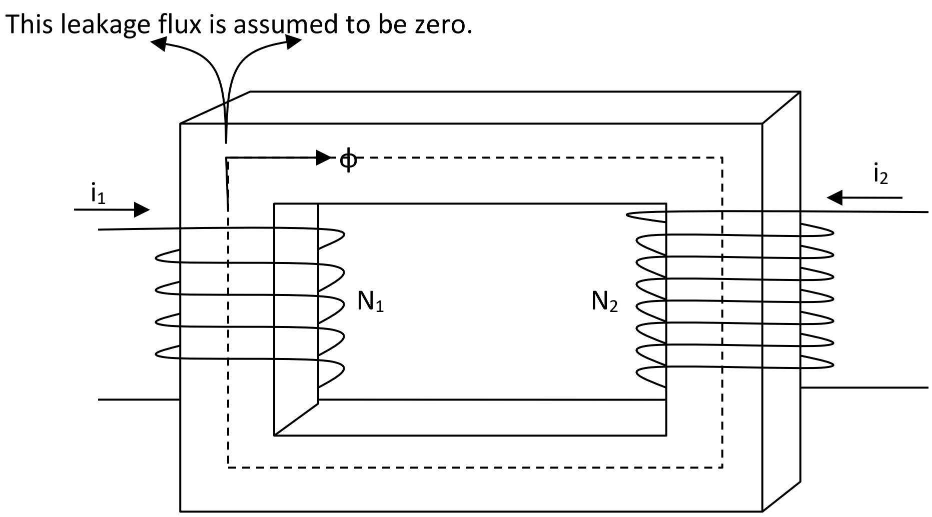

Fig. A2

Although in reality there is some leakage flux, it is quite small because the iron has much less reluctance than the air. With this assumption, then we can write:

- the flux from coil 2 linking with coil 1 is equal to the flux from coil 2 linking with coil 2, i.e.,

$$\phi_{12} = \phi_{22} = \frac{\mu A}{l} N_2 i_2 \tag{A6a}$$

- the flux from coil 1 linking with coil 2 is equal to the flux from coil 1 linking with coil 1, i.e.,

$$\phi_{21} = \phi_{11} = \frac{\mu A}{l} N_1 i_1 \tag{A6b}$$

Substitution of (A6a) and (A6b) into (A4) and (A5b), respectively, results in:

$$L_{12} = \frac{N_1 \varphi_{12}}{i_2} = \frac{N_1 \frac{\mu A}{l} N_2 i_2}{i_2} = N_1 N_2 \frac{\mu A}{l} = \frac{N_1 N_2}{\mathcal{R}} \tag{A7a}$$

$$L_{21} = \frac{N_2 \varphi_{21}}{i_1} = \frac{N_2 \frac{\mu A}{l} N_1 i_1}{i_1} = N_2 N_1 \frac{\mu A}{l} = \frac{N_2 N_1}{\mathcal{R}} \tag{A7b}$$

Examination of (A7a) and (A7b) leads to

$$L_{21} = L_{12} = \frac{N_1 N_2}{\mathcal{R}} \tag{A8}$$

Also recall

$$L = \frac{N^2}{\mathcal{R}}$$

or in subscripted notation

$$L_{11} = \frac{N_1^{\,2}}{\mathcal{R}} \tag{A9a}$$

$$L_{22} = \frac{N_2^{\,2}}{\mathcal{R}} \tag{A9b}$$

Solving for N_1 and N_2in (A9a) and (A9b) results in

$$N_1 = \sqrt{L_{11} \mathcal{R}} \tag{A10a}$$

$$N_2 = \sqrt{L_{22}\mathscr{R}} \tag{A10b}$$

Now substitute (A10a) and (A10b) into (A8) to obtain

$$L_{21} = L_{12} = \frac{\sqrt{L_{11}\mathscr{R}}\sqrt{L_{22}\mathscr{R}}}{\mathscr{R}} = \sqrt{L_{11}L_{22}} \tag{21}$$

Definition: L_{12}=L_{21} is the mutual inductance and is often denoted M.

Mutual inductance gives the ratio of

- flux from coil k linking with coil j, λ_{jk}
- to the current in coil k, i_k,

That is,

$$M = \begin{cases} \dfrac{\lambda_{12}}{i_2} \\ \dfrac{\lambda_{21}}{i_1} \end{cases}$$

---

[i] http://geothermal.marin.org/GEOpresentation/
[ii] A. Fitzgerald, C. Kingsley, and A. Kusko, “Electric Machinery, Processes, Devices, and Systems of Electromechanical Energy Conversion,” 3rd edition, 1971, McGraw Hill.