Load Equations

Throughout all of chapter 4, our focus is on the machine itself, therefore we will only perform a very simple treatment of the network in order to see a complete model.

So let’s look at a single machine connected to an infinite bus, as illustrated in Fig. 1 below.

![Fig. 1](image)

From KVL, we have

\[
\begin{bmatrix}
    v_a \\
    v_b \\
    v_c
\end{bmatrix} =
\begin{bmatrix}
    v_{\alpha,a} \\
    v_{\alpha,b} \\
    v_{\alpha,c}
\end{bmatrix} +
\begin{bmatrix}
    1 & 0 & 0 \\
    0 & 1 & 0 \\
    0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
    i_a \\
    i_b \\
    i_c
\end{bmatrix} +
\begin{bmatrix}
    1 & 0 & 0 \\
    0 & 1 & 0 \\
    0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
    i_a \\
    i_b \\
    i_c
\end{bmatrix}
\]

\[
\Rightarrow v_{abc} = v_{\alpha,abc} + R_e \bar{U} \bar{i}_{abc} + L_e \bar{U} \dot{i}_{abc}
\]

Now use Park’s transformation to obtain:
We would like to express \( v_d \) and \( v_q \) as a function of state variables (the 0dq currents for the current model or the 0dq flux linkages for the flux linkage model). Let’s consider each term.

**TERM1:**

\[
v_{\infty,0dq} = \frac{P}{N} v_{\infty,abc}
\]

So what is \( v_{\infty,abc} \)?

A good assumption for purposes of stability assessment is that they are a set of balanced voltages having rms value of \( V_{\infty} \), i.e.,

\[
\begin{bmatrix}
  v_{\infty,a} \\
  v_{\infty,b} \\
  v_{\infty,c}
\end{bmatrix} =
\begin{bmatrix}
  \sqrt{2} V_{\infty} \cos(\omega_{Re} t + \alpha) \\
  \sqrt{2} V_{\infty} \cos(\omega_{Re} t + \alpha - 120) \\
  \sqrt{2} V_{\infty} \cos(\omega_{Re} t + \alpha + 120)
\end{bmatrix}
\]

Hit the above with Park’s transformation matrix to obtain:

\[
v_{\infty,0dq} = \frac{P}{N} v_{\infty,abc} = \sqrt{3} V_{\infty} \begin{bmatrix}
  0 \\
  -\sin(\delta - \alpha) \\
  \cos(\delta - \alpha)
\end{bmatrix}
\]

And so we see that we can the balanced AC voltages transform to a set of DC voltages, as we have observed before.

**TERM2:** This one is easy as it is already written in terms of the 0dq currents.

**TERM3:** We must be a little careful here. It is tempting to use \( \dot{i}_{0dq} = \frac{P}{N} i_{abc} \). But is this true?

Let’s back up and recall that

\[
\dot{i}_{0dq} = \frac{P}{N} i_{abc}
\]

Taking the derivative of the left-hand-side, we obtain:
\[ \dot{i}_{0dq} = P_{abc} \dot{i}_{abc} + \dot{P}i_{abc} \]  
(2)

And this proves that \( \dot{i}_{0dq} \neq P_{abc} \).

But we know that \( \dot{i}_{abc} = P^{-1} \dot{i}_{0dq} \), and using this in (2) results in
\[ \dot{i}_{0dq} = P_{abc} \dot{i}_{abc} + \dot{PP}^{-1}i_{0dq} \]

Isolating the first term on the right results in
\[ \dot{P}i_{abc} = \dot{i}_{0dq} - \dot{PP}^{-1}i_{0dq} \]

Recalling that term3 is \( L_e \dot{P}_{abc} \), we multiple the above by \( L_e \) to obtain term3:
\[ L_e \dot{P}_{abc} = L_e \left( \dot{i}_{0dq} - \dot{PP}^{-1}i_{0dq} \right) \]

You may recall now that in Section 4.4 (notes on “macheqts”, pp. 22-23) that we found
\[ \dot{PP}^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\omega \\ 0 & \omega & 0 \end{bmatrix} \]

So term3 becomes
\[ L_e \dot{P}_{abc} = L_e \left( \dot{i}_{0dq} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\omega \\ 0 & \omega & 0 \end{bmatrix} \dot{i}_{0dq} \right) \]

Or
\[ L_e \dot{P}_{abc} = L_e \left( \begin{bmatrix} \dot{i}_0 \\ \dot{i}_d \\ \dot{i}_q \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\omega \\ 0 & \omega & 0 \end{bmatrix} \begin{bmatrix} \dot{i}_0 \\ \dot{i}_d \\ \dot{i}_q \end{bmatrix} \right) = L_e \left( \begin{bmatrix} \dot{i}_0 \\ \dot{i}_d \\ \dot{i}_q \end{bmatrix} - \omega \begin{bmatrix} 0 \\ -\dot{i}_q \\ \dot{i}_d \end{bmatrix} \right) \]

Substitution of our terms 1, 2, and 3 back into eq. (1) results in
Now we need to incorporate this into our state-space model.

We have three different models.
A. Current state-space model
B. Flux-linkage-state-space model with $\lambda_{AD}$ and $\lambda_{AQ}$ (for modeling saturation)
C. Flux-linkage-state-space model with $\lambda_{AD}$ and $\lambda_{AQ}$ eliminated (and so without the ability to modeling saturation)

I have hand-written notes where I went through the details of this for models (A) and (C), although I did not include the G-winding. I want to do that but have just not had time to do it. And so I simply provide the results for the model without the G-winding.

A. Current state-space model (See section 4.13.2)

Recall that the current state-space model is

$$
\begin{bmatrix}
    i_d \\
    i_F \\
    i_D \\
    i_q \\
    i_Q \\
    i_G \\
    \dot{\delta}
\end{bmatrix}
= \begin{bmatrix}
    -L_d i_d \\
    -k M_F i_q \\
    -k M_D i_d \\
    L_q i_d \\
    k M_Q i_d \\
    k M_O i_d \\
    0
\end{bmatrix} \\
\begin{bmatrix}
    \frac{3 \tau}{\tau} \\
    \frac{3 \tau}{\tau} \\
    \frac{3 \tau}{\tau} \\
    0 \\
    0 \\
    0 \\
    0
\end{bmatrix}
\begin{bmatrix}
    0 \\
    0 \\
    0 \\
    0 \\
    0 \\
    0 \\
    0
\end{bmatrix}
\begin{bmatrix}
    0 \\
    0 \\
    0 \\
    0 \\
    0 \\
    0 \\
    0
\end{bmatrix}
\begin{bmatrix}
    -D \\
    1 \\
    0 \\
    \tau \\
    1 \\
    0 \\
    \omega
\end{bmatrix}
\begin{bmatrix}
    i_d \\
    i_F \\
    i_D \\
    i_q \\
    i_Q \\
    i_G \\
    \dot{\delta}
\end{bmatrix}
+ \begin{bmatrix}
    \frac{T_m}{\tau} \\
    \frac{T_m}{\tau} \\
    \frac{T_m}{\tau} \\
    \frac{T_m}{\tau} \\
    \frac{T_m}{\tau} \\
    \frac{T_m}{\tau} \\
    \frac{T_m}{\tau}
\end{bmatrix}
\begin{bmatrix}
    \frac{L^{-1}}{N} \\
    \frac{L^{-1}}{N} \\
    \frac{L^{-1}}{N} \\
    \frac{L^{-1}}{N} \\
    \frac{L^{-1}}{N} \\
    \frac{L^{-1}}{N} \\
    \frac{L^{-1}}{N}
\end{bmatrix}
\begin{bmatrix}
    i_d \\
    i_F \\
    i_D \\
    i_q \\
    i_Q \\
    i_G \\
    \omega
\end{bmatrix}
$$

(4.103’)

where the submatrices are given by
\[ R = \begin{bmatrix} r & 0 & 0 & 0 & 0 & 0 \\ 0 & r_F & 0 & 0 & 0 & 0 \\ 0 & 0 & r_D & 0 & 0 & 0 \\ 0 & 0 & 0 & r_Q & 0 \\ 0 & 0 & 0 & 0 & r_G \end{bmatrix} ; \quad N = \begin{bmatrix} 0 & 0 & 0 & L_q & kM_Q & kM_G \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -L_d & -kM_F & -kM_D & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ L = \begin{bmatrix} L_d & kM_F & kM_D & 0 & 0 & 0 \\ kM_F & L_F & M_R & 0 & 0 & 0 \\ kM_D & M_R & L_D & 0 & 0 & 0 \\ 0 & 0 & 0 & L_q & kM_Q & kM_G \\ 0 & 0 & 0 & kM_Q & L_Q & M_Y \\ 0 & 0 & 0 & kM_G & M_Y & L_G \end{bmatrix} \]

\[ v = \begin{bmatrix} v_d \\ -v_F \\ 0 \\ v_q \\ 0 \\ 0 \end{bmatrix} ; \quad i = \begin{bmatrix} i_d \\ i_F \\ i_D \\ i_q \\ i_Q \\ i_G \end{bmatrix} \]

Incorporating into our load equations, eq. (4.149), into our state-space current model, (4.103'), results in

\[
\begin{bmatrix} i_d \\ i_F \\ i_D \\ i_q \\ i_Q \\ i_G \end{bmatrix} = \begin{bmatrix} \hat{L}^{-1} \left( R + \omega \hat{N} \right) \\ 0 \end{bmatrix}
\]

\[
\begin{bmatrix} -\omega \\ \frac{L_d i_d}{3\tau_j} - \frac{kM_F i_d}{3\tau_j} \frac{kM_D i_d}{3\tau_j} - \frac{L_d i_d}{3\tau_j} - \frac{kM_F i_d}{3\tau_j} \frac{kM_D i_d}{3\tau_j} \frac{D}{\tau_j} \omega \\ 0 \\ 0 \\ 0 \\ \delta \end{bmatrix} + \begin{bmatrix} -K\sin\gamma \\ -v_F \\ 0 \\ -K\cos\gamma \\ 0 \end{bmatrix}
\]

(4.154)
where the matrices with the hats above them, i.e., $\hat{L}, \hat{R}, \hat{N}$, are exactly as the unhat-ed versions above, except that

- Wherever you see $r$, replace it with $r + R_e$
- Wherever you see $L_d$, replace it with $L_d + L_e$
- Wherever you see $L_q$, replace it with $L_q + L_e$

Note that:

$K = \sqrt{3} \ V_\infty$ (not the same $K$ as used in the saturation notes), and $\gamma = \delta - \alpha$.

Your text makes a useful remark (pg. 117) in saying that,

“The system described by (4.154) is now in the form of …

$\dot{x} = f(x, u, t)$, where $x^T = [i_d \ i_F \ i_D \ i_q \ i_Q \ \omega \ \delta]$.”

(and, of course, $i_G$)…

“The function $f$ is a nonlinear function of the state variables and $t$, and $u$ contains the system driving functions, which are $v_F$ and $T_m$. The loading effect of the transmission line is incorporated in the matrices $\hat{L}, \hat{R}, \hat{N}$. The infinite bus voltage $V_\infty$ appears in the terms $Ksin\gamma$ and $Kcos\gamma$. Note also that these latter terms are not driving functions, but rather nonlinear functions of the state variable $\delta$.\"
C. Flux-linkage-state-space model with $\lambda_{AD}$, $\lambda_{AQ}$ eliminated (so without ability to modeling saturation) (See section 4.13.3 of text).

Recall the state-space model of eq. (4.138)

\[
\begin{bmatrix}
\dot{\lambda}_d \\
\dot{\lambda}_q \\
\dot{\lambda}_r \\
\dot{\lambda}_0 \\
\dot{\omega} \\
\dot{\delta}
\end{bmatrix}
= \\
\begin{bmatrix}
\frac{r}{\tau_s} (1 - \frac{L_{MD}}{\tau_s}) & \frac{r}{\tau_r} & \frac{r}{\tau_0} & -\omega & 0 & 0 \\
\frac{r}{\tau_s} & -\frac{r}{\tau_r} (1 - \frac{L_{MD}}{\tau_r}) & \frac{r}{\tau_0} & 0 & 0 & 0 \\
\frac{r}{\tau_s} & \frac{r}{\tau_r} & -\frac{r}{\tau_0} (1 - \frac{L_{MQ}}{\tau_0}) & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{r}{\tau_0} (1 - \frac{L_{MQ}}{\tau_0}) & \frac{r}{\tau_0} & 0 \\
0 & 0 & 0 & \frac{r}{\tau_0} & \frac{r}{\tau_0} (1 - \frac{L_{MQ}}{\tau_0}) & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{r}{\tau_0}
\end{bmatrix}
\begin{bmatrix}
\lambda_d \\
\lambda_q \\
\lambda_r \\
\lambda_0 \\
\omega \\
\delta
\end{bmatrix}
= \\
\begin{bmatrix}
-v_d \\
v_q
\end{bmatrix}
\]

We see we need to incorporate the load equations, (4.149), through the $v_d$, $v_q$ terms. These equations are repeated here for convenience:

\[
v_{0dq} = \sqrt{3} V_{\infty} \begin{bmatrix} 0 \\ -\sin(\delta - \alpha) \\ \cos(\delta - \alpha) \end{bmatrix} + R_e i_{0dq} + L_e i_{0dq} - L_e \omega - i_q
\]

However, this time we need the load equations in terms of flux linkages. This takes some work, which I have done in detailed hand-written notes (will be happy to provide if you want them).

This results in eqts. 4.57, 4.58 in your text, repeated here.

\[
\begin{align*}
\dot{\lambda}_d &= \frac{L_e}{\ell_d} \left(1 - \frac{L_{MD}}{\ell_d}\right) \dot{\lambda}_d - \frac{L_e L_{MD}}{\ell_d \ell_F} \lambda_F - \frac{L_e L_{MD}}{\ell_d \ell_D} \lambda_D - \frac{\hat{R}}{\ell_d} \left(1 - \frac{L_{MD}}{\ell_D}\right) \lambda_d + \frac{\hat{R} L_{MD}}{\ell_D \ell_F} \lambda_F \\
&\quad + \frac{\hat{R} L_{MD}}{\ell_D \ell_D} \lambda_D - \omega \left[1 + \frac{L_e}{\ell_q} \left(1 - \frac{L_{MQ}}{\ell_q}\right)\right] \lambda_q + \frac{\omega L_e L_{MQ}}{\ell_q \ell_Q} \lambda_Q + \sqrt{3} V_\omega \sin(\delta - \alpha)
\end{align*}
\]

Similarly, we combine (4.156) with (4.136) to get

\[
\begin{align*}
\dot{\lambda}_q &= \frac{L_e}{\ell_q} \left(1 - \frac{L_{MQ}}{\ell_q}\right) \dot{\lambda}_q - \frac{L_e L_{MQ}}{\ell_q \ell_Q} \lambda_Q - \frac{\hat{R}}{\ell_q} \left(1 - \frac{L_{MQ}}{\ell_Q}\right) \lambda_q + \frac{\hat{R} L_{MQ}}{\ell_q \ell_Q} \lambda_Q \\
&\quad + \omega \left[1 + \frac{L_e}{\ell_d} \left(1 - \frac{L_{MD}}{\ell_d}\right)\right] \lambda_d - \frac{\omega L_e L_{MD}}{\ell_d \ell_F} \lambda_F - \frac{\omega L_e L_{MD}}{\ell_d \ell_D} \lambda_D - \sqrt{3} V_\omega \cos(\delta - \alpha)
\end{align*}
\]
Note in these two equations that there are several derivative terms and so we cannot “cleanly” use these equations to simply replace the derivatives on $\lambda_d$ and $\lambda_q$ in the flux-linkage state-space model (we were able to do so with the current state-space model).

Rather, we have to create a pre-multiplier matrix $T$ such that

$$\dot{T} x = Cx + D$$

where

$$x = \begin{bmatrix} \lambda_d \\ \lambda_F \\ \lambda_D \\ \lambda_q \\ \lambda_Q \\ \omega \\ \delta \end{bmatrix}$$

And $T$, $C$, and $D$ are given by:

$$T = \begin{bmatrix}
1 + \frac{L_c}{\ell_d} \left(1 - \frac{L_{MD}}{\ell_d}\right) & -\frac{L_cL_{MD}}{\ell_d\ell_F} & -\frac{L_cL_{MD}}{\ell_d\ell_D} & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & \frac{1}{\ell_d} \left(1 - \frac{L_{MQ}}{\ell_q}\right) & -\frac{L_cL_{MQ}}{\ell_d\ell_Q} & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}$$

Note we need to include $\lambda_G$ here, and then augment the $T$, $C$, and $D$ matrices accordingly.
Then we can pre-multiple both sides by $T^{-1}$ to obtain

$$\dot{x} = T^{-1} C x + T^{-1} D$$  \hspace{1cm} (4.163)

Equation (4.163) describes the complete system of interest to us at this point, i.e., the system of Fig. 1 at the beginning of these notes. To use it, we need the initial states $x(0)$ which are found by solving $T \dot{x} = C x + D = 0$, via $x = C^{-1} D$ where vector $D$ provides system loading information.

Then, if we perturb the system by setting, for example, $V_\infty = 0$ for a few cycles, then the response can be obtained by solving eq. (4.163) using numerical integration.