Brief Review of Linear System Theory

The following information is typically covered in a course on linear system theory. At ISU, EE 577 is one such course and is highly recommended for power system engineering students.

We have developed a model that appears as

$$ \Delta \dot{x} = A \Delta x $$

We may write this more compactly as

$$ \dot{x} = Ax $$

where the "Δ" is implied.

Taking the LaPlace transform, with initial conditions \(x(0)\), we have:

$$ sX(s) - x(0) = AX(s) $$

$$ \Rightarrow sX(s) - AX(s) = x(0) $$

Factoring out the vector \(X(s)\) results in:

$$ [sI - A]X(s) = x(0) $$

where \(I\) is the identity matrix of same dimension as \(A\).

Pre-multiplying both sides by \([sI - A]^{-1}\), we get:

$$ X(s) = [sI - A]^{-1}x(0) $$

(\text{L-1})

and taking the inverse-LaPlace transform leads to

$$ x(t) = L^{-1}\left\{ [sI - A]^{-1}x(0) \right\} $$

(\text{L-2a})

Note that in the above, by expressing \([sI - A]^{-1}\), we implicitly assume that it is invertible and therefore non-singular (this requires that our system has non-zero determinant).

Recall that a matrix inverse is the adjoint divided by the determinant, i.e., \(K^{-1} = \text{Adj}(K)/\text{det}(K)\).

Applying this to eq. (L-1), we have:
\[ X(s) = \frac{\text{Adj}\{[sI - A]\}}{\det\{[sI - A]\}} x(0) \]

The determinant of a matrix is a scalar quantity, and in this case, it is a scalar polynomial in the Laplace variable \( s \) so that:

\[
\det\{[sI - A]\} = a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0
\]

Such a polynomial may always be factored in the form:

\[
\det\{[sI - A]\} = a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0 = (s - \lambda_1)(s - \lambda_2)\ldots(s - \lambda_n)
\]

where the \( \lambda_k \), \( k=1, \ldots, n \) are the roots of the polynomial. Therefore,

\[
X(s) = \frac{\text{Adj}\{[sI - A]\}}{\det\{[sI - A]\}} x(0) = \frac{\text{Adj}\{[sI - A]\} x(0)}{(s - \lambda_1)(s - \lambda_2)\ldots(s - \lambda_n)} \quad \text{(L-3)}
\]

Eq. (L-3) expresses the \( n \)-dimensional vector \( X(s) \) as a function of

1. The \( n \times n \) matrix \( \text{Adj}[sI - A] \),
2. The \( n \times 1 \) vector \( x(0) \)
3. The factored polynomial \( (s - \lambda_1)(s - \lambda_2)\ldots(s - \lambda_n) \)

Note that the numerator is the product of an \( n \times n \) matrix and an \( n \times 1 \) vector and therefore it is \( n \times 1 \), which is the dimension of the right-hand-side and thus the vector \( X(s) \). This is as it should be, since \( X(s) \) is the vector of states, and there should be \( n \) states.

If none of the roots \( \lambda_k \), \( k=1, \ldots, n \) are repeated, it will be possible to use partial fraction expansion to express eq. (L-3) in the following way:

\[
X(s) = \frac{R_1(s)}{(s - \lambda_1)} + \frac{R_2(s)}{(s - \lambda_2)} + \ldots + \frac{R_n(s)}{(s - \lambda_n)} \quad \text{(L-4)}
\]
where each \( \mathbf{R}_k(s) \) is an \( n \times 1 \) vector. The inverse LaPlace transform will then appear as:
\[
\mathbf{x}(t) = R_1(t)e^{\lambda_1 t} + R_2(t)e^{\lambda_2 t} + \ldots + R_n(t)e^{\lambda_n t}
\]
The \( \lambda_k, \ k=1,\ldots,n \) are, in general, complex, such that \( \lambda_k=\sigma_k+j\omega_k \).

The \( \lambda_k, \ k=1,\ldots,n \) are called the system eigenvalues.

We see that the system eigenvalues \( \lambda_k, \ k=1,\ldots,n \) dictate the nature of the system in terms of the system modal response, where each \( \lambda_k \) corresponds to a system mode. These modes may be oscillatory or non-oscillatory, damped or undamped.

1. Oscillatory:
   - Any mode with \( \omega_k \neq 0 \) is oscillatory. If there exists an \( \lambda_k=\sigma_k+j\omega_k \) such that \( \omega_k \neq 0 \), then there will exist a corresponding \( \lambda_k=\sigma_k-j\omega_k \). These two eigenvalues correspond to the same system mode.
   - Any mode with \( \omega_k=0 \) is non-oscillatory.

2. Damping: Any mode \( \lambda_k=\sigma_k \pm j\omega_k \),
   - a. if \( \sigma_k > 0 \), the mode is negatively damped (unstable)
   - b. if \( \sigma_k < 0 \), the mode is positively damped (stable)
   - c. if \( \sigma_k = 0 \), the mode is marginally damped.

If repeated roots occur in the factorization of \( (L-2b) \), then these roots will have time-domain expressions like \( t^{r-1}e^{-\lambda t} \), and will therefore have the following effects:
   - a. if \( \sigma_k > 0 \), the mode is negatively damped (unstable)
   - b. if \( \sigma_k < 0 \), the mode is positively damped (stable); however, the effects of the “t” coefficient might initially dominate the effects of the exponential and cause very large oscillations that could disrupt the system.
   - c. with \( \sigma_k = 0 \), the effects of the “t” coefficient will result in growing response (unstable)

In practice, it is very unlikely to see repeated roots for power systems. Therefore, we safely assume there are no repeated roots.
Right eigenvectors:
For each eigenvalue, $\lambda_k$, k=1,…,n, there exists an n-element column vector $p_k$, called a right eigenvector, such that
$$A p_k = \lambda_k p_k$$
Since there are n eigenvalues, there are n right eigenvectors.
We may form a matrix of these n right eigenvectors as follows:
$$P = \begin{bmatrix} p_1 & \cdots & p_n \end{bmatrix}$$
The above matrix, $P$, is called the modal matrix.

Left eigenvectors:
For each eigenvalue, $\lambda_k$, k=1,…,n, there exists an n-element column vector $q_k$, called a left eigenvector, such that
$$q_k^T A = \lambda_k q_k^T$$
Since there are n eigenvalues, there are n left eigenvectors.
We may form a matrix of these n left eigenvectors as follows:
$$Q^T = \begin{bmatrix} q_1^T \\ \vdots \\ q_n^T \end{bmatrix}$$

Some properties:
For any two eigenvalues, $\lambda_j$, $\lambda_k$, then
- For $j \neq k$, $q_j$ and $p_k$ are orthogonal, i.e., their dot product is 0:
  $$q_j^T p_k = 0$$
- For $j = k$,
  $$q_j^T p_j = c_j$$
  where $c_j$ is a constant. A simple scaling of either the right or the left eigenvector will provide that
  $$q_j^T p_j = 1$$
Now consider, based on the above properties, we will get:

\[
\begin{bmatrix}
q_1^T P_1 & q_2^T P_2 & q_3^T P_3 & \cdots & q_n^T P_n
\end{bmatrix}
\begin{bmatrix}
p_1 & \cdots & p_n
\end{bmatrix}
= \begin{bmatrix}
q_1^T P_1 & 0 & \cdots & 0
0 & q_2^T P_2 & \cdots & 0
\vdots & \vdots & \ddots & \vdots
0 & 0 & \cdots & q_n^T P_n
\end{bmatrix}
\begin{bmatrix}
p_1 & \cdots & p_n
\end{bmatrix}
\]

We can go a step further if the scaling is performed:

\[
Q^T P = I
\]

Post-multiplying both sides by \(P^{-1}\) results in

\[
Q^T = P^{-1}
\]

Note that:

- \(PP^{-1} = I\)
- \((Q^T)^{-1} Q^T = I\)

We can illustrate calculation of the right and left eigenvectors using the sample system given in the book (fig. 2.19, and example 3.2), having state-space model of

\[
\begin{bmatrix}
\Delta \dot{\delta}_{13} \\
\Delta \dot{\delta}_{23} \\
\Delta \dot{\omega}_{13} \\
\Delta \dot{\omega}_{23}
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 1 & 0
0 & 0 & 0 & 1
-104.096 & -59.524 & 0 & 0
-33.841 & -153.460 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\Delta \delta_{13} \\
\Delta \delta_{23} \\
\Delta \omega_{13} \\
\Delta \omega_{23}
\end{bmatrix}
\]

Observe the eigenvalues in Table 3.2.
Also observe the relative rotor angle plots of fig. 3.3-b, for the case when a small load was added to bus #8. Here we see that one mode can be clearly observed having a period of about 0.7 sec (f=1.4Hz, ω=2πf=8.8 rad/sec).

The other mode (2.1Hz) is not readily observable, although its presence is probably responsible for the distortion seen in the δ₃₁ plot.

Using matlab, we use

$[P,D]=\text{eig}(A)$ where $A$ is the matrix given above.
Then the matrix of eigenvalues $D$ is given by

$$
\begin{pmatrix}
+13.4164i & 0 & 0 & 0 \\
0 & -13.4164i & 0 & 0 \\
0 & 0 & +8.8067i & 0 \\
0 & 0 & 0 & -8.8067i
\end{pmatrix}
$$

And the matrix of right eigenvectors $P$ is given by

$$
\begin{pmatrix}
-0.0459 - 0.0000i & -0.0459 + 0.0000i & -0.0585 - 0.0000i & 0.0459 + 0.0000i & 0.0459 - 0.0000i \\
0.0000 + 0.6154i & 0.0000 - 0.6154i & 0.0000 + 0.0000i & 0.0000 - 0.0000i & 0.0000 + 0.0000i \\
-0.1030 - 0.0000i & 0.1030 + 0.0000i & 0.0000 + 0.0000i & 0.0000 - 0.0000i & 0.0000 + 0.0000i \\
-0.0459 + 0.0000i & -0.0459 - 0.0000i & 0.0000 + 0.0000i & 0.0000 - 0.0000i & 0.0000 + 0.0000i \\
-0.0459 - 0.0000i & -0.0459 + 0.0000i & 0.0000 - 0.0000i & 0.0000 + 0.0000i & 0.0000 - 0.0000i
\end{pmatrix}
$$

And the matrix of left eigenvectors $Q^T$ is given by $P^{-1}$, which is:

$$
\begin{pmatrix}
-2.8240 + 0.0000i & -6.3340 + 0.0000i & 0.0000 + 0.2105i & 0.0000 + 0.4721i \\
-2.8240 - 0.0000i & -6.3340 - 0.0000i & 0.0000 - 0.2105i & 0.0000 - 0.4721i \\
-3.5951 + 0.0000i & 2.8194 - 0.0000i & 0.0000 + 0.4082i & -0.0000 - 0.3201i \\
-3.5951 - 0.0000i & 2.8194 + 0.0000i & 0.0000 - 0.4082i & -0.0000 + 0.3201i
\end{pmatrix}
$$

Note that here, the eigenvectors are along the rows. Taking transpose, we get $Q$, which is:

$$
\begin{pmatrix}
-2.8240 + 0.0000i & -2.8240 - 0.0000i & -3.5951 + 0.0000i & -3.5951 - 0.0000i \\
-6.3340 + 0.0000i & -6.3340 - 0.0000i & 2.8194 - 0.0000i & 2.8194 + 0.0000i \\
0.0000 + 0.2105i & 0.0000 - 0.2105i & 0.0000 + 0.4082i & 0.0000 - 0.4082i \\
0.0000 + 0.4721i & 0.0000 - 0.4721i & -0.0000 - 0.3201i & -0.0000 + 0.3201i
\end{pmatrix}
$$

In the above, the left eigenvectors are the columns.

Note also that the columns of right (or left) eigenvectors corresponding to complex conjugate eigenvalues are complex conjugate eigenvectors.

The numerators of eq. (L-4)

Let’s return to eq. (L-4), which is restated here for convenience:

$$\dot{X}(s) = \frac{R_1(s)}{s + \lambda_1} + \frac{R_2(s)}{s + \lambda_2} + \ldots + \frac{R_n(s)}{s + \lambda_n}$$
What are these $R_k$, $k=1,...,n$?

To answer this, let’s return to eq. (L-1), which is:

$$X(s) = [sI - A]^{-1} x(0)$$

Let’s pre-multiply the right-hand side by $PP^{-1}$ and post-multiply the right-hand side by $[Q^T]^{-1} Q^T$. This is acceptable, since both of these products yield the identity. This results in:

$$X(s) = PP^{-1}[sI - A]^{-1}[Q^T]^{-1} Q^T x(0)$$

Bracket the inner products:

$$X(s) = P\{P^{-1}[sI - A]^{-1}[Q^T]^{-1} Q^T x(0)$$

We can show that:

$$[sI - A]^{-1} = P^{-1}[sI - A]^{-1}[Q^T]^{-1}$$

where

$$\Lambda = \text{diag}(\lambda_k)$$

The proof is below:

---

Since $A$ is square and assumed to have distinct eigenvalues, it has full rank. Given that $P$ is the matrix of $N$ independent right eigenvectors, $A$ may be diagonalized by

$$P^{-1}AP = \Lambda = \text{diag}(\lambda_i)$$

Since the right and left eigenvector matrices $P$ and $Q^T$ are orthogonal, $P = [Q^T]^{-1}$, and

$$P^{-1}A[Q^T]^{-1} = \Lambda = \text{diag}(\lambda_i)$$

If $P$ and $Q^T$ are the right and left eigenvector matrices for $A$, then they are also the right and left eigenvector matrices for $[sI - A]$. Therefore

$$P^{-1}[sI - A][Q^T]^{-1} = [sI - A]$$

Recalling $[BCD]^{-1} = D^{-1}C^{-1}B^{-1}$, both sides of the previous equation may be inverted to yield

$$Q^T[sI - A]^{-1}[P] = [sI - A]^{-1}$$

Again using the orthogonality condition $P = [Q^T]^{-1}$,

$$P^{-1}[sI - A]^{-1}[Q^T]^{-1} = [sI - A]^{-1}$$

---

Then, we have that:
\[
\begin{align*}
\begin{bmatrix} n \times 1 \end{bmatrix} X(s) &= \begin{bmatrix} n \times n \end{bmatrix} P \begin{bmatrix} n \times n \end{bmatrix} \left[ \begin{bmatrix} n \times 1 \end{bmatrix} Q^T \right] \begin{bmatrix} n \times 1 \end{bmatrix} x(0) \\
&= \begin{bmatrix} n \times 1 \end{bmatrix} \left( s I - \Lambda \right)^{-1} \begin{bmatrix} n \times 1 \end{bmatrix} x(0) \\
&= \begin{bmatrix} n \times 1 \end{bmatrix} \begin{bmatrix} n \times n \end{bmatrix} \begin{bmatrix} n \times n \end{bmatrix} \begin{bmatrix} n \times 1 \end{bmatrix} x(0) \\
&= \begin{bmatrix} n \times 1 \end{bmatrix} \begin{bmatrix} n \times n \end{bmatrix} \begin{bmatrix} n \times n \end{bmatrix} \begin{bmatrix} n \times 1 \end{bmatrix} x(0)
\end{align*}
\]

Two comments are relevant at this point:

1. The matrix being inverted is a diagonal matrix. Therefore, the matrix inverse is obtained by inverting each diagonal element.

2. Recall the orthogonality property \( p_i q_j = 0 \) for \( i \neq j \).

Using these comments, we can manipulate \((*)\) to obtain:

\[
\begin{align*}
X(s) &= \frac{1}{s - \lambda_k} \sum_{k=1}^{n} p_k \left( q_k^T x(0) \right) \\
&= \frac{1}{s - \lambda_k} \sum_{k=1}^{n} \left( q_k^T x(0) \right) p_k
\end{align*}
\]

Taking the inverse LaPlace transform, we obtain:

\[
\begin{align*}
x(t) &= \sum_{k=1}^{n} \left[ q_k^T x(0) e^{\lambda_k t} \right] p_k \\
&= \sum_{k=1}^{n} \left[ q_k^T x(0) e^{\lambda_k t} \right] p_k \tag{L-5}
\end{align*}
\]

This is a very important relationship. It shows how we can use the right eigenvalue to determine the \textit{shape} of the \( k \text{th} \) mode.

Inspecting eq. (L-5), we see that the right eigenvector \( p_k \) determines the relative distribution of the mode through the state variables. To see this, note that

- \( q_k, p_k, \) and \( x(t) \) are all \( n \times 1 \) vectors, with element \( i \) corresponding to the \( i \text{th} \) state variable.
- \( q_k^T x(0) e^{\lambda_k t} \) is scalar and multiplies every element of \( p_k \); therefore it does not distinguish any state differently than another state.
- \( p_k \) is therefore the only thing that distinguishes one state from another in terms of the mode \( k \) dynamics.

If the states are limited to only the generator inertial states \( \Delta \delta \) and \( \Delta \omega \), then each element of \( p_k \) gives the relative distribution of the mode in a particular generator’s angle or speed.

One caution: The right eigenvector does NOT tell you how much the state influences the mode.
The right eigenvector does tell you the **relative phase** of each state in that mode. If you “plot” each element (a complex number and thus interpretable as a vector) corresponding to each $\Delta \omega$ state (one for each generator) in the right eigenvector $p_k$, you can see which generators are swinging against one another. This is called **mode shape**. The relative phases can be observed in the time domain simulations.

Some interesting ways of illustrating the relative phase of each $\Delta \omega_k$ as determined by the $p_k$’s are shown in the following.

Klein, Rogers, and Kundur, “A fundamental study of interarea oscillations in power systems.” See page 915-916, attached below. Fig. 2 shows the mode shape where gens 1,2 swing against gens 11,12, and in the time domain simulation, Fig. 3.
3.0 METHODS OF ANALYSIS

Both small signal stability analysis and transient stability analysis were used, in a complementary way, in our study of inter-area oscillations. Small signal stability analysis, using modal techniques [7], is most appropriate for determining the nature of inter-area modes in power systems. In this case, the system studied was small enough to allow the analysis of all system modes, using MASS computer program [8]. The system eigenvalues, eigenvectors, and participation factors [7] were computed for a number of different system conditions and configurations.

In some instances, in particular in our investigation of the effects of loads, we found it useful to augment the small signal stability analysis with transient stability runs. The graphic nature of the output of the transient stability program aids in picturing the pattern of voltage oscillations, and their relationship with the eigenvectors calculated using modal analysis.

4.0 EFFECTS OF TIE LINE IMPEDANCE AND FLOW

In these tests, all four generating units were represented identically by detailed generator and fast static exciter models. All loads were represented as constant impedances. The tie line impedance was varied by changing the number of tie circuits in service. Power transfers between the two areas were created, either by an uneven distribution of generation between the areas, or by an uneven split of the total system load.

4.1 Effect on Frequency and Damping

The frequency and damping ratio of the inter-area mode for various combinations of tie line power flow and number of tie circuits in service are given in Table 1. As is to be expected; the frequency and damping ratio, of the inter-area mode, drop as the tie line impedances or power flow is increased.

<table>
<thead>
<tr>
<th>POWER FLOW</th>
<th>GENERATION/LOAD</th>
<th>FREQUENCY</th>
<th>DAMPING</th>
</tr>
</thead>
<tbody>
<tr>
<td>AREA 1 to 2</td>
<td>TIES 1/S AREA 1</td>
<td>AREA 1 AREA 2</td>
<td>Hz</td>
</tr>
<tr>
<td>0 3</td>
<td>1400/1367</td>
<td>1400/1367</td>
<td>0.748</td>
</tr>
<tr>
<td>2 1</td>
<td>1400/1367</td>
<td>0.661</td>
<td>0.011</td>
</tr>
<tr>
<td>1 1</td>
<td>1400/1367</td>
<td>0.514</td>
<td>0.002</td>
</tr>
<tr>
<td>400 3</td>
<td>1400/1367</td>
<td>1450/1376</td>
<td>0.732</td>
</tr>
<tr>
<td>400 1</td>
<td>1400/1367</td>
<td>1450/1767</td>
<td>0.683</td>
</tr>
<tr>
<td>380 1</td>
<td>1400/1367</td>
<td>1800/1367</td>
<td>0.395</td>
</tr>
</tbody>
</table>

4.2 Effect on Mode Shape

The normalized eigenvector components, corresponding to rotor speeds, of the inter-area mode, for various tie line impedances and power flows, are shown in Figure 2. The results lead to the following conclusions.

5.0 EFFECT OF EXCITATION SYSTEMS

5.1 Effect on Frequency and Damping

To test the effect of the excitation systems on the frequency and damping of the inter-area mode, we carried out two sets of tests: one set with identical excitors on all four units and the other set with one fast exciter and three slow or manually controlled excitors.

5.1.1 Tests with Identical Excitors

In this set of tests we explored the effect of the following four types of excitors on the inter-area mode:

- Manually controlled excitors
- Slow dc excitors
- Fast static excitors with and without transient gain reduction (TGR).

Only the automatic voltage regulator effects were investigated. Other controls, such as power system stabilizers, were not considered.

We considered two operating conditions: a stressed system with only one exciter and 400 MW power transfer from Area 1 to 2, and an unstressed system with no power transfer between the areas.

Constant impedance loads were assumed in these tests.

The results, summarized in Table 2, show that the inter-area mode is best damped with manually controlled excitors, and worst damped with fast excitors with TGR. The frequency is highest for fast excitors without TGR and lowest for slow excitors.
Wang, Howell, Kundur, Chung, and Xu, “A tool for small-signal security assessment of power systems.” See mode shape, Fig. 5.
Y. Mansour, “Application of eigenanalysis to the Western North American Power system,” Tables 4, 5, and 6, each table for a certain condition, give eigenvector elements for speed deviation at each of a number of generators. Figures 1, 2, and 3 show, for three conditions, geographical plots of the mode shapes for 4 different modes.
Table 1

<table>
<thead>
<tr>
<th>Eigen Value</th>
<th>Frequency (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.036 + j 0.73</td>
<td>0.272</td>
</tr>
<tr>
<td>-0.124 + j 0.88</td>
<td>0.474</td>
</tr>
<tr>
<td>-0.085 + j 0.77</td>
<td>0.268</td>
</tr>
<tr>
<td>-0.055 + j 0.82</td>
<td>0.706</td>
</tr>
<tr>
<td>-0.035 + j 0.87</td>
<td>0.78</td>
</tr>
<tr>
<td>-0.052 + j 0.77</td>
<td>0.823</td>
</tr>
<tr>
<td>-0.037 + j 0.93</td>
<td>0.697</td>
</tr>
<tr>
<td>-0.037 + j 0.84</td>
<td>0.865</td>
</tr>
<tr>
<td>-0.052 + j 0.69</td>
<td>0.858</td>
</tr>
<tr>
<td>-0.035 + j 0.67</td>
<td>0.086</td>
</tr>
</tbody>
</table>

Table 2

<table>
<thead>
<tr>
<th>Case 1</th>
<th>Case 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.026 + j 0.78</td>
<td>0.61</td>
</tr>
<tr>
<td>-0.120 + j 0.68</td>
<td>0.765</td>
</tr>
<tr>
<td>-0.112 + j 0.48</td>
<td>0.741</td>
</tr>
<tr>
<td>-0.128 + j 0.48</td>
<td>0.765</td>
</tr>
<tr>
<td>-0.055 + j 0.75</td>
<td>0.064</td>
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<tr>
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<td>0.706</td>
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</table>

Table 3

<table>
<thead>
<tr>
<th>Case 1</th>
<th>Case 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.026 + j 0.38</td>
<td>0.61</td>
</tr>
<tr>
<td>-0.120 + j 0.68</td>
<td>0.765</td>
</tr>
<tr>
<td>-0.112 + j 0.48</td>
<td>0.741</td>
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<td>-0.128 + j 0.48</td>
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<td>0.858</td>
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<td>0.086</td>
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</table>

Table 4

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<thead>
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<th>Case 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.026 + j 0.78</td>
<td>0.61</td>
</tr>
<tr>
<td>-0.120 + j 0.68</td>
<td>0.765</td>
</tr>
<tr>
<td>-0.112 + j 0.48</td>
<td>0.741</td>
</tr>
<tr>
<td>-0.128 + j 0.48</td>
<td>0.765</td>
</tr>
<tr>
<td>-0.055 + j 0.75</td>
<td>0.064</td>
</tr>
<tr>
<td>-0.055 + j 0.82</td>
<td>0.706</td>
</tr>
<tr>
<td>-0.035 + j 0.77</td>
<td>0.823</td>
</tr>
<tr>
<td>-0.037 + j 0.93</td>
<td>0.697</td>
</tr>
<tr>
<td>-0.037 + j 0.84</td>
<td>0.865</td>
</tr>
<tr>
<td>-0.052 + j 0.69</td>
<td>0.858</td>
</tr>
<tr>
<td>-0.035 + j 0.67</td>
<td>0.086</td>
</tr>
</tbody>
</table>

Table 5

<table>
<thead>
<tr>
<th>Case 1</th>
<th>Case 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.026 + j 0.38</td>
<td>0.61</td>
</tr>
<tr>
<td>-0.120 + j 0.68</td>
<td>0.765</td>
</tr>
<tr>
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</tr>
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<td>0.086</td>
</tr>
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Table 6

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<tbody>
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Table 8

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</tr>
</thead>
<tbody>
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Finally, below is some work recently done reflecting mode shape in the Southwestern WECC system for a certain mode.