Multi-machine Systems

1.0 Introduction and assumptions

We have so far studied the one-machine-infinite bus and the two-machine system. Now we extend analysis to a multi-machine system. We will revisit this topic later in the course when we have more fully developed the synchronous machine model. *This chapter 2 material provides a preliminary, simplified view multi-machine modeling*; a deeper view is provided in Chapters 5 and 7.

Section 2.9 provides 5 assumptions, four of which amount to use of the classical machine model (P_m is constant, no damping, constant internal voltage magnitude, and mechanical rotor angle coincides with angle of internal voltage).

We will relieve the assumption of no damping. Therefore our swing equation (with per-unit power) will appear as:

$$\frac{2H}{\omega_{\mathrm{Re}}}\dot{\omega}(t) = P_m - P_e - D\omega \qquad (1)$$

A conservative estimate for damping frequently used in industry is D=2.0 when power is in per-unit.

Another assumption is that all loads are represented by passive impedances, i.e., all load is assumed to be constant impedance load. This is not accurate because loads respond to voltage and frequency deviation in a variety of ways that impedances do not capture. In general, stability performance is improved with constant impedance loads because the power decreases as voltage declines with the square of the voltage. Therefore, representing loads as constant impedance results in optimistic results (stability limits, for example, will be identified higher than they should be).

Use of the constant impedance load model allows us to catch a first glimpse into what a full-blown stability program looks like, with relatively little work, and so we will use it temporarily. In Chapter 6, we will look at load modeling in great detail.

2.0 Accounting for the network

We will have a swing equation for every machine in the network, i.e., for machine i, we will have

$$\frac{2H_i}{\omega_{\text{Re}}}\dot{\omega}_i(t) = P_{mi} - P_{ei} - D_i\omega_i \tag{2}$$

We have previously accounted for the network effects in our one-machine-infinite-bus system by expressing

$$P_{ei} = \frac{EV}{X_k}\sin\delta, \qquad k = 1,2,3 \tag{3}$$

where k=1,2,3 denotes pre-fault, fault-on, and post-fault systems, respectively; X_k is the reactance of the transfer impedance between the internal machine voltage and the infinite bus.

For the simple networks considered so far, we have been able to easily determine X_k. It is not possible to do so for general networks, however, because they are typically large & complex, & because there is no true infinite bus.

Now, we will need to utilize the real power flow equation, which is expressed by eq. (2.55) of your text as

$$P_{ei} = E_i^2 G_{ii} + \sum_{\substack{j=1 \\ j\neq i}} E_i E_j Y_{ij} \cos(\theta_{ij} - \delta_i + \delta_j) \tag{4}$$

where

$$G_{ii} = \mathrm{Re}\left\{\overline{Y}_{ii}\right\}, \qquad Y_{ij} = \left|\overline{Y}_{ij}\right| \qquad \theta_{ij} = \angle \overline{Y}_{ij}$$

and $\overline{Y}_{ii}$ and $\overline{Y}_{ij}$ are elements of the network's Y-bus matrix.

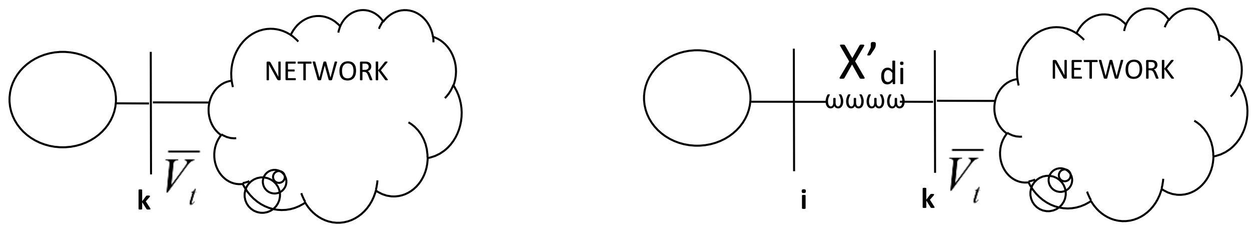
The Y-bus matrix needed here differs from the Y-bus matrix of the standard power flow model in three important ways:

1. Machine internal buses must be included;
2. Loads must be represented as constant impedance instead of constant power;
3. Change in network configuration between pre-fault, fault-on, and post-fault conditions must be modeled.

We will address each one of these changes in what follows.

2.1 Machine internal buses

Relative to the power flow model, we must form an extended network by inserting a new bus between E and the reactance X'_d. This is illustrated in Fig. 1 below.



Power flow representation Stability representation

Fig. 1

So we must increase the Y-bus matrix dimension by 1 for each generator in the system. For every additional internal machine node i connected to existing terminal bus k, we will need to modify the Y-bus according to

- Add new row and column for node i with diagonal element $Y_{ii}=y_{ik}=-j/X'_{di}$ and off-diagonal elements all zero except for $Y_{ik}=Y_{ki}=-y_{ik}=j/X'_{di}$ (upper case "Y" denotes Y-bus element; lower case "y" denotes admittance).
- Modify diagonal element Y_{kk} by adding to it $y_{ik}=-j/X'_{di}$.

We can visualize the above changes via the matrix below.

$$
\begin{array}{c}
\begin{array}{cc} \text{col i} & \text{col k} \\ \downarrow & \downarrow \end{array} \\
\begin{array}{r}
\\ \\ \text{row i} \rightarrow \\ \text{row k} \rightarrow \\ \\ \\
\end{array}
\begin{bmatrix}
\overline{Y}_{11} & \cdots & 0 & \overline{Y}_{1k} & \cdots & \overline{Y}_{1N} \\
\vdots & \cdots & 0 & \vdots & \cdots & \vdots \\
0 & 0 & \overline{Y}_{ii} & \overline{Y}_{ik} & 0 & 0 \\
\overline{Y}_{k1} & \cdots & \overline{Y}_{ki} & \overline{Y}_{kk} & \cdots & \overline{Y}_{kN} \\
\vdots & \cdots & 0 & \vdots & \cdots & \vdots \\
\overline{Y}_{N1} & \cdots & 0 & \overline{Y}_{Nk} & \cdots & \overline{Y}_{NN}
\end{bmatrix}
\end{array}
$$

The above modifications, when performed for all new generator nodes i, result in what we will call the *extended Y-bus matrix*.

Your text uses Fig. 2.17, shown below, to visualize the addition of the internal nodes. It represents the connecting impedance as r_i+jx'_{di} (instead of jX'_{di}); it uses n to denote number of machines (I use N to denote number of buses); and it shows loads as constant impedances, an issue we address in the next subsection.

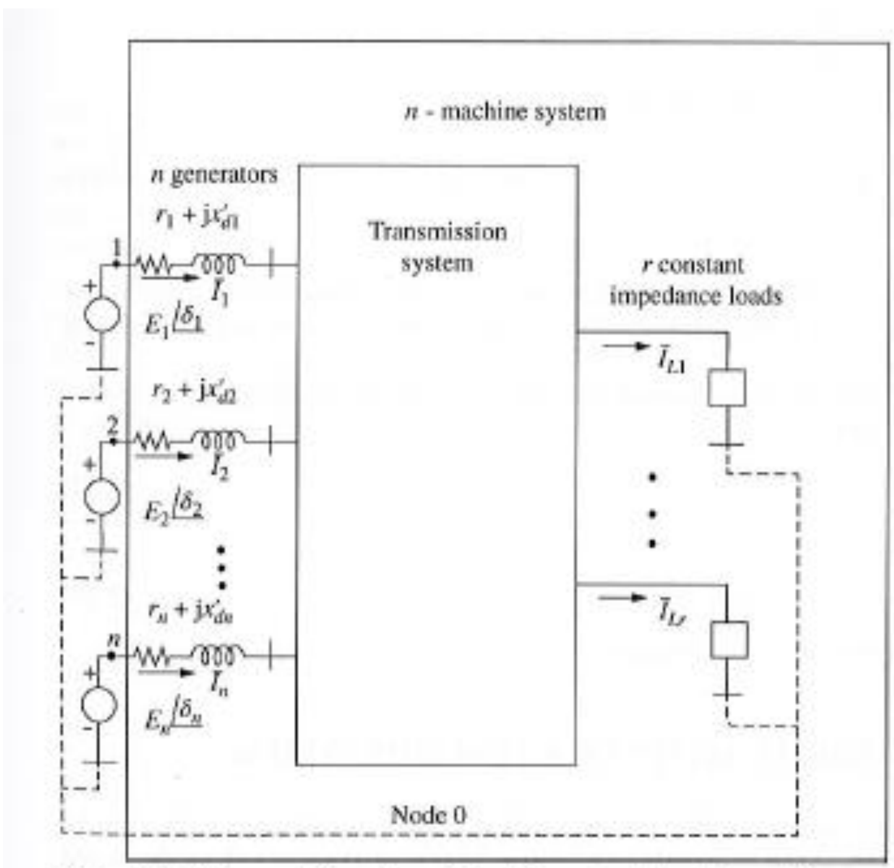


Figure 2.17 Representation of a multimachine system (classical model).

2.2 Representing loads as constant impedances

Loads are generally represented in the power flow model as constant power, P_L+jQ_L. We may convert such loads into constant impedance representation according to the following development:

$$P_L + jQ_L = V_L I_L^* \qquad \text{(in pu)} \tag{5}$$

where I_L is the current flowing into the load. To obtain the equivalent admittance Y_L for this load, assuming constant voltage, we have

$$I_L = V_L Y_L \Rightarrow I_L^* = V_L^* Y_L^* \tag{6}$$

Substituting (6) into (5) results in

$$P_L + jQ_L = V_L V_L^* Y_L^* = |V_L|^2 Y_L^* \tag{7}$$

Solving for Y_L results in

$$Y_L = \frac{P_L - jQ_L}{|V_L|^2} \tag{8}$$

This is eq. (2.60) in your text. Recall the issue mentioned on page 2 above: under transient simulation, the voltage $|V_L|$ is not constant, so the power drawn under voltage conditions that differ from the power flow voltage conditions will change, and this model enforces that the new power drawn will be proportional to the square of the voltage magnitude. We will accept this for now.

Now a question arises: Assume we have calculated Y_{Lk} for each load bus k.

➔Which Y-bus elements are affected by inclusion of these admittances? Off diagonal, diagonal, or both?

The answer is "diagonal" because the constant impedance loads are actually shunts from a bus to ground (accounted for only in diagonal Y-bus elements) and not from a bus to another bus (accounted for in both diagonal and off-diagonal elements).

2.3 Change in network configuration

Remember we will have at least 3 configurations to model: pre-fault, fault-on, post-fault. To address this, we will form 3 different Y-bus matrices, denoted as:

- Pre-fault: Y-bus matrix $\underline{Y}_1$
- Fault-on: Y-bus matrix $\underline{Y}_2$
- Post-fault: Y-bus matrix $\underline{Y}_3$

where each $\underline{Y}_i$ includes the "load shunts" discussed in Section 2.2 above.

Let's consider that there are n generator buses, numbered first, and r load buses, numbered last (buses without load or generation are also considered load buses). Then the Y-bus relation can be written as

$$\begin{bmatrix} \underline{I}_n \\ \underline{I}_r \end{bmatrix} = \begin{bmatrix} \underline{Y}_{nn} & \underline{Y}_{nr} \\ \underline{Y}_{rn} & \underline{Y}_{rr} \end{bmatrix} \begin{bmatrix} \underline{V}_n \\ \underline{V}_r \end{bmatrix} \tag{9}$$

However, if the loads are modeled as shunts, then they are included in the Y-bus matrix, and their corresponding current injections should be 0. (Loads may be modeled as constant currents in which case Kron distributes them to retained buses, in contrast to what is stated in the middle of VMAF, p.46). Therefore, (9) becomes:

$$\begin{bmatrix} \underline{I}_n \\ \underline{0} \end{bmatrix} = \begin{bmatrix} \underline{Y}_{nn} & \underline{Y}_{nr} \\ \underline{Y}_{rn} & \underline{Y}_{rr} \end{bmatrix} \begin{bmatrix} \underline{V}_n \\ \underline{V}_r \end{bmatrix} \tag{10}$$

(If gen buses have load, that load is modeled as constant impedance in the Y-bus as at any load bus; although current injection corresponding to load at that gen bus is 0, current injection corresponding to the gen at that bus is not 0).

Now we perform Kron reduction[1] on load bus voltages $\underline{V}_r$ from (10). First, multiply the bottom row by $-\underline{Y}_{nr}\underline{Y}_{rr}^{-1}$:

$$\begin{bmatrix} \underline{I}_n \\ \underline{0} \end{bmatrix} = \begin{bmatrix} \underline{Y}_{nn} & \underline{Y}_{nr} \\ -\underline{Y}_{nr}\underline{Y}_{rr}^{-1}\underline{Y}_{rn} & -\underline{Y}_{nr}\underline{Y}_{rr}^{-1}\underline{Y}_{rr} \end{bmatrix} \begin{bmatrix} \underline{V}_n \\ \underline{V}_r \end{bmatrix} \tag{11}$$

Note the term in row 2, column 2 is -$\underline{Y}_{nr}$, so that

$$\begin{bmatrix} \underline{I}_n \\ \underline{0} \end{bmatrix} = \begin{bmatrix} \underline{Y}_{nn} & \underline{Y}_{nr} \\ \underline{-\underline{Y}_{nr}\underline{Y}_{rr}^{-1}Y}_{rn} & -\underline{Y}_{nr} \end{bmatrix} \begin{bmatrix} \underline{V}_n \\ \underline{V}_r \end{bmatrix} \tag{12}$$

Now add the second row to the first row to obtain:

$$\begin{bmatrix} \underline{I}_n \\ \underline{0} \end{bmatrix} = \begin{bmatrix} \underline{Y}_{nn} - \underline{Y}_{nr}\underline{Y}_{rr}^{-1}\underline{Y}_{rn} & \underline{0} \\ -\underline{Y}_{nr}\underline{Y}_{rr}^{-1}\underline{Y}_{rn} & -\underline{Y}_{nr} \end{bmatrix} \begin{bmatrix} \underline{V}_n \\ \underline{V}_r \end{bmatrix} \tag{13}$$

And we see that the first row provides an equation that is independent of the load bus voltages V_r, resulting in

$$\underline{I}_n = \left[\underline{Y}_{nn} - \underline{Y}_{nr}\underline{Y}_{rr}^{-1}\underline{Y}_{rn}\right]\left[\underline{V}_n\right] \tag{14}$$

This is eqt. (2.65) in your text. The implication of (14) is we have reduced the network to generator nodes only.

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[1] Kron reduction is named after its inventor, GE power engineer Gabriel Kron (1901-1968). It is heavily used in power systems to obtain reduced equivalent networks. Kron reduction may be thought of as the matrix equivalent of Gaussian elimination. Kron reduction is illustrated in Section 2.10.2 of VMAF (step 5) but not called "Kron reduction" there. It is called "Kron reduction" on pg. 249 at the beginning of Sec 7.7 and mentioned there that what is illustrated in Section 2.10.2 is Kron reduction. In Kron reduction, the analyst designates retained buses, and all other buses are eliminated. If there are load buses with load modeled as current injections, then the formulation is as follows:

$$\begin{bmatrix} I_1 \\ 0 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} 0 \\ I_S \end{bmatrix}$$

where Y_{11} is the square submatrix of retained buses, Y_{22} is the square submatrix of buses to be eliminated, I_1 are current injections from loads on retained buses, and I_s are current injections from loads on buses to be eliminated. The Y-bus of the reduced network is found from

$$[Y_{11}'] = [Y_{11}] - [Y_{12}][Y_{22}]^{-1}[Y_{21}]$$

Load is distributed from eliminated buses to retained buses according to

$$[I_S'] = -[Y_{12}][Y_{22}]^{-1}[I_S]$$

Gen is typically not included in this reduction process but rather moved to retained buses according to heuristic rules to retain gen unit identity.

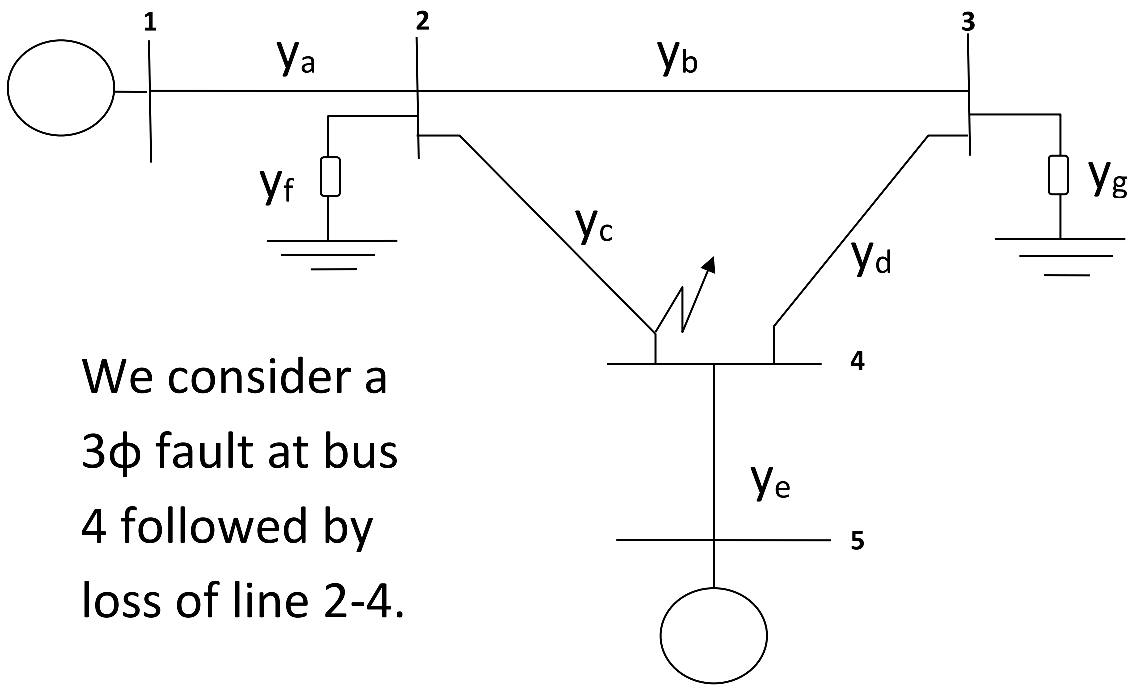
We introduce additional Y-bus notation; in all cases we assume we already included X'_d to obtain extended Y-bus.

Subscripts 1, 2, 3 indicate pre-fault, fault-on, and post fault, respectively. Double prime means unreduced, without load shunts; prime means unreduced, with load shunts; unprimed means reduced, with load shunts. Arrows show transitions from one condition to another.

Notation	**Condition**	**Reduction**	**Load shunts?**
$\underline{Y}_1''$	Pre-fault	Unreduced	Without
$\underline{Y}_1'$	Pre-fault	Unreduced	With
$\underline{Y}_1$	Pre-fault	Reduced	With
$\underline{Y}_2'$	Fault-on	Unreduced	With
$\underline{Y}_2$	Fault-on	Reduced	With
$\underline{Y}_3'$	Post-fault	Unreduced	With
$\underline{Y}_3$	Post-fault	Reduced	With

There are seven basic steps to perform in preparing to run a stability simulation. We will show these steps using a simple example, illustrated in the 5-bus systems of Fig. 2.

Notice that y_f and y_g are load shunts, and buses 1 and 5 are internal machine nodes.



We consider a 3ϕ fault at bus 4 followed by loss of line 2-4.

Fig. 2

We begin from $\underline{Y}_1''$, the pre-fault, unreduced Y-bus without load shunts.

1. Include load shunts using modification matrix $\underline{Y}_L$, where we add it to $\underline{Y}''_1$ since we are adding elements.

$$\underline{Y}_1{}' = \underline{Y}_1{}'' + \underline{Y}_L$$

where $\underline{Y}_L$ contains Y_{Li} in the diagonal element corresponding to load bus i. So reference to Fig. 2 indicates we need to include y_f and y_g in diagonal elements corresponding to buses 2 and 3.

$$\underline{Y}_1{}' = \underline{Y}_1{}'' + Y_L = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} & Y_{15} \\ Y_{21} & Y_{22} & Y_{23} & Y_{24} & Y_{25} \\ Y_{31} & Y_{32} & Y_{33} & Y_{34} & Y_{35} \\ Y_{41} & Y_{42} & Y_{43} & Y_{44} & Y_{44} \\ Y_{51} & Y_{52} & Y_{53} & Y_{54} & Y_{55} \end{bmatrix}$$

$$= \begin{bmatrix} y_a & -y_a & 0 & 0 & 0 \\ -y_a & y_a + y_b + y_c & -y_b & -y_c & 0 \\ 0 & -y_b & y_b + y_d & -y_d & 0 \\ 0 & -y_c & -y_d & y_c + y_d + y_e & -y_e \\ 0 & 0 & 0 & -y_e & y_e \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & y_f & 0 & 0 & 0 \\ 0 & 0 & y_g & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} y_a & -y_a & 0 & 0 & 0 \\ -y_a & y_a + y_b + y_c + y_f & -y_b & -y_c & 0 \\ 0 & -y_b & y_b + y_d + y_g & -y_d & 0 \\ 0 & -y_c & -y_d & y_c + y_d + y_e & -y_e \\ 0 & 0 & 0 & -y_e & y_e \end{bmatrix}$$

Note that we could reduce $\underline{Y}_1{}'$ to obtain $\underline{Y}_1$ at this point, and we would need to do that if we were going to wait and apply the fault at t=1 sec and thus have to simulate prefault conditions for a second (why would we want to do that?), but this is unnecessary if the fault is applied at t=0. We skip this step here.

2. Modify $\underline{Y}_1'$ to obtain $\underline{Y}_2'$, the fault-on Y-bus. Note that you should not obtain $\underline{Y}'_2$ from $\underline{Y}_1$ (the Y-bus for reduced pre-fault network) if $\underline{Y}_1$ does not include the faulted bus.

To see how to do this, let's consider a simpler case of faulting bus 2 in the two-bus system of Fig. 3.

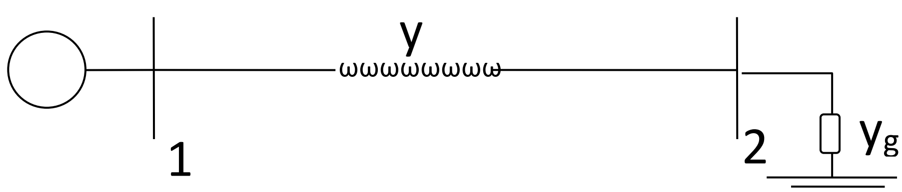


Fig. 3

The corresponding Y-bus relation is

$$\begin{bmatrix} I_1 \\ 0 \end{bmatrix} = \begin{bmatrix} y & -y \\ -y & y+y_g \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

The fault at bus 2 causes V_2=0. Therefore the first equation becomes $I_1=yV_1$ which shows the fault-on Y-bus is $Y'_2=y$. How did we obtain this? We

- zeroed the voltage variable corresponding to the faulted bus, V_2;
- removed the second column of Y-bus since those variables multiply V_2 which is now V_2=0;
- removed the current injection equation corresponding to the faulted bus since this bus has been shorted to ground and can have no current injection into the network (relations from this equation are embedded in remaining equations).

Since we have faulted bus 4 in our small network, we remove the fourth column and the fourth equation:

$$\underline{Y}_1{}' = \begin{bmatrix} y_a & -y_a & 0 & 0 & 0 \\ -y_a & y_a + y_b + y_c + y_f & -y_b & -y_c & 0 \\ 0 & -y_b & y_b + y_d + y_g & -y_d & 0 \\ 0 & -y_c & -y_d & y_c + y_d + y_e & -y_e \\ 0 & 0 & 0 & -y_e & y_e \end{bmatrix}$$

$$\Rightarrow \underline{Y}_2{}' = \begin{bmatrix} y_a & -y_a & 0 & 0 \\ -y_a & y_a + y_b + y_c + y_f & -y_b & 0 \\ 0 & -y_b & y_b + y_d + y_g & 0 \\ 0 & 0 & 0 & y_e \end{bmatrix}$$

3. To obtain $\underline{Y}_2$, we perform Kron reduction on $\underline{Y}_2'$ to eliminate buses 2, and 3, leaving only buses 1 and 5. We will skip this step here as it is a bit messy using just symbols.

4. To obtain $\underline{Y}_3'$, we should start from $\underline{Y}_1'$

   - Not $\underline{Y}_1$ because $\underline{Y}_1$ is reduced and may not have the buses in it terminating the outaged circuit;

   - Not $\underline{Y}_2'$ because the fault is cleared and the faulted bus is no longer at zero voltage.

In our case, the outaged circuit is cct 2-4. To obtain $\underline{Y}_3'$, we define a modification matrix Y_{jk} containing terms corresponding to the necessary changes to $\underline{Y}_1'$ In the case of our example, we will call this matrix $\underline{Y}_{24}=\underline{Y}_{42}$, given by

$$\underline{Y}_{24} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & y_c & 0 & -y_c & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -y_c & 0 & y_c & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We subtract it since we are removing elements:

$$
\underline{Y}_3' = \underline{Y}_1' - \underline{Y}_{24}
$$

$$
= \begin{bmatrix} y_a & -y_a & 0 & 0 & 0 \\ -y_a & y_a + y_b + y_c + y_f & -y_b & -y_c & 0 \\ 0 & -y_b & y_b + y_d + y_g & -y_d & 0 \\ 0 & -y_c & -y_d & y_c + y_d + y_e & -y_e \\ 0 & 0 & 0 & -y_e & y_e \end{bmatrix}
$$

$$
- \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & y_c & 0 & -y_c & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -y_c & 0 & y_c & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} y_a & -y_a & 0 & 0 & 0 \\ -y_a & y_a + y_b + y_f & -y_b & 0 & 0 \\ 0 & -y_b & y_b + y_d + y_g & -y_d & 0 \\ 0 & 0 & -y_d & y_d + y_e & -y_e \\ 0 & 0 & 0 & -y_e & y_e \end{bmatrix}
$$

5. To obtain $\underline{Y}_3$, we Kron reduce $\underline{Y}_3'$, eliminating buses 2, 3, and 4, leaving only buses 1 and 5.

6. A last thing that is necessary before initiating the stability simulation is to obtain the internal voltages for all of the machines. Consider Fig. 4 below.

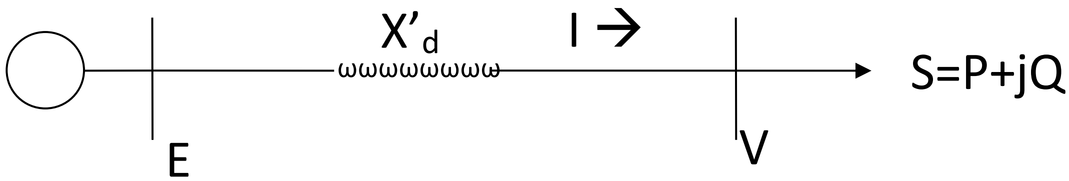


Fig. 4

From the power flow solution, we know V and S, and so our problem now is to obtain E.

We can express complex power (in per unit) as

$$S = VI^* = P + jQ \tag{15}$$

Solving for I, we obtain

$$I = \left(\frac{P + jQ}{V}\right)^* = \frac{P - jQ}{V^*} \tag{16}$$

Then, from Fig. 4, we can express the internal voltage as

$$E = V + I(jX_d{}') = V + \left(\frac{P - jQ}{V^*}\right) jX_d{}' \tag{17}$$

Or:

$$E = V + \frac{QX_d{}'}{V^*} + \frac{jPX_d{}'}{V^*} \tag{18}$$

Equation (18) is almost the same as (2.61) in VMAF, which is

$$E = V + \frac{QX_d{}'}{V} + \frac{jPX_d{}'}{V} \qquad \text{(2.61, text)}$$

The difference between (18) and (2.61) is that in (18), the voltage in the denominator is conjugated, whereas in (2.61), it is not. The two equations are the same under the condition that the terminal voltage V is the reference, i.e., that it has angle 0 degrees.

However, in the multi-machine case, there will be only one machine that satisfies this condition. Therefore VMAF suggests adjusting for this by adding the "pre-transient voltage angle" to the angle provided by (2.61). This is fine, but equation (18) makes this additional step unnecessary.

We are now in a position to consider ways of solving the swing equation accounting for the network, via numerical integration.