Linearization of the Swing Equation

We will cover sections 2.5.2-2.6 and beginning of Section 3.3 in these notes.

1.0 Single machine-infinite bus case

Consider a single machine connected to an infinite bus, as shown in Fig. 1 below.

The admittance matrix is given by

\[
Y = \begin{bmatrix}
Y_{11} & Y_{12} \\
Y_{21} & Y_{22}
\end{bmatrix} = \begin{bmatrix}
y_{12} & -y_{12} \\
-y_{12} & y_{12}
\end{bmatrix}
\]  

(1)

Let’s assume the machine is modeled by the swing equation with damping, given by

\[
\frac{2H}{\omega_{re}} \frac{d^2 \delta}{dt^2} + \frac{D}{\omega_{re}} \frac{d\delta}{dt} = P_m - P_e = P_m - P_M \sin(\delta - \gamma)
\]  

(2)

where

- \( P_M = |E\|V\|Y_{12}| \)
- \( Y_{12} = |Y_{12}| \angle \theta_{12} \)
- \( \gamma = \theta_{12} - \pi/2 \) (enables use of sin instead of cos-see p.27 of A&F)

Now let the angle \( \delta \) change by a **small amount**. Then

\[
\delta = \delta_0 + \Delta \delta \Rightarrow \frac{d\delta}{dt} = \frac{d\Delta \delta}{dt}, \quad \frac{d^2 \delta}{dt^2} = \frac{d^2 \Delta \delta}{dt^2}
\]  

(3)

Also recall that by Taylor series,

\[
\sin x = \sin(x_0 + \Delta x) \approx \sin x_0 + \frac{d\sin x}{dx}\bigg|_{x_0} \Delta x = \sin x_0 + (\cos x_0)\Delta x
\]  

(4)
Then we also see that
\[
\sin(\delta - \gamma) = \sin(\delta_0 - \gamma + \Delta \delta) \approx \sin(\delta_0 - \gamma) + (\cos(\delta_0 - \gamma))\Delta \delta
\]  
(Eqt. 3.3)

Applying (3) to the left-hand-side of (2) and (5) to the right-hand-side of (2), we obtain

\[
\frac{2H}{\omega_{Re}} \frac{d^2 \Delta \delta}{dt^2} + \frac{D}{\omega_{Re}} \frac{d \Delta \delta}{dt} = P_m - P_M \sin(\delta_0 - \gamma + \Delta \delta)
\]

\[
= P_m - P_M [\sin(\delta_0 - \gamma) + (\cos(\delta_0 - \gamma))\Delta \delta]
\]

\[
= P_m - P_M \sin(\delta_0 - \gamma) - P_M (\cos(\delta_0 - \gamma))\Delta \delta
\]

But

\[
P_m = P_M \sin(\delta_0 - \gamma)
\]

Therefore,

\[
\frac{2H}{\omega_{Re}} \frac{d^2 \Delta \delta}{dt^2} + \frac{D}{\omega_{Re}} \frac{d \Delta \delta}{dt} = -P_M (\cos(\delta_0 - \gamma))\Delta \delta
\]  
(7)

Or

\[
\frac{2H}{\omega_{Re}} \frac{d^2 \Delta \delta}{dt^2} + \frac{D}{\omega_{Re}} \frac{d \Delta \delta}{dt} + P_M (\cos(\delta_0 - \gamma))\Delta \delta = 0
\]  
(8)

Now define

\[
P_S = P_M \cos(\delta_0 - \gamma)
\]  
(9)

What is it?

To answer this question, observe:

\[
P_e = P_M \sin(\delta - \gamma)
\]  
(10)

\[
\frac{dP_e}{d\delta} = P_M \cos(\delta - \gamma)
\]  
(11)

\[
\left. \frac{dP_e}{d\delta} \right|_{\delta_0} = P_M \cos(\delta_0 - \gamma)
\]  
(12)
Therefore,

\[ P_S = \left. \frac{dP_e}{d\delta} \right|_{\delta_0} = P_M \cos(\delta_0 - \gamma) \quad (13) \]

\( P_S \) is called the **synchronizing power coefficient**.

In regards to early swing instability (which is a nonlinear phenomena), the larger \( P_S \) is, the more stable will be the generator for a given disturbance.

This is true because \( P_S \) indicates the slope of the power-angle curve, and the higher this slope, the more decelerating energy is available to the machine for a given fault. This idea is illustrated in Fig. 2.

![Fig. 2](image-url)

But let’s see what it means for “small signal instability,” which is characterized by the eigenvalues (roots) of the system differential equation transformed to the s-domain through LaPlace transforms.

Substituting (13) into (8) results in

\[ \frac{2H}{\omega_{Re}} \frac{d^2 \Delta \delta}{dt^2} + \frac{D}{\omega_{Re}} \frac{d \Delta \delta}{dt} + P_S \Delta \delta = 0 \quad (14) \]

Taking the LaPlace transform (assuming all initial conditions are 0), we obtain
\[
\frac{2H}{\omega_{Re}} s^2 \Delta \delta(s) + \frac{D}{\omega_{Re}} s \Delta \delta(s) + P_S \Delta \delta(s) = 0
\]

(15)

Eliminating \( \Delta \delta(s) \), we obtain the system’s characteristic equation:

\[
\frac{2H}{\omega_{Re}} s^2 + \frac{D}{\omega_{Re}} s + P_S = 0
\]

(16)

(Eqt. 3.7)

Solving using the quadratic formula, we get

\[
s = -\frac{D}{4H} \pm \frac{1}{2} \sqrt{\frac{D^2}{4H^2} - \frac{2P_S \omega_{Re}}{H}}
\]

(17)

Pulling \( \omega_{Re}/2H \) out of the radical, we have

\[
s = -\frac{D}{4H} \pm \frac{\omega_{Re}}{4H} \sqrt{\left(\frac{D}{\omega_{Re}}\right)^2 - \frac{8P_S H}{\omega_{Re}}} \]

(18)

(Eqt. 3.8)

We can make some observations about (18), as follows:

1. No damping: If \( D=0 \), then

\[
s = \pm \frac{\omega_{Re}}{4H} \sqrt{-\frac{8P_S H}{\omega_{Re}}} = \pm \sqrt{-\frac{\omega_{Re} P_S}{2H}}, \quad (19a)
\]

or

\[
s = \pm j \sqrt{\frac{\omega_{Re} P_S}{2H}} \]

(19b)

a. Observe in (19b) that if \( P_S > 0 \), (a) any response to a small disturbance will be oscillatory, and (b) the oscillatory frequency becomes lower as \( H \) becomes larger.

b. Observe in (19a) that if \( P_S < 0 \), then

\[
s = \pm \sqrt{-\frac{\omega_{Re} \left| P_S \right|}{2H}} = \pm \sigma \]

(20)

and any response is unstable.
Figures 3, 4 illustrate, for both situations $P_s>0$, $P_s<0$, respectively, the pole (eigenvalue) locations in the $s$-plane and the operating point location on the power-angle curve.

In Fig. 3, the oscillatory system is characterized by purely imaginary poles (left) and a stable operating point (right). In Fig. 4, the unstable system is characterized by the RHP-pole (left) and an unstable equilibrium point (right).
2. With damping: If D≠0, then

\[ s = \pm \frac{D}{4H} \sqrt{\left( \frac{D}{\omega_{Re}} \right)^2 - \frac{8P_S H}{\omega_{Re}}} \]  

(18)

Let’s look at the most positive root (and so we will use “+” sign before the radical, and we ensure the contribution from the second term inside the radical is positive, i.e., \( P_S < 0 \)) and ask what are the conditions under which it can be in the right-half-plane, that is:

\[ s = -\frac{D}{4H} + \frac{\omega_{Re}}{4H} \sqrt{\left( \frac{D}{\omega_{Re}} \right)^2 + \frac{8 |P_S| H}{\omega_{Re}}} > 0 \]

\[ \Rightarrow \frac{\omega_{Re}}{4H} \sqrt{\left( \frac{D}{\omega_{Re}} \right)^2 + \frac{8 |P_S| H}{\omega_{Re}}} > \frac{D}{4H} \]

\[ \Rightarrow \sqrt{\left( \frac{D}{\omega_{Re}} \right)^2 + \frac{8 |P_S| H}{\omega_{Re}}} > \frac{D}{\omega_{Re}} \]

\[ \Rightarrow \left( \frac{D}{\omega_{Re}} \right)^2 + \frac{8 |P_S| H}{\omega_{Re}} > \left( \frac{D}{\omega_{Re}} \right)^2 \]

\[ \Rightarrow \frac{8 |P_S| H}{\omega_{Re}} > 0 \]

The above relation must be true. Because the above relation is independent of damping, we conclude that if \( P_S < 0 \), the system must be unstable, independent of how much damping exists.

**2.0 Multi-machine case (Section 3.4)**

(We will come back to sections 3.2 and 3.3.1)

Recall that for a generator connected to an infinite bus, we found that the swing equation is
\[
\frac{2H}{\omega_{Re}} \frac{d^2 \delta}{dt^2} + \frac{D}{\omega_{Re}} \frac{d \delta}{dt} = P_m - P_e
\]  
(21)

where

- \( P_e = P_M \sin(\delta - \gamma) \)
- \( P_M = |E||V||Y_{12}| \)
- \( Y_{12} = |Y_{12}| \angle \theta_{12} \)

Letting \( \delta = \delta_0 + \Delta \delta \) and linearizing, we find that

\[
\frac{2H}{\omega_{Re}} \frac{d^2 \Delta \delta}{dt^2} + \frac{D}{\omega_{Re}} \frac{d \Delta \delta}{dt} - P_S \Delta \delta = 0
\]  
(22)

where

\[
P_S = \left. \frac{dP_e}{d\delta} \right|_{\delta_0} = P_M \cos(\delta_0 - \gamma)
\]  
(13)

Let’s now consider the multi-machine system assuming:
- Classical models
- Network reduced to only internal generator nodes

For generator \( i \), we have that the swing equation is

\[
\frac{2H_i}{\omega_{Re}} \frac{d^2 \Delta \delta_i}{dt^2} + \frac{D_i}{\omega_{Re}} \frac{d \Delta \delta_i}{dt} = P_{m_i} - P_{e_i}
\]  
(23)

where

\[
P_{e_i} = E_i^2 G_{ii} + \sum_{j=1}^{n} E_i E_j Y_{ij} \cos(\theta_{ij} - \delta_i + \delta_j)
\]

\[
= E_i^2 G_{ii} + \sum_{j=1, j \neq i}^{n} E_i E_j Y_{ij} \cos(\theta_{ij} - \delta_{ij})
\]  
(24)
where $\delta_{ij} = \delta_i - \delta_j$.

In (24), all voltages $E_i$, $E_j$, and all Y-bus elements $Y_{ij}$ are magnitudes.

Now let’s consider a small change in the angle of machine i: $\delta_i = \delta_{i0} + \Delta \delta_i$.

The left-hand-side of (24) is precisely as in the case of the single generator vs. infinite buses case. But what happened to the right-hand-side? Now the right-hand-side is, by (23), $P_{m_i} - P_{e_i}$.

- $P_{m_i}$ is unaffected by $+\Delta \delta_j$, but
- $P_{e_i}$ is affected by it.

Recall $\delta_{ij} = \delta_i - \delta_j$. We consider a small change in rotor angle at generator i. To be more general, we also allow a small change in generator j. However, generator j will not change as a result of the generator i change; they are independent changes and we could just as well have only one of them.

\[
\begin{align*}
\delta_i &= \delta_{i0} + \Delta \delta_i \\
\delta_j &= \delta_{j0} + \Delta \delta_j
\end{align*}
\]

Recalling that

\[
P_{ei} = E_i^2 G_{ii} + \sum_{j=1, j \neq i}^{n} E_i E_j Y_{ij} \cos(\theta_{ij} - \delta_{ij})
\]

we need to see what happens to the cos term for the small change in angle.

We know from trigonometry that

\[
\cos(x - y) = \sin x \sin y + \cos x \cos y
\]

Then

\[
\cos(\theta_{ij} - \delta_{ij}) = \sin \theta_{ij} \sin \delta_{ij} + \cos \theta_{ij} \cos \delta_{ij}
\]

Application of (26) to (25) yields:
\[ P_{ei} = E_i^2 G_{ii} + \sum_{j=1}^{n} E_i E_j \left\{ Y_{ij} \sin \theta_{ij} \sin \delta_{ij} + Y_{ij} \cos \theta_{ij} \cos \delta_{ij} \right\} \]

\[ = E_i^2 G_{ii} + \sum_{j=1}^{n} E_i E_j \left\{ B_{ij} \sin \delta_{ij} + G_{ij} \cos \delta_{ij} \right\} \]  

(Eqt. 3.21)

Now we need to linearize the \( \cos \delta_{ij} \) and \( \sin \delta_{ij} \) terms using \( \delta_{ij} = \delta_{ij0} + \Delta \delta_{ij} \).

From Taylor series with first order term only,
\[
\sin \delta_{ij} = \sin(\delta_{ij0} + \Delta \delta_{ij}) = \sin \delta_{ij0} + \Delta \delta_{ij} \cos \delta_{ij0} \quad (28)
\]
\[
\cos \delta_{ij} = \cos(\delta_{ij0} + \Delta \delta_{ij}) = \cos \delta_{ij0} - \Delta \delta_{ij} \sin \delta_{ij0} \quad (29)
\]

Substituting (28) and (29) into (27), we get
\[
P_{ei} = E_i^2 G_{ii} + \sum_{j=1}^{n} E_i E_j \left\{ B_{ij} \left( \sin \delta_{ij0} + \Delta \delta_{ij} \cos \delta_{ij0} \right) \right. \\
+ \left. G_{ij} \left( \cos \delta_{ij0} - \Delta \delta_{ij} \sin \delta_{ij0} \right) \right\} \]

Now collect terms in \( \Delta \delta_{ij} \):
\[ P_{ei} = E_i^2 G_{ii} + \sum_{j=1, j\neq i}^{n} E_i E_j \left\{ B_{ij} \sin \delta_{ij0} + G_{ij} \cos \delta_{ij0} \right\} \]

\[ + \sum_{j=1, j\neq i}^{n} E_i E_j \left\{ B_{ij} \cos \delta_{ij0} - G_{ij} \sin \delta_{ij0} \right\} \Delta \delta_{ij} \]

\[ = P_{mi} + \sum_{j=1, j\neq i}^{n} E_i E_j \left\{ B_{ij} \cos \delta_{ij0} - G_{ij} \sin \delta_{ij0} \right\} \Delta \delta_{ij} \]

(31a)

Recall that the right-hand-side of the swing equation is \( P_{mi} - P_{ei} \).
Equation (31a) can be rewritten then as (31b)

\[ P_{ei} - P_{mi} = \sum_{j=1, j\neq i}^{n} E_i E_j \left\{ B_{ij} \cos \delta_{ij0} - G_{ij} \sin \delta_{ij0} \right\} \Delta \delta_{ij} \]

(31b)

therefore the swing equation (23), which is

\[ \frac{2H_i}{\omega_{Re}} \frac{d^2 \Delta \delta_i}{dt^2} + \frac{D_i}{\omega_{Re}} \frac{d \Delta \delta_i}{dt} = P_{mi} - P_{ei} \]

becomes

\[ \frac{2H_i}{\omega_{Re}} \frac{d^2 \Delta \delta_i}{dt^2} + \frac{D_i}{\omega_{Re}} \frac{d \Delta \delta_i}{dt} = -\sum_{j=1, j\neq i}^{n} E_i E_j \left\{ B_{ij} \cos \delta_{ij0} - G_{ij} \sin \delta_{ij0} \right\} \Delta \delta_{ij} \]

(32)

Define everything inside the expression within the summation of (32), except \( \Delta \delta_{ij} \), as \( P_{Sij} \), that is
\[ P_{Sij} = E_i E_j \left\{ B_{ij} \cos \delta_{ij0} - G_{ij} \sin \delta_{ij0} \right\} \]  

(33)

Then (32) becomes

\[ \frac{2H_i}{\omega_{Re}} \frac{d^2 \Delta \delta_i}{dt^2} + \frac{D_i}{\omega_{Re}} \frac{d \Delta \delta_i}{dt} = - \sum_{j=1}^{n} P_{Sij} \Delta \delta_{ij} \]  

(34)

Given the mechanical power is constant, the right-hand-side of (34) gives the negative of the change in electric power out of the machine due to the small changes \( \Delta \delta_{ij} \), that is

\[ \Delta P_{ei} = \sum_{j=1}^{n} P_{Sij} \Delta \delta_{ij} \]  

(35)

(Eqt. 3.23)

What is \( P_{Sij} \)? We answer this question by observing that the power flowing from generator internal node \( i \) to generator internal node \( j \) is

\[ P_{ij} = E_i E_j \left\{ B_{ij} \sin \delta_{ij} + G_{ij} \cos \delta_{ij} \right\} \]  

(36)

Differentiating, we get

\[ \frac{\partial P_{ij}}{\partial \delta_{ij}} = E_i E_j \left\{ B_{ij} \cos \delta_{ij} - G_{ij} \sin \delta_{ij} \right\} \]  

(37)

Evaluating at \( \delta_{ij0} \), we get

\[ \frac{\partial P_{ij}}{\partial \delta_{ij}} \bigg|_{\delta_{ij0}} = E_i E_j \left\{ B_{ij} \cos \delta_{ij0} - G_{ij} \sin \delta_{ij0} \right\} = P_{Sij} \]  

(Eqt. 3.24)

Note that if bus \( j \) is the infinite bus, neglecting resistance, we have:

\[ P_{Sij} = E_i E_j B_{ij} \cos \delta_{ij0} \]

which is the same as the synchronizing power coefficient in the infinite bus case (we called it \( P_s \)).

We will look at multimachine systems, but before we do, we consider something very important... response to load changes...