HOMEWORK #2, Due Monday, Jan 26.

SOLUTIONS

1. Use the equations at the bottom of slide 7, which are:

\[
\begin{align*}
P_D &= |V_1| |V_2| B \sin \theta_1 \sin \theta_2 \\
Q_D &= -|V_2|^2 B + |V_1| |V_2| B \cos \theta_1 \cos \theta_2
\end{align*}
\]

2. Now, just bring the right hand side of these 2 equations over to the left-hand side, and you have the 2 equations that correspond to \( G(y,p) = 0 \).

\[
\begin{align*}
P_D - |V_1| |V_2| B \sin \theta_1 &= 0 \\
Q_D + |V_2|^2 B - |V_1| |V_2| B \cos \theta_1 &= 0
\end{align*}
\]

3. Solve these equations to get the corresponding power flow solution (but you do not need Newton-Raphson to do this – you can just use the equation at the bottom of slide 10). Use \( V1=1 \), \( PD=0.4 \), \( pf=0.97 \) lagging, \( B=1 \) as the operating conditions.

\[
\text{Solving using below code, we get } |V2| = 0.9220.
\]

```python
beta=.25;
B=2;
v1=1.0;
pdn=[0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.85 0.9];
v2n=sqrt((v1^2-(2/B)*beta.*pdn - sqrt(v1^4-(4/B)*pdn.*(1/B)*pdn+beta*v1^2)))/2);
pdp=[0.85 0.8 0.75 0.7 0.6 0.5 0.4 0.3 0.2 0.1 0];
v2p=sqrt((v1^2-(2/B)*beta.*pdp + sqrt(v1^4-(4/B)*pdp.*(1/B)*pdp+beta*v1^2)))/2);
pd4=[pdp pdn];
v24=[v2n v2p];
```

Then we obtain \( \theta \) from

\[
\begin{align*}
P_D &= |V_1| |V_2| B \sin \theta_1 \\
\sin \theta_1 &= \frac{P_D}{|V_1| |V_2| B} = 0.4/(1)(0.9220)(2) = 0.2169 \\
\Rightarrow \theta_1 &= \arcsin(0.2169) = 0.2186 rad = 12.527 degrees
\end{align*}
\]

4. Now you need to replace the value specified in the equations for \( PD \) (assuming that the initial load is 0.4) with 0.4*lambda. This gives you the equations in the form of slide 49: \( 0=G(\theta, V, \text{lambda}) \). Note, however, that \( G \) is really two equations: G1 and G2. Again, use \( V1=1.0 \), \( pf=0.97 \) lagging, and \( B=1.0 \).

\[
\text{With pf=0.97 lagging, then } Beta=0.25, \text{ and } Q_0=Beta*P_0. \text{ Since } P_0=0.4\lambda, \text{ then }
Q_0=0.25*0.4\lambda=0.1\lambda. \text{ Substituting these into equations of #2 above, we get: }
\]

\[
\begin{align*}
0.4\lambda - |V_1| |V_2| B \sin \theta_1 &= 0 \\
0.1\lambda + |V_2|^2 B - |V_1| |V_2| B \cos \theta_1 &= 0
\end{align*}
\]

With \( B=2 \) and \( |V_1|=1 \), and dropping the subscripts from \( \theta_1 \), the subscript and absolute value sign from \( |V_2| \), we get
\[ G_1(\theta) = 0.4 \lambda - 2V \sin \theta = 0 \]
\[ G_2(\theta) = 0.1 \lambda + 2V^2 - 2V \cos \theta = 0 \]

5. Now you need to formulate the equations on the slide 55. This is a matter of taking derivatives and then evaluating those derivatives at the solution that you obtained above. Note, however, that each element in the matrix of slide 55 actually represents 2 elements. That is:
\[
\begin{vmatrix}
  dG_1/d\theta & dG_1/dV & dG_1/d\lambda \\
  dG_2/d\theta & dG_2/dV & dG_2/d\lambda \\
  0 & 0 & 1 \\
\end{vmatrix}
\]
\[ \Rightarrow \text{we obtain} \begin{bmatrix} 2V \sin \theta & 4V - 2 \cos \theta & 0.4 \\
  -2V \cos \theta & -2 \sin \theta & 0.4 \\
  0 & 0 & 1 \end{bmatrix} \begin{bmatrix} d\theta \\
  dV \\
  d\lambda \end{bmatrix} = \begin{bmatrix} 0 \\
  0 \\
  0 \end{bmatrix} \]

6. Evaluate each of the above matrix elements at the solution obtained in step 3.
\[ \begin{bmatrix} -2V \cos \theta & -2 \sin \theta & 0.4 \\
  2V \sin \theta & 4V - 2 \cos \theta & 0.1 \\
  0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.3999 & 1.7356 & 0.1 \\
  -1.8001 & -0.4337 & 0.4 \\
  0 & 0 & 1 \end{bmatrix} \]

7. Then solve these equations for the tangent vector. You can do this by inverting the above matrix (use matlab or a calculator to do this) and then multiply the right-hand-side by this inverted matrix.
\[ \begin{bmatrix} d\theta \\
  dV \\
  d\lambda \end{bmatrix} = \begin{bmatrix} 0.3999 & 1.7356 & 0.1 \\
  -1.8001 & -0.4337 & 0.4 \\
  0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\
  0 \\
  0 \end{bmatrix} = \begin{bmatrix} 0.2186 \\
  0.9220 \\
  1 \end{bmatrix} \]

8. Then take a “step” using an appropriately chosen step size per the equation on slide 56.
\[ \Rightarrow \text{A reasonable step size is } \sigma=0.1 \text{ or } 0.2. \text{ We will choose } 0.2. \text{ Then the predicted point becomes:} \]
\[ \begin{bmatrix} \theta^{(2,p)} \\
  V^{(2,p)} \\
  \lambda^{(2,p)} \end{bmatrix} = \begin{bmatrix} \theta^{(1)} \\
  V^{(1)} \end{bmatrix} + \sigma \begin{bmatrix} d\theta \\
  dV \\
  d\lambda \end{bmatrix} = \begin{bmatrix} 0.2186 \\
  0.9220 \\
  1 \end{bmatrix} + 0.2 \times \begin{bmatrix} 0.2500 \\
  -0.1152 \\
  1 \end{bmatrix} = \begin{bmatrix} 0.2686 \\
  0.8990 \\
  1.2 \end{bmatrix} \]
9. Beginning from your predicted point that you identified in step 8 of #2a, develop equations for approach a, solve them, and identify the resulting corrected point in terms of voltage and power.

⇒ We need to develop the following equations:

\[ 0 = G(y^{(i+1)}_v, \lambda^{(i+1)}) \Rightarrow \begin{align*}
0.4\lambda - 2V \sin \theta &= 0 \\
0.1\lambda + 2V^2 - 2V \cos \theta &= 0
\end{align*} \]

\[ \left\{ y^{(i+1)}_v - y^{(i+1, p)} \right\} \cdot \mathbf{I} = 0 \Rightarrow \begin{bmatrix} \theta - 0.2686 & V - 0.899 & \lambda - 1.2 \end{bmatrix} \cdot \begin{bmatrix} 0.2500 \\
-0.1152 \\
1 \end{bmatrix} = 0 \]

Performing the multiplication associated with the last equation results in

\[ 0.25\theta - 0.0712 - 0.1152V + 0.1036 + \lambda - 1.2 = 0 \Rightarrow 0.25\theta - 0.1152V + \lambda - 1.1676 = 0 \]

Writing the three equations together

\[ 0.4\lambda - 2V \sin \theta = 0 \]
\[ 0.1\lambda + 2V^2 - 2V \cos \theta = 0 \]
\[ 0.25\theta - 0.1152V + \lambda - 1.1676 = 0 \]

Define:

\[ \mathbf{x} = \begin{bmatrix} \theta \\ V \\ \lambda \end{bmatrix}, \quad \mathbf{f}(\mathbf{x}) = \begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ f_3(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} 0.4\lambda - 2V \sin \theta \\ 0.1\lambda + 2V^2 - 2V \cos \theta \\ 0.25\theta - 0.1152V + \lambda - 1.1676 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \]

Writing \( \mathbf{f}(\mathbf{x}) = 0 \) as a Taylor series, we get

\[ \mathbf{f}(\mathbf{x}^{(0)} + \Delta \mathbf{x}^{(0)}) = \mathbf{f}(\mathbf{x}^{(0)}) + \mathbf{f}'(\mathbf{x}^{(0)})\Delta \mathbf{x}^{(0)} + \frac{1}{2} \mathbf{f}''(\mathbf{x}^{(0)})(\Delta \mathbf{x}^{(0)})^2 + \ldots = 0 \]

We will make a guess at the solution, and hopefully our guess will be a good one such that \( \Delta \mathbf{x}^{(0)} \) is small, then the higher order terms in the above relation are also small and we can write

\[ \mathbf{f}(\mathbf{x}^{(0)} + \Delta \mathbf{x}^{(0)}) = \mathbf{f}(\mathbf{x}^{(0)}) + \mathbf{f}'(\mathbf{x}^{(0)})\Delta \mathbf{x}^{(0)} = 0 \]

The above relation can be manipulated to get

\[ \Delta \mathbf{x}^{(0)} = -\left[ \mathbf{f}'(\mathbf{x}^{(0)}) \right]^{-1} \mathbf{f}(\mathbf{x}^{(0)}) = -\mathbf{J}^{-1} \mathbf{f}(\mathbf{x}^{(0)}) \]

where \( \mathbf{J} \) is the Jacobian evaluated at our guess. Then we will correct our guess according to

\[ \mathbf{x}^{(i)} = \mathbf{x}^{(0)} + \Delta \mathbf{x}^{(0)} = \mathbf{x}^{(0)} - \mathbf{J}^{-1} \mathbf{f}(\mathbf{x}^{(0)}) \]

Once the “mismatch” vector \( \mathbf{f}(\mathbf{x}) \) is within a small tolerance of the zero vector \( \mathbf{0} \), we will assume that we have obtained the solution.

The relation will be
\[
\dot{x}^{(1)} = x^{(0)} - J^{-1} f(x^{(0)}) = \begin{bmatrix} \theta^{(0)} \\ V^{(0)} \\ \lambda^{(0)} \end{bmatrix} - \begin{bmatrix} -2V \cos \theta & -2 \sin \theta & 0.4 \\ 2V \sin \theta & 4V - 2 \cos \theta & 0.1 \\ 0.25 & -0.1152 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0.4 \lambda - 2V \sin \theta \\ 0.1 \lambda + 2V^2 - 2V \cos \theta \\ 0.25 \theta - 0.1152V + \lambda - 1.1676 \end{bmatrix}
\]

Now we will use as our “initial guess” the predicted solution, which is
\[
\begin{bmatrix} \theta^{(2,p)} \\ V^{(2,p)} \\ \lambda^{(2,p)} \end{bmatrix} = \begin{bmatrix} 0.2686 \\ 0.8990 \\ 1.2 \end{bmatrix}
\]

The above expression was coded in the following, using Matlab. Note that the stopping criterion is that the maximum absolute value of \( f(x) \) must be less than tol, which was set to 0.00001.

```matlab
xold=[0.2686; 0.8990; 1.2];
f=[10; 10; 10];
tol=0.00001;
count=0;
while max(abs(f)) > tol,
t=xold(1);
v=xold(2);
lam=xold(3);
J=[-2*v*cos(t) -2*sin(t)   0.4;
   2*v*sin(t)  4*v-2*cos(t)  0.1;
   0.25 -0.1152     1.0];
f=[0.4*lam-2*v*sin(t);
   0.1*lam+2*v^2-2*v*cos(t);
   0.25*t-0.1152*v+lam-1.1676];
xnew=xold-inv(J)*f;
xold=xnew;
count=count+1;
end

```

count, xnew, f,
The above converged on the third iteration to the following solution:
\[
\begin{bmatrix} \theta^{(1)} \\ V^{(1)} \\ \lambda^{(1)} \end{bmatrix} = \begin{bmatrix} 0.2718 \\ 0.8962 \\ 1.2029 \end{bmatrix}
\]

10. Repeat #9 except implement approach b.

\( \Rightarrow \) Noting that the largest element in the tangent vector is d\( \lambda \), we choose \( \lambda \) for the continuation parameter. We then need to develop the following equations:
\[
0 = G(V^{(i+1)}, \lambda^{(i+1)}) \ \Rightarrow \ 0.4 \lambda - 2V \sin \theta = 0
\]
\[
0.1 \lambda + 2V^2 - 2V \cos \theta = 0
\]
\[
\{ y_k - \eta \} = 0 \ \Rightarrow \ \lambda - 1.2 = 0
\]

Writing the three equations together
\[ 0.4\lambda - 2V\sin\theta = 0 \]
\[ 0.1\lambda + 2V^2 - 2V\cos\theta = 0 \]
\[ \lambda - 1.2 = 0 \]

Define:
\[
\begin{bmatrix}
\theta \\
V \\
\lambda
\end{bmatrix}
= 
\begin{bmatrix}
0.4\lambda - 2V\sin\theta \\
0.1\lambda + 2V^2 - 2V\cos\theta \\
\lambda - 1.2
\end{bmatrix} = 
\begin{bmatrix} 0 \\
0 \\
0
\end{bmatrix}
\]

The Newton-Raphson procedure is
\[
x^{(1)} = x^{(0)} - J^{-1}f(x^{(0)}) = 
\begin{bmatrix}
\theta^{(0)} \\
V^{(0)} \\
\lambda^{(0)}
\end{bmatrix} - 
\begin{bmatrix}
-2V\cos\theta & -2\sin\theta & 0.4 \\
2V\sin\theta & 4V - 2\cos\theta & 0.1 \\
0 & 0 & 1
\end{bmatrix}^{-1} 
\begin{bmatrix} 0.4\lambda - 2V\sin\theta \\
0.1\lambda + 2V^2 - 2V\cos\theta \\
\lambda - 1.2
\end{bmatrix}
\]

We again use as our “initial guess” the predicted solution, which is
\[
x^{(2)} = 
\begin{bmatrix}
\theta^{(2,p)} \\
V^{(2,p)} \\
\lambda^{(2,p)}
\end{bmatrix} = 
\begin{bmatrix} 0.2686 \\
0.8990 \\
1.2
\end{bmatrix}
\]

The following Matlab code was used, again with stopping criterion that the maximum absolute value of \( f(x) \) must be less than \( \text{tol} \), which was set to 0.00001.

```matlab
xold=[0.2686; 0.8990; 1.2];
f=[10;10;10];
tol=0.00001;
count=0;
while max(abs(f)) > tol,
    t=xold(1);
v=xold(2);
lam=xold(3);
J=[-2*v*cos(t) -2*sin(t)   0.4;   
   2*v*sin(t)  4*v-2*cos(t) 0.1;   
   0           0            1.0];
f=[0.4*lam-2*v*sin(t);   
   0.1*lam+2*v^2-2*v*cos(t);   
   lam-1.2];
xnew=xold-inv(J)*f;
xold=xnew;
count=++count;
end

count, xnew, f,
```

The above converged on the third iteration to the following solution:
\[
\begin{bmatrix}
\theta^{(1)} \\
V^{(1)} \\
\lambda^{(1)}
\end{bmatrix} = 
\begin{bmatrix} 0.2710 \\
0.8966 \\
1.2
\end{bmatrix}
\]

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