Load Equations (Section 4.13) and
Flux Linkage State Space Model (Section 4.12)

Throughout all of chapter 4, our focus is on the machine itself, therefore we will only perform a very simple treatment of the network in order to see a complete model. We do that here, but realize that we will return to this issue in Chapter 7.

So let's look at a single machine connected to an infinite bus, as illustrated in Fig. 1 below. We neglect line charging in this work.

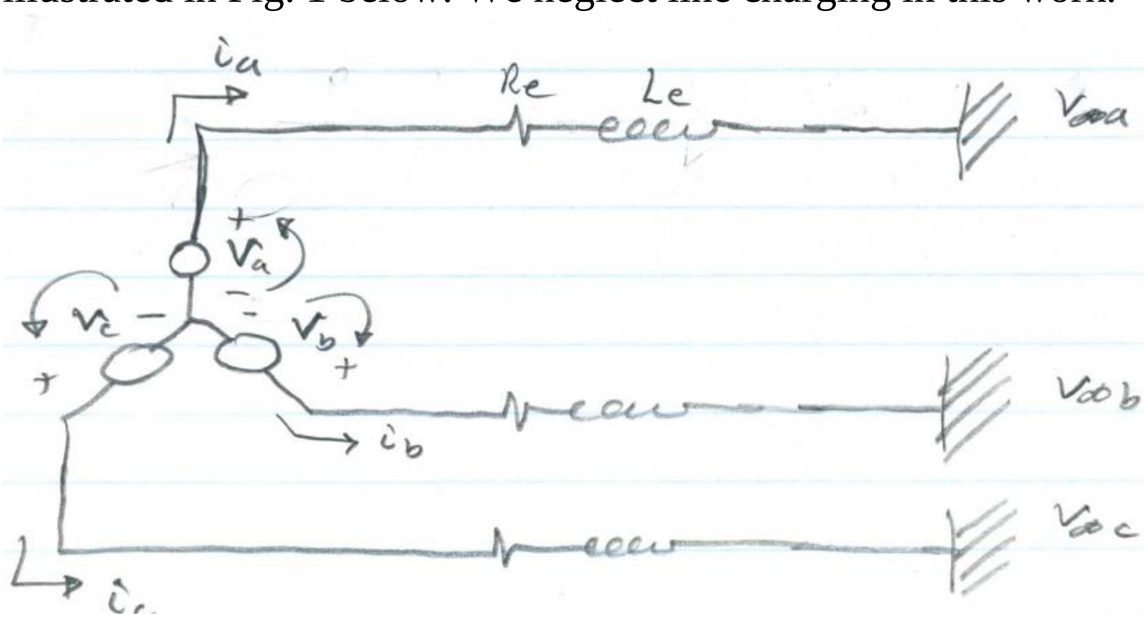

Fig. 1

From KVL, we have

$$\begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} = \begin{bmatrix} v_{\infty,a} \\ v_{\infty,b} \\ v_{\infty,c} \end{bmatrix} + R_e \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\underline{U}} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix} + L_e \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\underline{U}} \begin{bmatrix} \dot{i}_a \\ \dot{i}_b \\ \dot{i}_c \end{bmatrix}$$

$$\Rightarrow \underline{v}_{abc} = \underline{v}_{\infty,abc} + R_e \underline{U}\,\underline{i}_{abc} + L_e \underline{U}\,\dot{\underline{i}}_{abc} \qquad (4.144)$$

Now use Park's transformation to obtain (in either volts or in pu):

$$\underline{v}_{0dq} = \underline{P}\underline{v}_{abc} = \underline{P}\underline{v}_{\infty,abc} + R_e \underbrace{\underline{P}\underline{U}}_{\underline{P}} \underline{i}_{abc} + L_e \underbrace{\underline{P}\underline{U}}_{\underline{P}} \dot{\underline{i}}_{abc}$$

$$\Rightarrow \underline{v}_{0dq} = \underbrace{\underline{v}_{\infty,0dq}}_{TERM\,1} + \underbrace{R_e \underline{i}_{0dq}}_{TERM\,2} + \underbrace{L_e \underline{P}\dot{\underline{i}}_{abc}}_{TERM\,3} \qquad (1)$$

We would like to express v_d and v_q as a function of state variables - the 0dq currents for the current model (or, we will see, the 0dq flux linkages for the flux linkage model). Let's consider each term.

<u>TERM1</u>:

$$\underline{v}_{\infty,0dq} = \underline{P}\underline{v}_{\infty,abc}$$

So what is $\underline{v}_{\infty,abc}$?

A good assumption for purposes of stability assessment is that they are a set of balanced voltages having, in volts or pu, an rms value of V_∞, so that the peak value (in volts or pu) is $\sqrt{2}\, V_\infty$, i.e.,

$$\underline{v}_{\infty,abc} = \begin{bmatrix} v_{\infty,a} \\ v_{\infty,b} \\ v_{\infty,c} \end{bmatrix} = \begin{bmatrix} \sqrt{2}V_\infty \cos(\omega_{\mathrm{Re}} t + \alpha) \\ \sqrt{2}V_\infty \cos(\omega_{\mathrm{Re}} t + \alpha - 120) \\ \sqrt{2}V_\infty \cos(\omega_{\mathrm{Re}} t + \alpha + 120) \end{bmatrix}$$

Note that in the "perunitization" notes, on p. 5, we expressed a balanced abc set of voltages in terms of sin functions. Here, we express the abc voltages in terms of cos functions. As a result, our Park's transformation will be slightly different.

Hit the above with Park's transformation matrix to obtain:

$$\underline{v}_{\infty,0dq} = \underline{P}\underline{v}_{\infty,abc} = \sqrt{3}V_\infty \begin{bmatrix} 0 \\ -\sin(\delta - \alpha) \\ \cos(\delta - \alpha) \end{bmatrix}$$

VMAF states, p. 125, "Using the identities in Appendix A and using $\theta=\omega_{\mathrm{Re}} + \delta + \pi/2$, we can show that…"

where, as we have previously seen, in radians, we have

$$\delta = \delta_0 + \int_0^t (\omega - \omega_{\mathrm{Re}})\, dt \qquad (4.150)$$

Differentiating, we get $\dot{\delta}(t) = \omega(t) - \omega_{\mathrm{Re}}$, which, in pu is $\dot{\delta} = \omega - 1$.

And so we see that the balanced AC voltages transform to a set of DC voltages, as we have observed before.

TERM2: This one is easy as it is already written in terms of the 0dq currents.

TERM3: $\underbrace{L_e \underline{P}\dot{\underline{i}}_{abc}}_{TERM3}$. We must be a little careful here.

It is tempting to use

$\dot{\underline{i}}_{0dq} = \underline{P}\dot{\underline{i}}_{abc}$. But is this true?

Let's back up and recall that

$\underline{i}_{0dq} = \underline{P}\underline{i}_{abc}$

Taking the derivative of the left-hand-side, we obtain:

$$\dot{\underline{i}}_{0dq} = \underline{P}\dot{\underline{i}}_{abc} + \dot{\underline{P}}\underline{i}_{abc} \qquad (2)$$

And this proves that $\dot{\underline{i}}_{0dq} \neq \underline{P}\dot{\underline{i}}_{abc}$.

But we know that $\underline{i}_{abc} = \underline{P}^{-1}\underline{i}_{0dq}$, and using this in (2) results in

$$\dot{\underline{i}}_{0dq} = \underline{P}\dot{\underline{i}}_{abc} + \dot{\underline{P}}\underline{P}^{-1}\underline{i}_{0dq}$$

Isolating the first term on the right results in

$$\underline{P}\dot{\underline{i}}_{abc} = \dot{\underline{i}}_{0dq} - \dot{\underline{P}}\underline{P}^{-1}\underline{i}_{0dq}$$

Recalling that term3 is $L_e \underline{P}\dot{\underline{i}}_{abc}$, we multiple the above by L_e to obtain term3:

$$L_e \underline{P}\dot{\underline{i}}_{abc} = L_e\left(\dot{\underline{i}}_{0dq} - \dot{\underline{P}}\underline{P}^{-1}\underline{i}_{0dq}\right)$$

You may recall now that in Section 4.4 (notes on "macheqts," p. 31) that we found

$$\dot{\underline{P}}\underline{P}^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\omega \\ 0 & \omega & 0 \end{bmatrix}$$

So term3 becomes

$$L_e \underline{P}\underline{i}_{abc} = L_e\left(\dot{\underline{i}}_{0dq} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\omega \\ 0 & \omega & 0 \end{bmatrix} \underline{i}_{0dq} \right)$$

Or

$$L_e \underline{P}\underline{i}_{abc} = L_e\left(\underline{\begin{bmatrix} \dot{i}_0 \\ \dot{i}_d \\ \dot{i}_q \end{bmatrix}} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\omega \\ 0 & \omega & 0 \end{bmatrix} \underline{\begin{bmatrix} i_0 \\ i_d \\ i_q \end{bmatrix}} \right) = L_e\left(\underline{\begin{bmatrix} \dot{i}_0 \\ \dot{i}_d \\ \dot{i}_q \end{bmatrix}} - \omega \underline{\begin{bmatrix} 0 \\ -i_q \\ i_d \end{bmatrix}} \right) = L_e\left(\dot{\underline{i}}_{0dq} - \omega \underline{\begin{bmatrix} 0 \\ -i_q \\ i_d \end{bmatrix}} \right)$$

Substitution of our terms 1, 2, and 3 back into eq. (1) results in

$$\underline{v}_{0dq} = \sqrt{3} V_\infty \begin{bmatrix} 0 \\ -\sin(\delta - \alpha) \\ \cos(\delta - \alpha) \end{bmatrix} + R_e \underline{i}_{0dq} + L_e \dot{\underline{i}}_{0dq} - L_e \omega \underline{\begin{bmatrix} 0 \\ -i_q \\ i_d \end{bmatrix}} \qquad (4.149)$$

Now we need to incorporate this into our state-space model.

We have (or will have) three different state-space models.
A. Current state-space model;
B. Flux-linkage-state-space model with λ_{AD} and λ_{AQ} - it is useful for modeling saturation;
C. Flux-linkage-state-space model with λ_{AD} and λ_{AQ} eliminated (and so without the ability to modeling saturation)

I have hand-written notes where I went through the details of this for models (A) and (C), although I did not include the G-winding. Here I do it with the G-winding.

A. Current state-space model (See section 4.13.2)

Recall that the current state-space model is

$$\begin{bmatrix} \dot{i}_d \\ \dot{i}_F \\ \dot{i}_D \\ \dot{i}_q \\ \dot{i}_G \\ \dot{i}_Q \\ \dot{\omega} \\ \dot{\delta} \end{bmatrix} = \left[\begin{array}{cccccc|cc} & & -\mathbf{L}^{-1}(\mathbf{R}+\omega\mathbf{N}) & & & & \mathbf{0} & \\ \hline \frac{-L_d i_q}{3\tau_j} & \frac{-kM_F i_q}{3\tau_j} & \frac{-kM_D i_q}{3\tau_j} & \frac{L_q i_d}{3\tau_j} & \frac{kM_G i_d}{3\tau_j} & \frac{kM_Q i_d}{3\tau_j} & \frac{-D}{\tau_j} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \begin{bmatrix} i_d \\ i_F \\ i_D \\ i_q \\ i_G \\ i_Q \\ \omega \\ \delta \end{bmatrix} + \begin{bmatrix} -\mathbf{L}^{-1}\mathbf{v} \\ \hline \frac{T_m}{\tau_j} \\ -1 \end{bmatrix} \qquad (4.103)$$

where the submatrices are given by

$$\mathbf{R} = \begin{bmatrix} r & 0 & 0 & 0 & 0 & 0 \\ 0 & r_F & 0 & 0 & 0 & 0 \\ 0 & 0 & r_D & 0 & 0 & 0 \\ 0 & 0 & 0 & r & 0 & 0 \\ 0 & 0 & 0 & 0 & r_G & 0 \\ 0 & 0 & 0 & 0 & 0 & r_Q \end{bmatrix}; \qquad \mathbf{N} = \begin{bmatrix} 0 & 0 & 0 & L_q & kM_G & kM_Q \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -L_d & -kM_F & -kM_D & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{L} = \begin{bmatrix} L_d & kM_F & kM_D & 0 & 0 & 0 \\ kM_F & L_F & M_R & 0 & 0 & 0 \\ kM_D & M_R & L_D & 0 & 0 & 0 \\ 0 & 0 & 0 & L_q & kM_G & kM_Q \\ 0 & 0 & 0 & kM_G & L_G & M_Y \\ 0 & 0 & 0 & kM_Q & M_Y & L_Q \end{bmatrix}$$

$$\mathbf{v} = \begin{bmatrix} v_d \\ -v_F \\ 0 \\ v_q \\ 0 \\ 0 \end{bmatrix}; \qquad \mathbf{i} = \begin{bmatrix} i_d \\ i_F \\ i_D \\ i_q \\ i_G \\ i_Q \end{bmatrix}$$

Incorporating our load equations, eq. (4.149), repeated here:

$$\underline{v}_{0dq} = \sqrt{3}V_\infty \begin{bmatrix} 0 \\ -\sin(\delta-\alpha) \\ \cos(\delta-\alpha) \end{bmatrix} + R_e \underline{i}_{0dq} + L_e \dot{\underline{i}}_{0dq} - L_e\omega \underline{\begin{bmatrix} 0 \\ -i_q \\ i_d \end{bmatrix}} \qquad (4.149)$$

into our state-space current model, (4.103), results in

$$\begin{bmatrix} \dot{i}_d \\ \dot{i}_F \\ \dot{i}_D \\ \dot{i}_q \\ \dot{i}_G \\ \dot{i}_Q \\ \hline \dot{\omega} \\ \dot{\delta} \end{bmatrix} = \left[\begin{array}{cccccc|cc} \multicolumn{6}{c|}{-\hat{\mathbf{L}}^{-1}(\hat{\mathbf{R}}+\omega\hat{\mathbf{N}})} & \multicolumn{2}{c}{\mathbf{0}} \\ \hline \frac{-L_d i_q}{3\tau_j} & \frac{-kM_F i_q}{3\tau_j} & \frac{-kM_D i_q}{3\tau_j} & \frac{L_q i_d}{3\tau_j} & \frac{kM_G i_d}{3\tau_j} & \frac{kM_Q i_d}{3\tau_j} & \frac{-D}{\tau_j} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \begin{bmatrix} i_d \\ i_F \\ i_D \\ i_q \\ i_G \\ i_Q \\ \hline \omega \\ \delta \end{bmatrix} + \left[\begin{array}{c|cc} -\hat{\mathbf{L}}^{-1} & \multicolumn{2}{c}{\mathbf{0}} \\ \hline \mathbf{0} & 1 & 0 \\ & 0 & 1 \end{array}\right] \begin{bmatrix} -K\sin\gamma \\ -v_F \\ 0 \\ \hdashline K\cos\gamma \\ 0 \\ 0 \\ \hdashline \frac{T_m}{\tau_j} \\ -1 \end{bmatrix} \qquad (4.154)$$

where the matrices with the hats above them, i.e., $\hat{\mathbf{L}}, \hat{\mathbf{R}}, \hat{\mathbf{N}}$, are exactly as the unhat-ed versions above, except that

- Wherever you see r, replace it with $r + R_e$
- Wherever you see L_d, replace it with $L_d + L_e$
- Wherever you see L_q, replace it with $L_q + L_e$

Note that:

- $K=\sqrt{3}\ V_\infty$ (**<u>not</u>** the same K as used in the saturation notes),
- $\gamma=\delta-\alpha$
- the speed deviation equation contains un-hatted parameters for L_d and L_q.

Your text remarks again on p. 126 (similar to the sentence at the end of Section 4.10 and noted on p. 12 of TorqueEquation notes):

"The system described by (4.154) is now in the form of …
$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\mathbf{u},t)$, where $\mathbf{x}^T = \begin{bmatrix} i_d & i_F & i_D & i_q & i_G & i_Q & \omega & \delta \end{bmatrix}$."

"The function **f** is a nonlinear function of the state variables and t, and **u** contains the system driving functions, which are v_F and T_m. The loading effect of the transmission line is incorporated in the matrices $\hat{\mathbf{L}}, \hat{\mathbf{R}}, \hat{\mathbf{N}}$. The infinite bus voltage V_∞ appears in the terms Ksinγ and Kcosγ. Note also that these latter terms are not driving functions, but rather nonlinear functions of the state variable δ."

Well, these latter terms are nonlinear functions of the state variable δ, but they are also functions of V_∞. For our model, V_∞ is a driving function (i.e., an independent input).

<u>C. Flux-linkage-state-space model with λ_{AD}, λ_{AQ} eliminated (so without ability to modeling saturation) (See section 4.13.3 of text).</u>

We will skip this part for now, which comes from 4.13.3, because we need to first go over section 4.12.

Recall the state-space model of eq. (4.138)

Without G-winding:

$$\begin{bmatrix} \dot{\lambda}_d \\ \dot{\lambda}_F \\ \dot{\lambda}_D \\ \hdashline \dot{\lambda}_q \\ \dot{\lambda}_Q \\ \hdashline \dot{\omega} \\ \dot{\delta} \end{bmatrix} = \begin{bmatrix} -\frac{r}{\ell_d}\left(1-\frac{L_{MD}}{\ell_d}\right) & \frac{r}{\ell_d}\frac{L_{MD}}{\ell_F} & \frac{r}{\ell_d}\frac{L_{MD}}{\ell_D} & -\omega & 0 & 0 & 0 \\ \frac{r_F}{\ell_F}\frac{L_{MD}}{\ell_d} & -\frac{r_F}{\ell_F}\left(1-\frac{L_{MD}}{\ell_F}\right) & \frac{r_F}{\ell_F}\frac{L_{MD}}{\ell_D} & 0 & 0 & 0 & 0 \\ \frac{r_D}{\ell_D}\frac{L_{MD}}{\ell_d} & \frac{r_D}{\ell_D}\frac{L_{MD}}{\ell_F} & -\frac{r_D}{\ell_D}\left(1-\frac{L_{MD}}{\ell_D}\right) & 0 & 0 & 0 & 0 \\ \omega & 0 & 0 & -\frac{r}{\ell_q}\left(1-\frac{L_{MQ}}{\ell_q}\right) & \frac{r}{\ell_q}\frac{L_{MQ}}{\ell_Q} & 0 & 0 \\ 0 & 0 & 0 & \frac{r_Q}{\ell_Q}\frac{L_{MQ}}{\ell_q} & -\frac{r_Q}{\ell_Q}\left(1-\frac{L_{MQ}}{\ell_Q}\right) & 0 & 0 \\ -\frac{L_{MD}}{3\tau_j\ell_d^2}\lambda_q & -\frac{L_{MD}}{3\tau_j\ell_d\ell_F}\lambda_q & -\frac{L_{MD}}{3\tau_j\ell_d\ell_D}\lambda_q & \frac{L_{MQ}}{3\tau_j\ell_q^2}\lambda_d & \frac{L_{MQ}}{3\tau_j\ell_q\ell_Q}\lambda_d & -\frac{D}{\tau_j} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \lambda_d \\ \lambda_F \\ \lambda_D \\ \hdashline \lambda_q \\ \lambda_Q \\ \hdashline \omega \\ \delta \end{bmatrix} + \begin{bmatrix} -\upsilon_d \\ -\upsilon_F \\ 0 \\ \hdashline -\upsilon_q \\ 0 \\ \hdashline \frac{T_m}{\tau_j} \\ -1 \end{bmatrix} \qquad (4.138)$$

With G-winding:

$$\begin{array}{cccccccc} \lambda_d & \lambda_F & \lambda_D & \lambda_q & \lambda_Q & \lambda_Q & \omega & \delta \end{array}$$

$$\begin{bmatrix} \dot{\lambda}_d \\ \dot{\lambda}_F \\ \dot{\lambda}_D \\ \hdashline \dot{\lambda}_q \\ \dot{\lambda}_G \\ \dot{\lambda}_Q \\ \hdashline \dot{\omega} \\ \dot{\delta} \end{bmatrix} = \left[\begin{array}{ccc:ccc:cc} -\frac{r}{\ell_d}\left(1-\frac{L_{MD}}{\ell_d}\right) & \frac{r}{\ell_d}\frac{L_{MD}}{\ell_F} & \frac{r}{\ell_d}\frac{L_{MD}}{\ell_D} & -\omega & 0 & 0 & 0 & 0 \\ \frac{r_F}{\ell_F}\frac{L_{MD}}{\ell_d} & -\frac{r_F}{\ell_F}\left(1-\frac{L_{MD}}{\ell_F}\right) & \frac{r_F}{\ell_F}\frac{L_{MD}}{\ell_D} & 0 & 0 & 0 & 0 & 0 \\ \frac{r_D}{\ell_D}\frac{L_{MD}}{\ell_d} & \frac{r_D}{\ell_D}\frac{L_{MD}}{\ell_F} & -\frac{r_D}{\ell_D}\left(1-\frac{L_{MD}}{\ell_D}\right) & 0 & 0 & 0 & 0 & 0 \\ \hdashline \omega & 0 & 0 & -\frac{r}{\ell_q}\left(1-\frac{L_{MQ}}{\ell_q}\right) & \frac{r}{\ell_q}\frac{L_{MQ}}{\ell_G} & \frac{r}{\ell_q}\frac{L_{MQ}}{\ell_Q} & 0 & 0 \\ 0 & 0 & 0 & \frac{r_G}{\ell_G}\frac{L_{MQ}}{\ell_q} & -\frac{r_G}{\ell_G}\left(1-\frac{L_{MQ}}{\ell_G}\right) & \frac{r_G}{\ell_G}\frac{L_{MQ}}{\ell_Q} & 0 & 0 \\ 0 & 0 & 0 & \frac{r_Q}{\ell_Q}\frac{L_{MQ}}{\ell_q} & \frac{r_Q}{\ell_Q}\frac{L_{MQ}}{\ell_G} & -\frac{r_Q}{\ell_Q}\left(1-\frac{L_{MQ}}{\ell_Q}\right) & 0 & 0 \\ \hdashline -\frac{L_{MD}}{3\tau_j\ell_d^2}\lambda_q & -\frac{L_{MD}}{3\tau_j\ell_d\ell_F}\lambda_q & -\frac{L_{MD}}{3\tau_j\ell_d\ell_D}\lambda_q & \frac{L_{MQ}}{3\tau_j\ell_q^2}\lambda_d & \frac{L_{MQ}}{3\tau_j\ell_G\ell_q}\lambda_d & \frac{L_{MQ}}{3\tau_j\ell_q\ell_Q}\lambda_d & -\frac{D}{\tau_j} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \begin{bmatrix} \lambda_d \\ \lambda_F \\ \lambda_D \\ \hdashline \lambda_q \\ \lambda_G \\ \lambda_Q \\ \hdashline \omega \\ \delta \end{bmatrix} + \begin{bmatrix} -\upsilon_d \\ -\upsilon_F \\ 0 \\ \hdashline -\upsilon_q \\ 0 \\ 0 \\ \hdashline \frac{T_m}{\tau_j} \\ -1 \end{bmatrix} \quad (4.138')$$

We see we need to incorporate the load equations, (4.149), through the v_d, v_q terms.These equations are repeated here for convenience:

$$\underline{v}_{0dq} = \sqrt{3}V_\infty \begin{bmatrix} 0 \\ -\sin(\delta-\alpha) \\ \cos(\delta-\alpha) \end{bmatrix} + R_e \underline{i}_{0dq} + L_e \dot{\underline{i}}_{0dq} - L_e\omega \underline{\begin{bmatrix} 0 \\ -i_q \\ i_d \end{bmatrix}} \quad (4.149)$$

Expressing v_d and v_q from (4.149), we have that

$$v_d = -\sqrt{3}V_\infty \sin(\delta-\alpha) + R_e i_d + L_e \dot{i}_d + \omega L_e i_q$$

$$v_q = \sqrt{3}V_\infty \cos(\delta-\alpha) + R_e i_q + L_e \dot{i}_q - \omega L_e i_d$$

But we need these in terms of flux linkages. Here, we go back to eqts (4.134) which give the currents as a function of flux linkages but with λ_{AD} and λ_{AQ} eliminated (we only need the i_d equation from (4.134))

$$i_d \;\;=\;\; \left(1-\frac{L_{MD}}{\ell_d}\right)\frac{\lambda_d}{\ell_d} - \frac{L_{MD}}{\ell_d}\frac{\lambda_F}{\ell_F} - \frac{L_{MD}}{\ell_d}\frac{\lambda_D}{\ell_D} \quad (4.134)$$

We also need the i_q equation which is derived as follows. Starting from (4.123), we have

$$i_q = \left(1/\ell_q\right)\left(\lambda_q - \lambda_{AQ}\right) \quad (4.123)$$

And then substitute λ_{AQ} from (4.121)

$$\lambda_{AQ} = \left(L_{MQ}/\ell_q\right)\lambda_q + \left(L_{MQ}/\ell_G\right)\lambda_G + \left(L_{MQ}/\ell_Q\right)\lambda_Q \quad (4.121)$$

to obtain:

$$i_q = \left(1/\ell_q\right)\left(\lambda_q - \left(L_{MQ}/\ell_q\right)\lambda_q - \left(L_{MQ}/\ell_G\right)\lambda_G - \left(L_{MQ}/\ell_Q\right)\lambda_Q\right)$$
$$= \left(\frac{\lambda_q}{\ell_q}\right)\left(1-\frac{L_{MQ}}{\ell_q}\right) - \left(\frac{L_{MQ}}{\ell_G\ell_q}\right)\lambda_G - \left(\frac{L_{MQ}}{\ell_Q\ell_q}\right)\lambda_Q$$

And so in summary we have:

$$i_d = \left(1-\frac{L_{MD}}{\ell_d}\right)\frac{\lambda_d}{\ell_d} - \left(\frac{L_{MD}}{\ell_d}\right)\frac{\lambda_F}{\ell_F} - \left(\frac{L_{MD}}{\ell_d}\right)\frac{\lambda_D}{\ell_D}$$
$$i_q = \left(1-\frac{L_{MQ}}{\ell_q}\right)\frac{\lambda_q}{\ell_q} - \left(\frac{L_{MQ}}{\ell_G}\right)\frac{\lambda_G}{\ell_q} - \left(\frac{L_{MQ}}{\ell_Q}\right)\frac{\lambda_Q}{\ell_q}$$

We also need current derivatives, obtained by differentiating the last two equations:

$$\dot{i}_d = \left(1-\frac{L_{MD}}{\ell_d}\right)\frac{\dot{\lambda}_d}{\ell_d} - \left(\frac{L_{MD}}{\ell_d}\right)\frac{\dot{\lambda}_F}{\ell_F} - \left(\frac{L_{MD}}{\ell_d}\right)\frac{\dot{\lambda}_D}{\ell_D}$$
$$\dot{i}_q = \left(1-\frac{L_{MQ}}{\ell_q}\right)\frac{\dot{\lambda}_q}{\ell_q} - \left(\frac{L_{MQ}}{\ell_G}\right)\frac{\dot{\lambda}_G}{\ell_q} - \left(\frac{L_{MQ}}{\ell_Q}\right)\frac{\dot{\lambda}_Q}{\ell_q}$$

Now substitute the last two equations into our expressions for v_d and v_q to obtain, for the v_d equation:

$$v_d = -\sqrt{3}V_\infty \sin(\delta-\alpha) + R_e\left(\left(1-\frac{L_{MD}}{\ell_d}\right)\frac{\lambda_d}{\ell_d} - \left(\frac{L_{MD}}{\ell_d}\right)\frac{\lambda_F}{\ell_F} - \left(\frac{L_{MD}}{\ell_d}\right)\frac{\lambda_D}{\ell_D}\right)$$
$$+L_e\left(\left(1-\frac{L_{MD}}{\ell_d}\right)\frac{\dot{\lambda}_d}{\ell_d} - \left(\frac{L_{MD}}{\ell_d}\right)\frac{\dot{\lambda}_F}{\ell_F} - \left(\frac{L_{MD}}{\ell_d}\right)\frac{\dot{\lambda}_D}{\ell_D}\right) + \omega L_e\left(\left(1-\frac{L_{MQ}}{\ell_q}\right)\frac{\lambda_q}{\ell_q} - \left(\frac{L_{MQ}}{\ell_G}\right)\frac{\lambda_G}{\ell_q} - \left(\frac{L_{MQ}}{\ell_Q}\right)\frac{\lambda_Q}{\ell_q}\right)$$

and for the v_q equation:

$$v_q = \sqrt{3}V_\infty \cos(\delta-\alpha) + R_e\left(\left(1-\frac{L_{MQ}}{\ell_q}\right)\frac{\lambda_q}{\ell_q} - \left(\frac{L_{MQ}}{\ell_G}\right)\frac{\lambda_G}{\ell_q} - \left(\frac{L_{MQ}}{\ell_Q}\right)\frac{\lambda_Q}{\ell_q}\right)$$
$$+L_e\left(\left(1-\frac{L_{MQ}}{\ell_q}\right)\frac{\dot{\lambda}_q}{\ell_q} - \left(\frac{L_{MQ}}{\ell_G}\right)\frac{\dot{\lambda}_G}{\ell_q} - \left(\frac{L_{MQ}}{\ell_Q}\right)\frac{\dot{\lambda}_Q}{\ell_q}\right) - \omega L_e\left(\left(1-\frac{L_{MD}}{\ell_d}\right)\frac{\lambda_d}{\ell_d} - \left(\frac{L_{MD}}{\ell_d}\right)\frac{\lambda_F}{\ell_F} - \left(\frac{L_{MD}}{\ell_d}\right)\frac{\lambda_D}{\ell_D}\right)$$

Now manipulate the above two equations:

$$v_d = -\sqrt{3}V_\infty \sin(\delta-\alpha) + \frac{R_e}{\ell_d}\left(1-\frac{L_{MD}}{\ell_d}\right)\lambda_d - \frac{R_eL_{MD}}{\ell_d\ell_F}\lambda_F - \frac{R_eL_{MD}}{\ell_d\ell_D}\lambda_D \tag{4.155}$$
$$+\frac{\omega L_e}{\ell_q}\left(1-\frac{L_{MQ}}{\ell_q}\right)\lambda_q - \frac{\omega L_eL_{MQ}}{\ell_q\ell_G}\lambda_G - \frac{\omega L_eL_{MQ}}{\ell_q\ell_Q}\lambda_Q + \frac{L_e}{\ell_d}\left(1-\frac{L_{MD}}{\ell_d}\right)\dot{\lambda}_d - \frac{L_eL_{MD}}{\ell_d\ell_F}\dot{\lambda}_F - \frac{L_eL_{MD}}{\ell_d\ell_D}\dot{\lambda}_D$$

$$v_q = \sqrt{3}V_\infty \cos(\delta-\alpha) + \frac{R_e}{\ell_q}\left(1-\frac{L_{MQ}}{\ell_q}\right)\lambda_q - \frac{R_eL_{MQ}}{\ell_q\ell_G}\lambda_G - \frac{R_eL_{MQ}}{\ell_q\ell_Q}\lambda_Q - \frac{\omega L_e}{\ell_d}\left(1-\frac{L_{MD}}{\ell_d}\right)\lambda_d \tag{4.156}$$
$$+\frac{\omega L_eL_{MD}}{\ell_d\ell_F}\lambda_F + \frac{\omega L_eL_{MD}}{\ell_d\ell_D}\lambda_D + \frac{L_e}{\ell_q}\left(1-\frac{L_{MQ}}{\ell_q}\right)\dot{\lambda}_q - \frac{L_eL_{MQ}}{\ell_q\ell_G}\dot{\lambda}_G - \frac{L_eL_{MQ}}{\ell_q\ell_Q}\dot{\lambda}_Q$$

Now recall the flux linkage state equations for λ_d from (4.135) and λ_q from (4.136), repeated here for convenience:

$$\dot{\lambda}_d = -r\left(1-\frac{L_{MD}}{\ell_d}\right)\frac{\lambda_d}{\ell_d} + r\frac{L_{MD}}{\ell_d}\frac{\lambda_F}{\ell_F} + r\frac{L_{MD}}{\ell_d}\frac{\lambda_D}{\ell_D} - \omega\lambda_q - \upsilon_d \qquad (4.135)$$

$$\dot{\lambda}_q = -r\left(1-\frac{L_{MQ}}{\ell_q}\right)\frac{\lambda_q}{\ell_q} + r\frac{L_{MQ}}{\ell_q}\frac{\lambda_G}{\ell_G} + r\frac{L_{MQ}}{\ell_q}\frac{\lambda_Q}{\ell_Q} + \omega\lambda_d - \upsilon_q \qquad (4.136)$$

Substituting (4.155) into (4.135) for v_d, we obtain:

$$\dot{\lambda}_d = -\frac{r}{\ell_d}\left(1-\frac{L_{MD}}{\ell_d}\right)\lambda_d + \frac{r}{\ell_d}\frac{L_{MD}}{\ell_F}\lambda_F + \frac{r}{\ell_d}\frac{L_{MD}}{\ell_D}\lambda_D - \omega\lambda_q$$
$$-\left[-\sqrt{3}V_\infty\sin(\delta-\alpha) + \frac{R_e}{\ell_d}\left(1-\frac{L_{MD}}{\ell_d}\right)\lambda_d - \frac{R_eL_{MD}}{\ell_d\ell_F}\lambda_F - \frac{R_eL_{MD}}{\ell_d\ell_D}\lambda_D + \frac{\omega L_e}{\ell_q}\left(1-\frac{L_{MQ}}{\ell_q}\right)\lambda_q\right.$$
$$\left.-\frac{\omega L_eL_{MQ}}{\ell_q\ell_G}\lambda_G - \frac{\omega L_eL_{MQ}}{\ell_q\ell_Q}\lambda_Q + \frac{L_e}{\ell_d}\left(1-\frac{L_{MD}}{\ell_d}\right)\dot{\lambda}_d - \frac{L_eL_{MD}}{\ell_d\ell_F}\dot{\lambda}_F - \frac{L_eL_{MD}}{\ell_d\ell_D}\dot{\lambda}_D\right]$$

Now gather terms in state variable derivative on the left and in each state variable on the right, to get

$$\left(1+\frac{L_e}{\ell_d}\left(1-\frac{L_{MD}}{\ell_d}\right)\right)\dot{\lambda}_d - \frac{L_eL_{MD}}{\ell_d\ell_F}\dot{\lambda}_F - \frac{L_eL_{MD}}{\ell_d\ell_D}\dot{\lambda}_D$$
$$= -\left(\frac{r+R_e}{\ell_d}\right)\left(1-\frac{L_{MD}}{\ell_d}\right)\lambda_d + \left(\frac{(r+R_e)L_{MD}}{\ell_d\ell_F}\right)\lambda_F + \frac{(r+R_e)L_{MD}}{\ell_d\ell_D}\lambda_D$$
$$-\omega\left(1+\frac{L_e}{\ell_q}\left(1-\frac{L_{MQ}}{\ell_q}\right)\right)\lambda_q + \frac{\omega L_eL_{MQ}}{\ell_q\ell_G}\lambda_G + \frac{\omega L_eL_{MQ}}{\ell_q\ell_Q}\lambda_Q + \sqrt{3}V_\infty\sin(\delta-\alpha)$$

Finally use $\hat{R} = r + R_e$ to obtain (4.157)

$$\left(1+\frac{L_e}{\ell_d}\left(1-\frac{L_{MD}}{\ell_d}\right)\right)\dot{\lambda}_d - \frac{L_eL_{MD}}{\ell_d\ell_F}\dot{\lambda}_F - \frac{L_eL_{MD}}{\ell_d\ell_D}\dot{\lambda}_D$$
$$= -\left(\frac{\hat{R}}{\ell_d}\right)\left(1-\frac{L_{MD}}{\ell_d}\right)\lambda_d + \left(\frac{\hat{R}L_{MD}}{\ell_d\ell_F}\right)\lambda_F + \frac{\hat{R}L_{MD}}{\ell_d\ell_D}\lambda_D \qquad (4.157)$$
$$-\omega\left(1+\frac{L_e}{\ell_q}\left(1-\frac{L_{MQ}}{\ell_q}\right)\right)\lambda_q + \frac{\omega L_eL_{MQ}}{\ell_q\ell_G}\lambda_G + \frac{\omega L_eL_{MQ}}{\ell_q\ell_Q}\lambda_Q + \sqrt{3}V_\infty\sin(\delta-\alpha)$$

Likewise, for the q-axis equation, substituting (4.156) into (4.136) for v_q, we obtain:

$$\dot{\lambda}_q = -r\left(1-\frac{L_{MQ}}{\ell_q}\right)\frac{\lambda_q}{\ell_q} + r\frac{L_{MQ}}{\ell_q}\frac{\lambda_G}{\ell_G} + r\frac{L_{MQ}}{\ell_q}\frac{\lambda_Q}{\ell_Q} + \omega\lambda_d$$
$$-\left[\sqrt{3}V_\infty\cos(\delta-\alpha) + \frac{R_e}{\ell_q}\left(1-\frac{L_{MQ}}{\ell_q}\right)\lambda_q - \frac{R_e L_{MQ}}{\ell_q\ell_G}\lambda_G - \frac{R_e L_{MQ}}{\ell_q\ell_Q}\lambda_Q - \frac{\omega L_e}{\ell_d}\left(1-\frac{L_{MD}}{\ell_d}\right)\lambda_d\right.$$
$$\left.+\frac{\omega L_e L_{MD}}{\ell_d\ell_F}\lambda_F + \frac{\omega L_e L_{MD}}{\ell_d\ell_D}\lambda_D + \frac{L_e}{\ell_q}\left(1-\frac{L_{MQ}}{\ell_q}\right)\dot{\lambda}_q - \frac{L_e L_{MQ}}{\ell_q\ell_G}\dot{\lambda}_G - \frac{L_e L_{MQ}}{\ell_q\ell_Q}\dot{\lambda}_Q\right]$$

Now gather terms in state variable derivatives on the left and in state variables on the right, to get

$$\left(1+\frac{L_e}{\ell_q}\left(1-\frac{L_{MQ}}{\ell_q}\right)\right)\dot{\lambda}_q - \frac{L_e L_{MQ}}{\ell_q\ell_G}\dot{\lambda}_G - \frac{L_e L_{MQ}}{\ell_q\ell_Q}\dot{\lambda}_Q$$
$$= -\frac{(r+R_e)}{\ell_q}\left(1-\frac{L_{MQ}}{\ell_q}\right)\lambda_q + \frac{(r+R_e)L_{MQ}}{\ell_q\ell_G}\lambda_G + \frac{(r+R_e)L_{MQ}}{\ell_q\ell_Q}\lambda_Q$$
$$+\omega\left(1+\frac{L_e}{\ell_d}\left(1-\frac{L_{MD}}{\ell_d}\right)\right)\lambda_d - \frac{\omega L_e L_{MD}}{\ell_d\ell_F}\lambda_F - \frac{\omega L_e L_{MD}}{\ell_d\ell_D}\lambda_D - \sqrt{3}V_\infty\cos(\delta-\alpha)$$

Finally use $\hat{R} = r + R_e$ to obtain (4.158)

$$\left(1+\frac{L_e}{\ell_q}\left(1-\frac{L_{MQ}}{\ell_q}\right)\right)\dot{\lambda}_q - \frac{L_e L_{MQ}}{\ell_q\ell_G}\dot{\lambda}_G - \frac{L_e L_{MQ}}{\ell_q\ell_Q}\dot{\lambda}_Q$$
$$= -\frac{\hat{R}}{\ell_q}\left(1-\frac{L_{MQ}}{\ell_q}\right)\lambda_q + \frac{\hat{R}L_{MQ}}{\ell_q\ell_G}\lambda_G + \frac{\hat{R}L_{MQ}}{\ell_q\ell_Q}\lambda_Q \qquad (4.158)$$
$$+\omega\left(1+\frac{L_e}{\ell_d}\left(1-\frac{L_{MD}}{\ell_d}\right)\right)\lambda_d - \frac{\omega L_e L_{MD}}{\ell_d\ell_F}\lambda_F - \frac{\omega L_e L_{MD}}{\ell_d\ell_D}\lambda_D - \sqrt{3}V_\infty\cos(\delta-\alpha)$$

Note in these two equations (4.157) and (4.158) that there are several derivative terms and so we cannot "cleanly" use these equations to simply replace the derivatives on λ_d and λ_q in the flux-linkage state-space model (we were able to do so with the current state-space model).

Rather, we have to create a pre-multiplier matrix $\underline{T}$ such that

$$\underline{T}\,\underline{\dot{x}} = \underline{C}\,\underline{x} + \underline{D}$$

where

$$\underline{x} = \begin{bmatrix} \lambda_d \\ \lambda_F \\ \lambda_D \\ \lambda_q \\ \lambda_G \\ \lambda_Q \\ \omega \\ \delta \end{bmatrix}$$

And $\underline{T}$, $\underline{C}$, and $\underline{D}$ are given by

$$\mathbf{T} = \left[\begin{array}{ccc:ccc:cc} 1+\frac{L_e}{\ell_d}\left(1-\frac{L_{MD}}{\ell_d}\right) & -\frac{L_e L_{MD}}{\ell_d \ell_F} & -\frac{L_e L_{MD}}{\ell_d \ell_D} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 1+\frac{L_e}{\ell_q}\left(1-\frac{L_{MQ}}{\ell_q}\right) & -\frac{L_e L_{MQ}}{\ell_q \ell_G} & -\frac{L_e L_{MQ}}{\ell_q \ell_Q} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \quad (4.160)$$

$$\mathbf{C} = \left[\begin{array}{ccc:ccc:cc} -\frac{\hat{R}}{\ell_d}\left(1-\frac{L_{MD}}{\ell_d}\right) & \frac{\hat{R}L_{MD}}{\ell_d \ell_F} & \frac{\hat{R}L_{MD}}{\ell_d \ell_D} & -\omega\left[1+\frac{L_e}{\ell_q}\left(1-\frac{L_{MQ}}{\ell_q}\right)\right] & \frac{\omega L_e L_{MQ}}{\ell_q \ell_G} & \frac{\omega L_e L_{MQ}}{\ell_q \ell_Q} & 0 & 0 \\ \frac{r_F L_{MD}}{\ell_F \ell_d} & -\frac{r_F}{\ell_F}\left(1-\frac{L_{MD}}{\ell_F}\right) & \frac{r_F L_{MD}}{\ell_F \ell_D} & 0 & 0 & 0 & 0 & 0 \\ \frac{r_D L_{MD}}{\ell_D \ell_d} & \frac{r_D L_{MD}}{\ell_D \ell_F} & -\frac{r_D}{\ell_D}\left(1-\frac{L_{MD}}{\ell_D}\right) & 0 & 0 & 0 & 0 & 0 \\ \hdashline \omega\left[1+\frac{L_e}{\ell_d}\left(1-\frac{L_{MD}}{\ell_d}\right)\right] & -\frac{\omega L_e L_{MD}}{\ell_d \ell_F} & -\frac{\omega L_e L_{MD}}{\ell_d \ell_D} & -\frac{\hat{R}}{\ell_q}\left(1-\frac{L_{MQ}}{\ell_q}\right) & \frac{\hat{R}L_{MQ}}{\ell_q \ell_G} & \frac{\hat{R}L_{MQ}}{\ell_q \ell_Q} & 0 & 0 \\ 0 & 0 & 0 & \frac{r_G}{\ell_G}\frac{L_{MQ}}{\ell_q} & -\frac{r_G}{\ell_G}\left(1-\frac{L_{MQ}}{\ell_G}\right) & \frac{r_G}{\ell_G}\frac{L_{MQ}}{\ell_Q} & 0 & 0 \\ 0 & 0 & 0 & \frac{r_Q L_{MQ}}{\ell_Q \ell_q} & \frac{r_Q}{\ell_Q}\frac{L_{MQ}}{\ell_G} & -\frac{r_Q}{\ell_Q}\left(1-\frac{L_{MQ}}{\ell_Q}\right) & 0 & 0 \\ \hdashline -\frac{L_{MD}}{3\tau_j \ell_d^2}\lambda_q & -\frac{L_{MD}}{3\tau_j \ell_d \ell_F}\lambda_q & -\frac{L_{MD}}{3\tau_j \ell_d \ell_D}\lambda_q & \frac{L_{MQ}}{3\tau_j \ell_d^2}\lambda_d & \frac{L_{MQ}}{3\tau_j \ell_G \ell_q}\lambda_d & \frac{L_{MQ}}{3\tau_j \ell_q \ell_Q}\lambda_d & -\frac{D}{\tau_j} & 0 \\ 0 & 0 & 0 & 0 & 0 & & 1 & 0 \end{array}\right]$$

(4.161)

$$\mathbf{D} = \begin{bmatrix} \sqrt{3}V_\infty \sin(\delta - \alpha) \\ \upsilon_F \\ 0 \\ -\sqrt{3}V_\infty \cos(\delta - \alpha) \\ 0 \\ T_m / \tau_j \\ -1 \end{bmatrix} \tag{4.162}$$

Then we can pre-multiple both sides by $\underline{T}^{-1}$ to obtain

$$\dot{\underline{x}} = \underline{T}^{-1}\underline{C}\underline{x} + \underline{T}^{-1}\underline{D} \tag{4.163}$$

Equation (4.163) describes the complete system of interest to us at this point, i.e., the system of Fig. 1 at the beginning of these notes. To use it, we need the initial states $\underline{x}(0)$ which are found by solving $\underline{T}\dot{\underline{x}} = \underline{C}\underline{x} + \underline{D} = \underline{0}$, via $\underline{x} = -\underline{C}^{-1}\underline{D}$ where vector $\underline{D}$ provides system loading information.

Then, if we perturb the system by setting, for example, $V_\infty=0$ for a few cycles, then the response can be obtained by solving eq. (4.163) using numerical integration.