Linearization of the Swing Equation

We will cover sections 2.5.2-2.6 and beginning of Section 3.3 in these notes.

1.0 Single machine-infinite bus case

Consider a single machine connected to an infinite bus, as shown in Fig. 1 below.

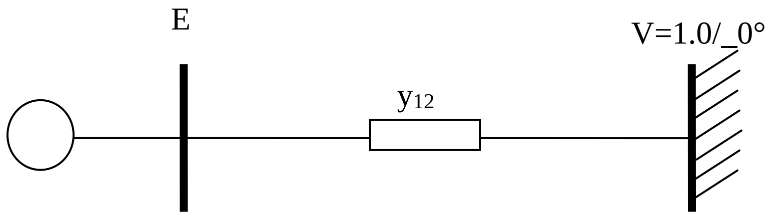

Fig. 1

The admittance matrix is given by

$$Y = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} y_{12} & -y_{12} \\ -y_{12} & y_{12} \end{bmatrix} \quad (1)$$

Let's assume the machine is modeled by the swing equation with damping (eq. 1 in our notes "multimachine" and (2.56) in VMAF).

$$\frac{2H}{\omega_{\text{Re}}} \frac{d^2\delta}{dt^2} + \frac{D}{\omega_{\text{Re}}} \frac{d\delta}{dt} = P_m - P_e = P_m - P_M \sin(\delta - \gamma) \quad (2)$$

where

- $P_M = |E||V||Y_{12}|$
- $Y_{12} = |Y_{12}| \angle\theta_{12}$
- $\gamma = \theta_{12} - \pi/2$ (enables use of sin instead of cos-see p. 33, VMAF)

Now let the angle δ change by a ***small amount***. Then

$$\delta = \delta_0 + \Delta\delta \Rightarrow \frac{d\delta}{dt} = \frac{d\Delta\delta}{dt}, \quad \frac{d^2\delta}{dt^2} = \frac{d^2\Delta\delta}{dt^2} \quad (3)$$

Also recall that by Taylor series,

$$\sin x = \sin\left(x_0 + \Delta x\right) \approx \sin x_0 + \left.\frac{d\sin x}{dx}\right|_{x_0} \Delta x = \sin x_0 + (\cos x_0)\Delta x \quad (4)$$

Then we also see that

$$\sin(\delta-\gamma)=\sin(\delta_0-\gamma+\Delta\delta)\approx\sin(\delta_0-\gamma)+(\cos(\delta_0-\gamma))\Delta\delta \quad (5)$$

(Eqt. 3.3)

Applying (3) to the left-hand-side of (2) and (5) to the right-hand-side of (2), we obtain

$$\begin{aligned}\frac{2H}{\omega_{\text{Re}}}\frac{d^2\Delta\delta}{dt^2}+\frac{D}{\omega_{\text{Re}}}\frac{d\Delta\delta}{dt} &= P_m - P_M\sin(\delta_0-\gamma+\Delta\delta)\\ &= P_m - P_M[\sin(\delta_0-\gamma)+(\cos(\delta_0-\gamma))\Delta\delta]\\ &= P_m - P_M\sin(\delta_0-\gamma)-P_M(\cos(\delta_0-\gamma))\Delta\delta\end{aligned} \quad (6)$$

But from steady-state conditions, we know the mechanical power is:

$$P_m = P_M\sin(\delta_0-\gamma)$$

Therefore,

$$\frac{2H}{\omega_{\text{Re}}}\frac{d^2\Delta\delta}{dt^2}+\frac{D}{\omega_{\text{Re}}}\frac{d\Delta\delta}{dt} = -P_M(\cos(\delta_0-\gamma))\Delta\delta \quad (7)$$

Or

$$\frac{2H}{\omega_{\text{Re}}}\frac{d^2\Delta\delta}{dt^2}+\frac{D}{\omega_{\text{Re}}}\frac{d\Delta\delta}{dt} + P_M(\cos(\delta_0-\gamma))\Delta\delta = 0 \quad (8)$$

Now define

$$P_S = P_M\cos(\delta_0-\gamma) \quad (9)$$

What is it?

To answer this question, observe:

$$P_e = P_M\sin(\delta-\gamma) \quad (10)$$

$$\frac{dP_e}{d\delta} = P_M\cos(\delta-\gamma) \quad (11)$$

$$\left.\frac{dP_e}{d\delta}\right|_{\delta_0} = P_M\cos(\delta_0-\gamma) \quad (12)$$

Therefore,

$$P_S = \left.\frac{dP_e}{d\delta}\right|_{\delta_0} = P_M \cos(\delta_0 - \gamma) \tag{13}$$

P_S is called the *synchronizing power coefficient*.
In regard to early swing instability (which is a nonlinear phenomena), the larger P_S is, the more stable will be the generator for a given disturbance.

This is true because P_S indicates the slope of the power-angle curve, and the higher this slope, the more decelerating energy is available to the machine for a given fault. This idea is illustrated in Fig. 2.

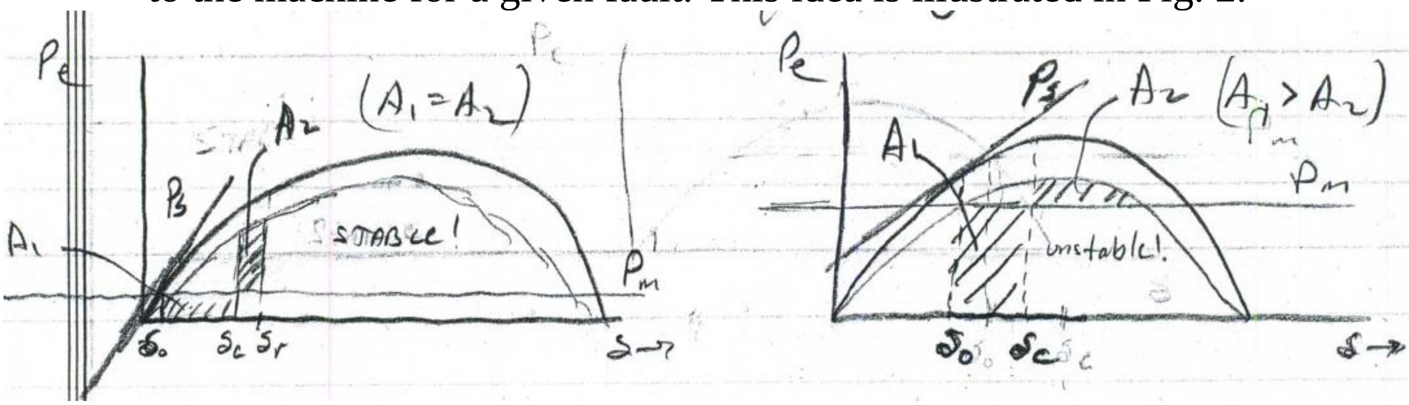

Fig. 2

But let’s see what it means for “small signal instability,” which is characterized by the eigenvalues (roots) of the system differential equation transformed to the s-domain through LaPlace transforms.

Substituting (13) into (8) results in

$$\frac{2H}{\omega_{\mathrm{Re}}}\frac{d^2\Delta\delta}{dt^2} + \frac{D}{\omega_{\mathrm{Re}}}\frac{d\Delta\delta}{dt} + P_S\Delta\delta = 0 \tag{14}$$

Taking the LaPlace transform (assuming all initial conditions are 0), we obtain

$$\frac{2H}{\omega_{\mathrm{Re}}} s^2 \Delta\delta(s) + \frac{D}{\omega_{\mathrm{Re}}} s\Delta\delta(s) + P_S \Delta\delta(s) = 0 \tag{15}$$

Eliminating $\Delta\delta(s)$, we obtain the system's characteristic equation:

$$\frac{2H}{\omega_{\mathrm{Re}}} s^2 + \frac{D}{\omega_{\mathrm{Re}}} s + P_S = 0 \tag{16}$$

(Eqt. 3.7)

Solving using the quadratic formula, we get

$$s = -\frac{D}{4H} \pm \frac{1}{2}\sqrt{\frac{D^2}{4H^2} - \frac{2P_S \omega_{\mathrm{Re}}}{H}} \tag{17}$$

Pulling $\omega_{Re}/2H$ out of the radical, we have

$$s = -\frac{D}{4H} \pm \frac{\omega_{\mathrm{Re}}}{4H}\sqrt{\left(\frac{D}{\omega_{\mathrm{Re}}}\right)^2 - \frac{8P_S H}{\omega_{\mathrm{Re}}}} \tag{18}$$

(Eqt. 3.8)

We can make some observations about (18), as follows:

1. No damping: If D=0, then

$$s = \pm \frac{\omega_{\mathrm{Re}}}{4H}\sqrt{-\frac{8P_S H}{\omega_{\mathrm{Re}}}} = \pm\sqrt{-\frac{\omega_{\mathrm{Re}} P_S}{2H}}, \tag{19a}$$

   or

$$s = \pm j\sqrt{\frac{\omega_{\mathrm{Re}} P_S}{2H}} \tag{19b}$$

   a. Observe in (19b) that if P_S>0, (a) any response to a small disturbance will be oscillatory, and (b) the oscillatory frequency becomes lower as H becomes larger.
   b. Observe in (19a) that if P_S<0, then

$$s = \pm\sqrt{\frac{\omega_{\mathrm{Re}} \mid P_S \mid}{2H}} = \pm\sigma \tag{20}$$

   and any response is unstable.

Figures 3, 4 illustrate, for both situations $P_S>0$, $P_S<0$, respectively, the pole (eigenvalue) locations in the s-plane and the operating point location on the power-angle curve.

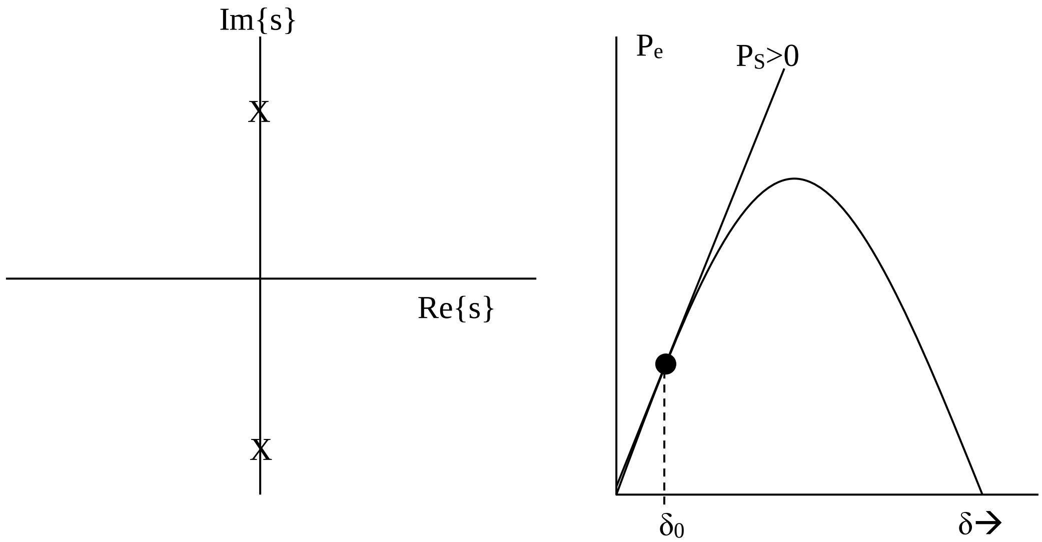

Fig. 3: $P_S>0$

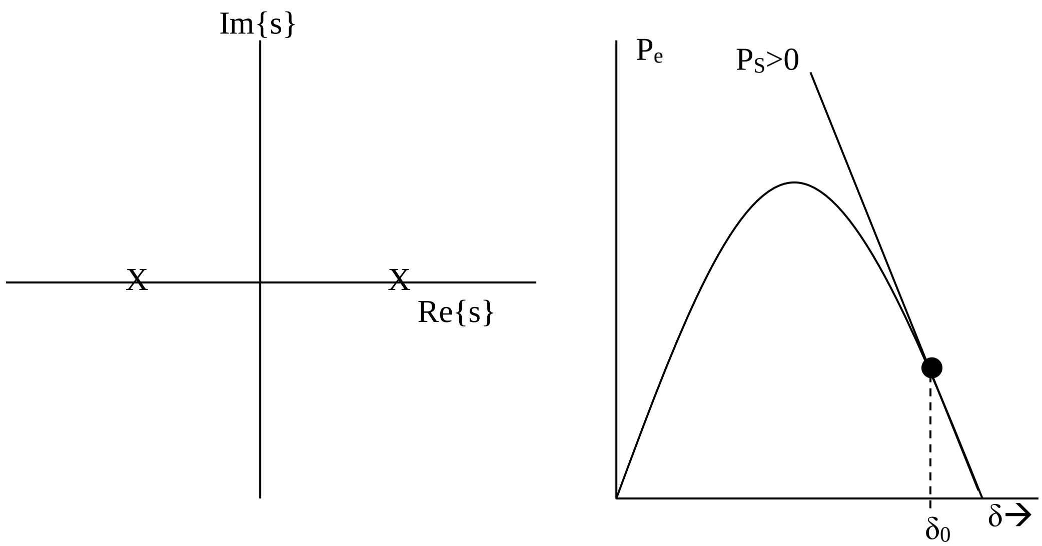

Fig. 4: $P_S<0$

In Fig. 3, the oscillatory system is characterized by purely imaginary poles (left) and a stable operating point (right). In Fig. 4, the unstable system is characterized by the RHP-pole (left) and an unstable equilibrium point (right). (Not shown: We may have a marginally stable system characterized by a pole at origin, equilibrium point δ_0=90°; it will have a constant amplitude response).

2. With damping: If D≠0, then

$$s = -\frac{D}{4H} \pm \frac{\omega_{\text{Re}}}{4H}\sqrt{\left(\frac{D}{\omega_{\text{Re}}}\right)^2 - \frac{8P_S H}{\omega_{\text{Re}}}} \tag{18}$$

Let's look at the most positive root (and so we will use "+" sign before the radical, and we ensure the contribution from the second term inside the radical is positive, i.e., $P_S<0$) and ask what are the conditions under which it can be in the right-half-plane, that is:

$$s = -\frac{D}{4H} + \frac{\omega_{\text{Re}}}{4H}\sqrt{\left(\frac{D}{\omega_{\text{Re}}}\right)^2 + \frac{8|P_S|H}{\omega_{\text{Re}}}} \overset{?}{>} 0$$

$$\Rightarrow \frac{\omega_{\text{Re}}}{4H}\sqrt{\left(\frac{D}{\omega_{\text{Re}}}\right)^2 + \frac{8|P_S|H}{\omega_{\text{Re}}}} \overset{?}{>} \frac{D}{4H}$$

$$\Rightarrow \sqrt{\left(\frac{D}{\omega_{\text{Re}}}\right)^2 + \frac{8|P_S|H}{\omega_{\text{Re}}}} \overset{?}{>} \frac{D}{\omega_{\text{Re}}}$$

$$\Rightarrow \left(\frac{D}{\omega_{\text{Re}}}\right)^2 + \frac{8|P_S|H}{\omega_{\text{Re}}} \overset{?}{>} \left(\frac{D}{\omega_{\text{Re}}}\right)^2$$

$$\Rightarrow \frac{8|P_S|H}{\omega_{\text{Re}}} \overset{?}{>} 0$$

The above relation must be true. Because the above relation is independent of damping, we conclude that if $P_S<0$, the system must be unstable, independent of how much damping exists.

On the other hand, if $P_S>0$, then one may show (see app. of these notes) from (18) there are 2 possible conditions, depending on how much damping there is: (i) small-signal stable and oscillatory (LHP poles on $j\omega$ axis); (ii) small-signal stable and non-oscillatory (LHP poles on real axis). It is not possible for the system to be small-signal unstable, a reflection of the fact that small excursions around a point having $P_S>0$ (left part of power-angle curve) must be stable.

2.0 Multi-machine case (Section 3.4)

(We will come back to sections 3.2 and 3.3.1)

Recall that for a generator connected to an infinite bus, we found that the swing equation is

$$\frac{2H}{\omega_{\mathrm{Re}}}\frac{d^2\delta}{dt^2}+\frac{D}{\omega_{\mathrm{Re}}}\frac{d\delta}{dt}=P_m-P_e \tag{21}$$

where

- $P_e = P_M \sin(\delta-\gamma)$
- $P_M = |E||V||Y_{12}|$
- $Y_{12} = |Y_{12}|\angle\theta_{12}$

Letting $\delta=\delta_0+\Delta\delta$ and linearizing, we find that

$$\frac{2H}{\omega_{\mathrm{Re}}}\frac{d^2\Delta\delta}{dt^2}+\frac{D}{\omega_{\mathrm{Re}}}\frac{d\Delta\delta}{dt}-P_S\Delta\delta=0 \tag{22}$$

where

$$P_S=\left.\frac{dP_e}{d\delta}\right|_{\delta_0}=P_M\cos(\delta_0-\gamma) \tag{13}$$

Let’s now consider the multi-machine system assuming:

- Classical models
- Network reduced to only internal generator nodes

For generator i, we have that the swing equation is

$$\frac{2H_i}{\omega_{\mathrm{Re}}}\frac{d^2\Delta\delta_i}{dt^2}+\frac{D_i}{\omega_{\mathrm{Re}}}\frac{d\Delta\delta_i}{dt}=P_{m_i}-P_{e_i} \tag{23}$$

where

$$P_{ei} = E_i^2 G_{ii} + \sum_{\substack{j=1 \\ j \neq i}}^{n} E_i E_j Y_{ij} \cos(\theta_{ij} - \delta_i + \delta_j)$$

$$= E_i^2 G_{ii} + \sum_{\substack{j=1 \\ j \neq i}}^{n} E_i E_j Y_{ij} \cos(\theta_{ij} - \delta_{ij}) \qquad (24)$$

where $\delta_{ij}=\delta_i-\delta_j$.

In (24), all voltages E_i, E_j, and all Y-bus elements Y_{ij} are magnitudes.

Now let's consider a small change in the angle of machine i: $\delta_i=\delta_{i0}+\Delta\delta_j$.

The left-hand-side of (23) is precisely as in the case of the single generator vs. infinite bus case. But what happened to the right-hand-side? Now the right-hand-side is, by (23), $P_{m_i} - P_{e_i}$.

- P_{mi} is unaffected by $+\Delta\delta_j$, but
- P_{ei} is affected by it, by (24).

Recall $\delta_{ij}=\delta_i-\delta_j$. We consider a small change in rotor angle at generator i. To be more general, we also allow a small change in generator j. However, in general, generator j does not change as a result of the generator i change; we consider they are independent changes and we could just as well have only one of them.

$\delta_i=\delta_{i0}+\Delta\delta_i \qquad\qquad \delta_j=\delta_{j0}+\Delta\delta_j$

Recalling from (24) that

$$P_{ei} = E_i^2 G_{ii} + \sum_{\substack{j=1 \\ j \neq i}}^{n} E_i E_j Y_{ij} \cos(\theta_{ij} - \delta_{ij}) \qquad (25)$$

we need to see what happens to the cos term for the small change in angle.

We know from trigonometry that

$$\cos(x-y)=\sin x \sin y+\cos x \cos y$$

Then

$$\cos(\theta_{ij}-\delta_{ij})=\sin\theta_{ij}\sin\delta_{ij}+\cos\theta_{ij}\cos\delta_{ij} \quad (26)$$

Application of (26) to (25) yields:

$$P_{ei}=E_i^2 G_{ii}+\sum_{\substack{j=1\\ j\neq i}}^{n} E_i E_j\left\{Y_{ij}\sin\theta_{ij}\sin\delta_{ij}+Y_{ij}\cos\theta_{ij}\cos\delta_{ij}\right\}$$

$$=E_i^2 G_{ii}+\sum_{\substack{j=1\\ j\neq i}}^{n} E_i E_j\left\{B_{ij}\sin\delta_{ij}+G_{ij}\cos\delta_{ij}\right\} \quad (27)$$

(Eqt. 3.21)

Now we need to linearize the cosδ_{ij} and sinδ_{ij} terms using $\delta_{ij}=\delta_{ij0}+\Delta\delta_{ij}$.

From Taylor series with first order term only,

$$\sin\delta_{ij}=\sin(\delta_{ij0}+\Delta\delta_{ij})=\sin\delta_{ij0}+\Delta\delta_{ij}\cos\delta_{ij0} \quad (28)$$

$$\cos\delta_{ij}=\cos(\delta_{ij0}+\Delta\delta_{ij})=\cos\delta_{ij0}-\Delta\delta_{ij}\sin\delta_{ij0} \quad (29)$$

Substituting (28) and (29) into (27), we get

$$P_{ei}=E_i^2 G_{ii}+\sum_{\substack{j=1\\ j\neq i}}^{n} E_i E_j\left\{B_{ij}\left(\sin\delta_{ij0}+\Delta\delta_{ij}\cos\delta_{ij0}\right)\right.$$

$$\left.+G_{ij}\left(\cos\delta_{ij0}-\Delta\delta_{ij}\sin\delta_{ij0}\right)\right\}$$

(30)

Now collect terms in $\Delta\delta_{ij}$:

$$P_{ei} = E_i^2 G_{ii} + \sum_{\substack{j=1 \\ j\neq i}}^{n} E_i E_j \left\{ B_{ij} sin\delta_{ij0} + G_{ij} cos\delta_{ij0} \right\}$$

$$+ \sum_{\substack{j=1 \\ j\neq i}}^{n} E_i E_j \left\{ B_{ij} cos\delta_{ij0} - G_{ij} sin\delta_{ij0} \right\} \Delta\delta_{ij} \qquad (31a)$$

But the top line of the RHS in (31a) is the steady-state power P_{mi}:

$$P_{mi} = E_i^2 G_{ii} + \sum_{\substack{j=1 \\ j\neq i}}^{n} E_i E_j \left\{ B_{ij} cos\delta_{ij0} - G_{ij} sin\delta_{ij0} \right\} \Delta\delta_{ij} \qquad (31b)$$

Recall that the right-hand-side of the swing equation is P_{mi}-P_{ei}. Substitution of (31a) and (31b) results in

$$P_{mi} - P_{ei} = E_i^2 G_{ii} + \sum_{\substack{j=1 \\ j\neq i}}^{n} E_i E_j \left\{ B_{ij} cos\delta_{ij0} - G_{ij} sin\delta_{ij0} \right\} \Delta\delta_{ij}$$

$$- E_i^2 G_{ii} - \sum_{\substack{j=1 \\ j\neq i}}^{n} E_i E_j \left\{ B_{ij} sin\delta_{ij0} + G_{ij} cos\delta_{ij0} \right\} - \sum_{\substack{j=1 \\ j\neq i}}^{n} E_i E_j \left\{ B_{ij} cos\delta_{ij0} - G_{ij} sin\delta_{ij0} \right\} \Delta\delta_{ij}$$

$$= - \sum_{\substack{j=1 \\ j\neq i}}^{n} E_i E_j \left\{ B_{ij} cos\delta_{ij0} - G_{ij} sin\delta_{ij0} \right\} \Delta\delta_{ij} \qquad (31c)$$

therefore the swing equation (23), which is

$$\frac{2H_i}{\omega_{\mathrm{Re}}} \frac{d^2 \Delta\delta_i}{dt^2} + \frac{D_i}{\omega_{\mathrm{Re}}} \frac{d\Delta\delta_i}{dt} = P_{m_i} - P_{e_i} \qquad (23)$$

becomes

$$\frac{2H_i}{\omega_{\mathrm{Re}}} \frac{d^2 \Delta\delta_i}{dt^2} + \frac{D_i}{\omega_{\mathrm{Re}}} \frac{d\Delta\delta_i}{dt} = -\sum_{\substack{j=1 \\ j\neq i}}^{n} E_i E_j \left\{ B_{ij} \cos\delta_{ij0} - G_{ij} \sin\delta_{ij0} \right\} \Delta\delta_{ij} \qquad (32)$$

Define everything inside the expression within the summation of (32), except $\Delta\delta_{ij}$, as P_{Sij}, that is

$$P_{Sij} = E_i E_j \{B_{ij} \cos\delta_{ij0} - G_{ij} \sin\delta_{ij0}\} \tag{33}$$

Then (32) becomes

$$\frac{2H_i}{\omega_{\text{Re}}} \frac{d^2\Delta\delta_i}{dt^2} + \frac{D_i}{\omega_{\text{Re}}} \frac{d\Delta\delta_i}{dt} = -\sum_{\substack{j=1 \\ j\neq i}}^{n} P_{Sij}\Delta\delta_{ij} \tag{34}$$

Given the mechanical power is constant, the right-hand-side of (34) gives the negative of the change in electric power out of the machine due to the small changes $\Delta\delta_{ij}$, that is

$$\Delta P_{ei} = \sum_{\substack{j=1 \\ j\neq i}}^{n} P_{Sij}\Delta\delta_{ij} \tag{35}$$

(Eqt. 3.23)

What is P_{Sij}? We answer this question by observing that the power flowing from generator internal node *i* to generator internal node *j* is

$$P_{ij} = E_i E_j \{B_{ij} \sin\delta_{ij} + G_{ij} \cos\delta_{ij}\} \tag{36}$$

Differentiating, we get

$$\frac{\partial P_{ij}}{\partial \delta_{ij}} = E_i E_j \{B_{ij} \cos\delta_{ij} - G_{ij} \sin\delta_{ij}\} \tag{37}$$

Evaluating at δ_{ij0}, we get

$$\left.\frac{\partial P_{ij}}{\partial \delta_{ij}}\right|_{\delta_{ij0}} = E_i E_j \{B_{ij} \cos\delta_{ij0} - G_{ij} \sin\delta_{ij0}\} = P_{Sij} \tag{38}$$

(Eqt. 3.24)

Note that if bus *j* is the infinite bus, neglecting resistance, we have:

$$P_{Sij} = E_i E_j B_{ij} \cos\delta_{ij0}$$

which is the same as the synchronizing power coefficient in the infinite bus case (we called it P_S).

We will look at multimachine systems; before we do, we consider something important in the next notes: response to load changes.

One last issue: what is the difference between synchronizing power coefficient, generation shift factor (GSF) and power transfer distribution factor? We answer this here.

- Synchronizing power coefficient (SPC):

$$P_{Sij} = \left.\frac{\partial P_{ij}}{\partial \delta_{ij}}\right|_{\delta_{ij0}} = E_i E_j \left\{ B_{ij} \cos \delta_{ij0} - G_{ij} \sin \delta_{ij0} \right\} \quad \text{(a1)}$$

  Observe that the SPC gives
  - change in flow on circuit $\{i,j\}$ with respect to
  - a change in angular separation across $\{i,j\}$.

- Generation shift factor (GSF):

$$t_{\{b\},i} = \left.\frac{\partial P_{\{b\}}}{\partial P_i}\right|_{\substack{\text{Reallocation}\\ \text{Policy}}} \quad \text{(b1)}$$

  Observe that the GSF gives
  - change in flow across any branch b with respect to
  - a change in injection at bus i, subject to a reallocation policy (i.e., how the bus i change in injection is compensated).
- Power transfer distribution factor (PTDF) for 1-bus injection change:

$$PTDF_{\{b\},i} = \left.\frac{\partial P_{\{b\}}}{\partial P_i}\right|_{\substack{\text{Reallocation}\\ \text{Policy}}} \quad \text{(c)}$$

  The PTDF for 1-bus injection change is the same as the GSF.
- Power transfer distribution factor for 2-bus injection change:

$$PTDF_{\{ij\},i,j} = \left.\frac{\partial P_{\{ij\}}}{\partial P_i}\right|_{\substack{\text{Reallocation}\\ \text{Policy}}} - \left.\frac{\partial P_{\{ij\}}}{\partial P_j}\right|_{\substack{\text{Reallocation}\\ \text{Policy}}} \qquad \text{(c)}$$

The upshot of the above is that relating SPC to GSF is enough to relate SPC to PTDF. We relate SPC to GSF as follows:
From (a1), we write that

$$P_{Sij} = \left.\frac{\Delta P_{ij}}{\Delta \delta_{ij}}\right|_{\delta_{ij0}} \Rightarrow \Delta P_{ij} = P_{Sij}\big|_{\delta_{ij0}} \Delta \delta_{ij} \qquad \text{(a2)}$$

From (b1), we write that

$$t_{\{b\},i} = \left.\frac{\Delta P_{\{b\}}}{\Delta P_i}\right|_{\substack{\text{Reallocation}\\ \text{Policy}}} \Rightarrow \Delta P_{\{b\}} = t_{\{b\},i}\big|_{\substack{\text{Reallocation}\\ \text{Policy}}} \Delta P_i\big|_{\substack{\text{Reallocation}\\ \text{Policy}}} \qquad \text{(b2)}$$

Now

1. Consider our power system is experiencing conditions such that the angular separation between buses i and j is δ_{ij0}.
2. Line b is terminated by buses i and j, i.e., b≡{i,j}.
3. We make a change in injected power at bus i equal to ΔP_i compensated by a "reallocation policy" where an equal and opposite change, ΔP_j, is made at bus j.

Then (a2) and (b2) are equivalent:

$$\Delta P_{ij} = P_{Sij}\big|_{\delta_{ij0}} \Delta \delta_{ij} = \Delta P_{\{b\}} = t_{\{b\},i}\big|_{\substack{\text{Reallocation}\\ \text{Policy: } \Delta P_j = -\Delta P_i}} \Delta P_i\big|_{\substack{\text{Reallocation}\\ \text{Policy: } \Delta P_j = -\Delta P_i}}$$

That is:

$$P_{Sij}\big|_{\delta_{ij0}} \Delta \delta_{ij} = t_{\{b\},i}\big|_{\substack{\text{Reallocation}\\ \text{Policy: } \Delta P_j = -\Delta P}} \Delta P_i\big|_{\substack{\text{Reallocation}\\ \text{Policy: } \Delta P_j = -\Delta P}}$$

which shows us that

$$t_{\{b\},i}\big|_{\substack{\text{Reallocation}\\ \text{Policy: } \Delta P_j = -\Delta P}} = P_{Sij}\big|_{\delta_{ij0}} \frac{\Delta \delta_{ij}}{\Delta P_i\big|_{\substack{\text{Reallocation}\\ \text{Policy: } \Delta P_j = -\Delta P}}}$$

Appendix

On p. 6 of these notes, we wrote

"On the other hand, if P_S>0, then one may show (see app. of these notes) from (18) there are 2 possible conditions, depending on how much damping there is: (i) small-signal stable and oscillatory (LHP poles on $j\omega$ axis); (ii) small-signal stable and non-oscillatory (LHP poles on real axis). It is not possible for the system to be small-signal unstable, a reflection of the fact that small excursions around a point having P_S>0 (left part of power-angle curve) must be stable."

Here, we prove the last statement, i.e., with P_S>0, that it is not possible for the system to be small-signal unstable. Starting from (18) (eq. 3.8 in VMAF):

$$s = -\frac{D}{4H} \pm \frac{\omega_{\mathrm{Re}}}{4H}\sqrt{\left(\frac{D}{\omega_{\mathrm{Re}}}\right)^2 - \frac{8P_S H}{\omega_{\mathrm{Re}}}} \tag{18}$$

If we assume that P_S>0, then (18) becomes

$$s = -\frac{D}{4H} \pm \frac{\omega_{Re}}{4H}\sqrt{\left(\frac{D}{\omega_{Re}}\right)^2 - \frac{8|P_S|H}{\omega_{Re}}} \tag{A-1}$$

If it is unstable, then the pole with the largest real part (and so we use the "+" sign in (A-1)) must be in RHP:

$$-\frac{D}{4H} + \frac{\omega_{Re}}{4H}\sqrt{\left(\frac{D}{\omega_{Re}}\right)^2 - \frac{8|P_S|H}{\omega_{Re}}} > 0 \tag{A-2}$$

$$\frac{\omega_{Re}}{4H}\sqrt{\left(\frac{D}{\omega_{Re}}\right)^2 - \frac{8|P_S|H}{\omega_{Re}}} > \frac{D}{4H} \Rightarrow \sqrt{\left(\frac{D}{\omega_{Re}}\right)^2 - \frac{8|P_S|H}{\omega_{Re}}} > \frac{D}{\omega_{Re}} \tag{A-3}$$

$$\Rightarrow \left(\frac{D}{\omega_{Re}}\right)^2 - \frac{8|P_S|H}{\omega_{Re}} > \left(\frac{D}{\omega_{Re}}\right)^2 \Rightarrow -\frac{8|P_S|H}{\omega_{Re}} > 0 \tag{A-4}$$

However, (A-4) is impossible. QED.