Flux-Linkage State-Space Model
(Section 4.12)

One limitation of the current state-space model is that the effects of saturation are difficult to represent.

This is because in the current state-space model, the effects of saturation cannot be isolated to a single current, whereas in the flux-linkage state-space model, it can. The reason saturation can be isolated to a single parameter in the flux-linkage state space model is because saturation occurs in the mutual fluxes, since mutual flux exists in the core material, which is quite saturable (i.e., the permeability decreases with high currents), whereas leakage flux exists largely external to the core material, i.e., in the air, which has permeability that is nearly constant with current.

So we will develop the flux-linkage state-space model, which uses the λ's as the state variables rather than the currents.

Our approach will follow four steps:

1. Develop d- and q- axis currents as function of λ's and mutual fluxes λ_{AD} and λ_{AQ} (we will define these shortly).

2. Develop state equations for λ's by substituting current expressions (from step 1) into voltage equations.

3. Develop the torque equation in terms of flux linkages.

4. Approximate the effects of saturation.

<u>Step 1: Develop d- and q- axis currents as function of λ's and mutual fluxes λ_{AD} and λ_{AQ}:</u>

(This comes partly from Section 4.11 and partly from 4.12 of VMAF)

We will take three sub-steps here.

- Step 1a: d-axis currents
- Step 1b: q-axis currents
- Step 1c: put them together

<u>Step 1a: d-axis currents</u>:

Recall that in per-unit, from eqs. (4.107-4.108) (see p. 30 of "perunitization" notes), all d-axis mutuals are equal, called L_{AD},

$$L_d\text{-}l_d=L_D\text{-}l_D=L_F\text{-}l_F=L_{AD}$$

We can replace the second and third terms to write (first term is L_{md})

$$L_d\text{-}l_d=kM_D=kM_F=L_{AD}$$

As we have seen, the d-axis mutual fluxes can be expressed as the total flux less the leakage flux, and we name it λ_{AD}:

$$\lambda_{AD}=\lambda_d\text{-}l_di_d=\lambda_F\text{-}l_Fi_F=\lambda_D\text{-}l_Di_D \qquad (4.110)$$

where λ_d, λ_F, λ_D are defined by eq. (4.20) (see p. 29 of "macheqts"). This says the following:

- If in each circuit on the d-axis (d, F, D),
- the pu leakage flux linkage is subtracted from the pu main flux linkage,
- the remaining pu flux linkage is the same for all other circuits coupled to it.

Solving (4.110) for each current in terms of flux linkages, we obtain:

$$\begin{aligned} i_d &= \frac{1}{l_d}(\lambda_d - \lambda_{AD}) \\ i_F &= \frac{1}{l_F}(\lambda_F - \lambda_{AD}) \\ i_D &= \frac{1}{l_D}(\lambda_D - \lambda_{AD}) \end{aligned} \qquad (4.118)$$

Now this appears to be enough to allow us to go to step 2 (develop state equations for λ's), by substituting the (4.118) expressions into the voltage equations to obtain voltage equations that are functions of flux linkages; however, we introduced an extra variable, λ_{AD}. But

we may address this, because λ_{AD} depends on the other fluxes (i.e., λ_d, λ_F, and λ_D).
So let's see if we can express λ_{AD} in terms of the other fluxes. From (4.110), we have

$$\lambda_{AD} = \lambda_d - l_d i_d \qquad \text{(d*)}$$

Expand the total flux term, λ_d, as a function of the currents. From eq. 4.20 (p. 29 of "macheqts.doc"), we have that:

$$\lambda_d = L_d i_d + kM_F i_F + kM_D i_D \qquad \text{(d**)}$$

Although eq. 4.20, p. 29 of macheqts, is written for MKS units, recall that we per-unitized so that relations have the same form in pu as they do when written in MKS units. Therefore, we may consider eq. (d**) to be pu.

Making the substitution of (d**) into (d*) results in

$$\lambda_{AD} = \left[L_d i_d + kM_F i_F + kM_D i_D\right] - l_d i_d$$

Now combining the two terms containing i_d results in:

$$\lambda_{AD} = [L_d - l_d] i_d + [kM_F] i_F + [kM_D] i_D \qquad \text{(d***)}$$

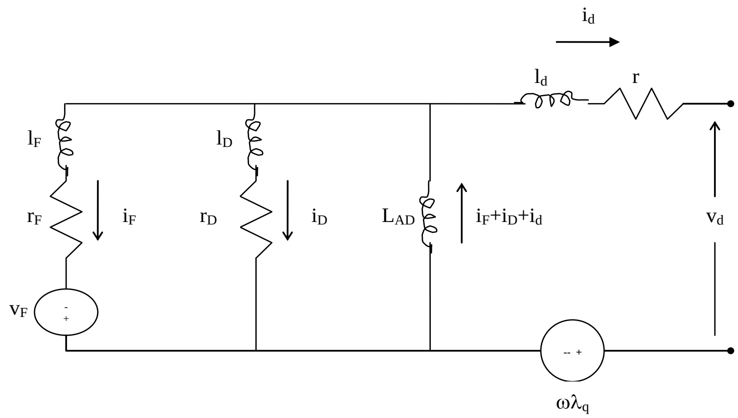
where we recognize each term in brackets as L_{AD}, which was defined in the "perunitization" notes (p. 30) as the per-unit value of any d-axis mutual inductance, i.e.,

$$L_{AD} \equiv L_{mdu} = kM_{Fu} = kM_{Du} = M_{Ru}$$

Paralleling what we did in the "perunitization" notes (p. 32) for the v_d voltage equation, we substitute L_{AD} into (d***), to obtain:

$$\lambda_{AD} = [L_{AD}] i_d + [L_{AD}] i_F + [L_{AD}] i_D = L_{AD}\left[i_d + i_F + i_D\right] \qquad (4.111)$$

Here, we recognize the flux linkage λ_{AD} as that seen by the "center-leg" inductance of our d-axis circuit model, as shown below (see "perunitization" notes, p. 33).



Substitution of (4.118),p.2, into (4.111),p.3, eliminates the currents:

$$\lambda_{AD} = L_{AD}\left[\frac{1}{l_d}(\lambda_d - \lambda_{AD}) + \frac{1}{l_F}(\lambda_F - \lambda_{AD}) + \frac{1}{l_D}(\lambda_D - \lambda_{AD})\right]$$

Now divide through by L_{AD} and then collect terms in λ_{AD}:

$$\lambda_{AD}\left[\frac{1}{L_{AD}} + \frac{1}{l_d} + \frac{1}{l_F} + \frac{1}{l_D}\right] = \frac{\lambda_d}{l_d} + \frac{\lambda_F}{l_F} + \frac{\lambda_D}{l_D} \qquad \text{(d****)}$$

Recognizing that the term in the brackets is the inverse of a parallel combination of the inductances, we define this term as:

$$\frac{1}{L_{MD}} = \left[\frac{1}{L_{AD}} + \frac{1}{l_d} + \frac{1}{l_F} + \frac{1}{l_D}\right] \qquad (4.119)$$

Making this substitution into (d****) results in:

$$\frac{\lambda_{AD}}{L_{MD}} = \frac{\lambda_d}{l_d} + \frac{\lambda_F}{l_F} + \frac{\lambda_D}{l_D} \Rightarrow \lambda_{AD} = \frac{L_{MD}}{l_d}\lambda_d + \frac{L_{MD}}{l_F}\lambda_F + \frac{L_{MD}}{l_D}\lambda_D \qquad (4.120)$$

<u>Step 1b: q-axis currents</u>:

We repeat the above procedure for the q-axis currents. We will actually do this, despite its similarity to step 1a, to be sure we correctly deal with the G-circuit.

Recall that in per-unit, all q-axis mutual flux linkages are equal.

As we have seen, these q-axis mutual fluxes can be expressed as the total flux less the leakage flux, and we call it λ_{AQ}:

$$\lambda_{AQ}=\lambda_q-l_qi_q=\lambda_Q-l_Qi_Q=\lambda_G-l_Gi_G \qquad (4.110\text{-Q})$$

where λ_q, λ_Q, λ_G are defined by eq. (4.20) (see p. 29 of "macheqts"). This says the following:

- If in each circuit on the q-axis,
- the pu leakage flux linkage is subtracted from the pu main flux linkage,
- the remaining pu flux linkage is the same for all other circuits coupled to it.

Solving (4.110-Q) for each of the currents, we obtain:

$$\begin{aligned} i_q &= \frac{1}{l_q}(\lambda_q - \lambda_{AQ}) \\ i_Q &= \frac{1}{l_Q}(\lambda_Q - \lambda_{AQ}) \\ i_G &= \frac{1}{l_G}(\lambda_G - \lambda_{AQ}) \end{aligned} \qquad (4.123)$$

Now this appears to be enough to allow us to go to step 2 (develop state equations for λ's), by substituting the (4.123) expressions into the voltage equations to obtain voltage equations that are functions of flux linkages; however, we introduced an extra variable, λ_{AQ}. But we may address this, because λ_{AQ} depends on the other fluxes (i.e., λ_q, λ_G, and λ_Q).

So let's see if we can express λ_{AQ} in terms of the other fluxes.

$$\lambda_{AQ} = \lambda_q - l_q i_q \qquad (q^*)$$

Expand the total flux term, λ_q, as a function of the currents. From eq. 4.20 (p. 29 of "macheqts.doc"), we have that:

$$\lambda_q = L_q i_q + kM_Q i_Q + kM_G i_G \qquad (q^{**})$$

Although eq. 4.20, p. 29 of macheqts, is written for MKS units, recall that we per-unitized so that relations have the same form in pu as they do when written in MKS units. Therefore, we may consider eq. (q**) to be pu.

Making this substitution of (q**) into (q*) results in

$$\lambda_{AQ} = \left[L_q i_q + kM_Q i_Q + kM_G i_G\right] - l_q i_q$$

Now combining the two terms containing i_q results in:

$$\lambda_{AQ} = [L_q - l_q]i_q + [kM_Q]i_Q + [kM_G]i_G \qquad \text{(q***)}$$

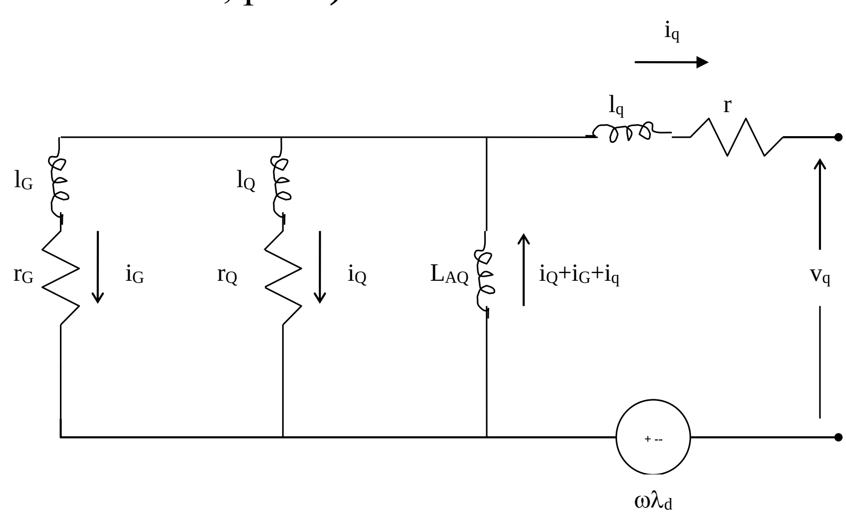
where we recognize each term in brackets as L_{AQ}, which was defined in the "perunitization" notes (pg 30) as the per-unit value of any q-axis mutual inductance, i.e.,

$$L_{AQ} \equiv L_{mqu} = kM_{Qu} = kM_{Gu} = M_{Yu}$$

Substitution of L_{AQ} into (q***) yields:

$$\lambda_{AQ} = [L_{AQ}]i_q + [L_{AQ}]i_Q + [L_{AQ}]i_G = L_{AQ}\left[i_q + i_Q + i_G\right] \qquad (4.112)$$

Here, we recognize the flux linkage λ_{AQ} as that seen by the "center-leg" inductance of our q-axis circuit model, as shown below. (See "perunitization" notes, p. 34)



Substitution of (4.123) into (4.112) eliminates the currents:

$$\lambda_{AQ} = L_{AQ}\left[\frac{1}{l_q}(\lambda_q - \lambda_{AQ}) + \frac{1}{l_Q}(\lambda_Q - \lambda_{AQ}) + \frac{1}{l_G}(\lambda_G - \lambda_{AQ})\right]$$

Now divide through by L_{AQ} and then collect terms in λ_{AQ}, resulting in:

$$\lambda_{AQ}\left[\frac{1}{L_{AQ}} + \frac{1}{l_q} + \frac{1}{l_Q} + \frac{1}{l_G}\right] = \frac{\lambda_q}{l_q} + \frac{\lambda_Q}{l_Q} + \frac{\lambda_G}{l_G} \qquad \text{(q****)}$$

Recognizing that the term in the brackets is the inverse of a parallel combination of the inductances, we define it as:

$$\frac{1}{L_{MQ}} = \left[\frac{1}{L_{AQ}} + \frac{1}{l_q} + \frac{1}{l_Q} + \frac{1}{l_G}\right] \qquad (4.122)$$

Making this substitution results in:

$$\frac{\lambda_{AQ}}{L_{MQ}} = \frac{\lambda_q}{l_q} + \frac{\lambda_Q}{l_Q} + \frac{\lambda_G}{l_G} \Rightarrow \lambda_{AQ} = \frac{L_{MQ}}{l_q}\lambda_q + \frac{L_{MQ}}{l_Q}\lambda_Q + \frac{L_{MQ}}{l_G}\lambda_G \qquad (4.121)$$

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We refer to eqs. (4.120) and (4.121) as the <u>*auxiliary equations*</u>.

$$\lambda_{AD} = \frac{L_{MD}}{l_d}\lambda_d + \frac{L_{MD}}{l_F}\lambda_F + \frac{L_{MD}}{l_D}\lambda_D \qquad (4.120)$$

$$\lambda_{AQ} = \frac{L_{MQ}}{l_q}\lambda_q + \frac{L_{MQ}}{l_Q}\lambda_Q + \frac{L_{MQ}}{l_G}\lambda_G \qquad (4.121)$$

Step 1c: Put them together:

Now combine equations 4.118 with 4.123, and write them all into a single matrix expression of currents in terms of the fluxes:

$$\begin{aligned} i_d &= \frac{1}{l_d}(\lambda_d - \lambda_{AD}) \\ i_F &= \frac{1}{l_F}(\lambda_F - \lambda_{AD}) \\ i_D &= \frac{1}{l_D}(\lambda_D - \lambda_{AD}) \end{aligned} \qquad (4.118); \qquad \begin{aligned} i_q &= \frac{1}{l_q}(\lambda_q - \lambda_{AQ}) \\ i_Q &= \frac{1}{l_Q}(\lambda_Q - \lambda_{AQ}) \\ i_G &= \frac{1}{l_G}(\lambda_G - \lambda_{AQ}) \end{aligned} \qquad (4.123)$$

$$\begin{bmatrix} i_d \\ i_F \\ i_D \\ i_q \\ i_G \\ i_Q \end{bmatrix} = \begin{bmatrix} \frac{1}{l_d} & 0 & 0 & -\frac{1}{l_d} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{l_F} & 0 & -\frac{1}{l_d} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{l_D} & -\frac{1}{l_D} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{l_q} & 0 & 0 & -\frac{1}{l_q} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{l_G} & 0 & -\frac{1}{l_G} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{l_Q} & -\frac{1}{l_Q} \end{bmatrix} \begin{bmatrix} \lambda_d \\ \lambda_F \\ \lambda_D \\ \lambda_{AD} \\ \lambda_q \\ \lambda_G \\ \lambda_Q \\ \lambda_{AQ} \end{bmatrix} \tag{4.124}$$

where we know from 4.120 and 4.121 that λ_{AD} and λ_{AQ} are "combination" states developed from the other flux-linkage states. Note 4.124 is "sort of" the inverse of (4.20) (p. 29 of "macheqts") except the variables are ordered differently, and (4.124) has $\underline{\lambda_{AD}}$.

Step 2: Develop state equations for λ's by substituting currents (from step 1) into voltage equations (see Section 4.12.1)

So we want derivatives on flux linkages. The procedure here is:

a. Start from a preliminary form of the voltage equation that contains flux linkage derivatives instead of current derivatives. This is convenient because it immediately provides us with the derivatives on the states we desire. The form that we will use is given by eq. 4.36 in the text, which is

$$\begin{bmatrix} v_d \\ v_q \\ -v_F \\ v_G = 0 \\ v_D = 0 \\ v_Q = 0 \end{bmatrix} = -\begin{bmatrix} r & 0 & 0 & 0 & 0 & 0 \\ 0 & r & 0 & 0 & 0 & 0 \\ 0 & 0 & r_F & 0 & 0 & 0 \\ 0 & 0 & 0 & r_G & 0 & 0 \\ 0 & 0 & 0 & 0 & r_D & 0 \\ 0 & 0 & 0 & 0 & 0 & r_Q \end{bmatrix} \begin{bmatrix} i_d \\ i_q \\ i_F \\ i_G \\ i_D \\ i_Q \end{bmatrix} + \begin{bmatrix} -\omega\lambda_q \\ \omega\lambda_d \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} \dot{\lambda}_d \\ \dot{\lambda}_q \\ \dot{\lambda}_F \\ \dot{\lambda}_G \\ \dot{\lambda}_D \\ \dot{\lambda}_Q \end{bmatrix} \tag{4.36}$$

b. Substitute for the currents using 4.124, and do some algebra.

We need to perform this procedure for all six equations, but the procedure is similar for all of them. Therefore, let's do it only for the d-axis flux linkage (as in the text) and the G-axis flux linkage.

d-axis flux linkage equation:

Step a: $$v_d = -ri_d - \dot{\lambda}_d - \omega\lambda_q$$

Step b: From eq. 4.124, we have that: $$i_d = \frac{\lambda_d}{l_d} - \frac{\lambda_{AD}}{l_d}$$

Substituting, we have:

$$v_d = -r(\frac{\lambda_d}{l_d} - \frac{\lambda_{AD}}{l_d}) - \dot{\lambda}_d - \omega\lambda_q = -r\frac{\lambda_d}{l_d} + r\frac{\lambda_{AD}}{l_d} - \dot{\lambda}_d - \omega\lambda_q$$

Solving for $d\lambda_d/dt$ results in: $$\dot{\lambda}_d = -r\frac{\lambda_d}{l_d} + r\frac{\lambda_{AD}}{l_d} - \omega\lambda_q - v_d$$

G-axis flux linkage equation:

Step a: $$v_G = 0 = -r_G i_G - \dot{\lambda}_G$$

Step b: From eq. 4.124, we have that: $$i_G = \frac{\lambda_G}{l_G} - \frac{\lambda_{AQ}}{l_G}$$

Substituting, we have:

$$0 = -r_G(\frac{\lambda_G}{l_G} - \frac{\lambda_{AQ}}{l_G}) - \dot{\lambda}_G = -r_G\frac{\lambda_G}{l_G} + r_G\frac{\lambda_{AQ}}{l_G} - \dot{\lambda}_G$$

Solving for $d\lambda_G/dt$ results in: $\dot{\lambda}_G = -r_G \dfrac{\lambda_G}{l_G} + r_G \dfrac{\lambda_{AQ}}{l_G}$

<u>Summarize all equations</u>

$$\dot{\lambda}_d = -\frac{r}{l_d}\lambda_d + \frac{r}{l_d}\lambda_{AD} - \omega\lambda_q - v_d \qquad (4.126)$$

$$\dot{\lambda}_F = -\frac{r_F}{l_F}\lambda_F + \frac{r_F}{l_F}\lambda_{AD} + v_F \qquad (4.128)$$

$$\dot{\lambda}_D = -\frac{r_D}{l_D}\lambda_D + \frac{r_D}{l_D}\lambda_{AD} \qquad (4.129)$$

$$\dot{\lambda}_q = -\frac{r}{l_q}\lambda_q + \frac{r}{l_q}\lambda_{AQ} + \omega\lambda_d - v_q \qquad (4.130)$$

$$\dot{\lambda}_G = -\frac{r_G}{l_G}\lambda_G + \frac{r_G}{l_G}\lambda_{AQ} \qquad (4.131a)$$

$$\dot{\lambda}_Q = -\frac{r_Q}{l_Q}\lambda_Q + \frac{r_Q}{l_Q}\lambda_{AQ} \qquad (4.131b)$$

<u>Step 3: Develop the torque equation in terms of flux linkages.</u>
We must also write the torque equation as a function of the flux linkages.

Here, we return to the expression we developed for the torque corresponding to the power crossing the air gap. This was given as:

$$T_{e\phi u} = -\lambda_q i_d + \lambda_d i_q$$

When developing the current state-space equations, you recall that at this point we substituted for the flux linkages, using 4.20. Now, when developing the flux-linkage state-space equations, we substitute for the currents, using 4.124 from above, which, for i_q and $i_{d,}$ is:

$$i_d = \frac{\lambda_d - \lambda_{AD}}{l_d} \qquad\qquad i_q = \frac{\lambda_q - \lambda_{AQ}}{l_q}$$

Substitution yields:

$$T_{e\phi u} = -\lambda_q \frac{\lambda_d - \lambda_{AD}}{l_d} + \lambda_d \frac{\lambda_q - \lambda_{AQ}}{l_q}$$

Performing the multiplication, and then gathering terms, results in

$$T_{e\phi u} = -\frac{1}{l_q}\lambda_d \lambda_{AQ} + \frac{1}{l_d}\lambda_q \lambda_{AD} + \left(\frac{1}{l_q} - \frac{1}{l_d}\right)\lambda_d \lambda_q \qquad (4.132a)$$

VMAF in 4.12.2 indicate that the leakage inductance is the same in the d- and q-axis, that is, l_q=l_d, and indicates it is called l_a by many references. Kundur, p. 153, calls it the "stator leakage inductance," and denotes it X_l. Although Kundur in his Table 4.2, p. 153, gives salient pole and round rotor machines the same range for X_l (0.1-0.2), and VMAF is consistent with this in Examples 4.1, 4.2, with l_d=l_q=0.15, and though I feel it is true for round rotor machines, I am unsure it is as true for salient pole machines – it be only approximate for salient-pole machines. Application of l_q=l_d to (4.132a) results in:

$$T_{e\phi u} = -\frac{1}{l_q}\lambda_d \lambda_{AQ} + \frac{1}{l_d}\lambda_q \lambda_{AD} \qquad (4.132b)$$

Recall the torque equation derived for the current-state-space model:

$$\dot{\omega} = \frac{T_m}{\tau_j} + \frac{1}{3\tau_j}[-T_{e\phi}] + \left[\frac{-D}{\tau_j}\right]\omega$$

Substitution of (4.132b) yields:

$$\dot{\omega} = \frac{T_m}{\tau_j} + \left[\frac{\lambda_{AQ}}{l_q 3\tau_j}\lambda_d - \frac{\lambda_{AD}}{l_d 3\tau_j}\lambda_q\right] + \left[\frac{-D}{\tau_j}\right]\omega \qquad (4.133)$$

And so we may gather our equations for the flux-linkage model as:

$$\dot{\lambda}_d = -\frac{r}{l_d}\lambda_d + \frac{r}{l_d}\lambda_{AD} - \omega\lambda_q - v_d \qquad (4.126)$$

$$\dot{\lambda}_F = -\frac{r_F}{l_F}\lambda_F + \frac{r_F}{l_F}\lambda_{AD} + v_F \qquad (4.128)$$

$$\dot{\lambda}_D = -\frac{r_D}{l_D}\lambda_D + \frac{r_D}{l_D}\lambda_{AD} \qquad (4.129)$$

$$\dot{\lambda}_q = -\frac{r}{l_q}\lambda_q + \frac{r}{l_q}\lambda_{AQ} + \omega\lambda_d - v_q \qquad (4.130)$$

$$\dot{\lambda}_G = -\frac{r_G}{l_G}\lambda_G + \frac{r_G}{l_G}\lambda_{AQ} \qquad (4.131a)$$

$$\dot{\lambda}_Q = -\frac{r_Q}{l_Q}\lambda_Q + \frac{r_Q}{l_Q}\lambda_{AQ} \qquad (4.131b)$$

$$\dot{\omega} = \frac{T_m}{\tau_j} + \left[\frac{\lambda_{AQ}}{l_q 3\tau_j}\lambda_d - \frac{\lambda_{AD}}{l_d 3\tau_j}\lambda_q\right] + \left[\frac{-D}{\tau_j}\right]\omega \qquad (4.133)$$

$$\dot{\delta} = \omega - 1 \qquad (4.102)$$

where circled terms given by *auxiliary equations* (p.7 these notes):

$$\lambda_{AD} = \frac{L_{MD}}{l_d}\lambda_d + \frac{L_{MD}}{l_F}\lambda_F + \frac{L_{MD}}{l_D}\lambda_D$$

$$\lambda_{AQ} = \frac{L_{MQ}}{l_q}\lambda_q + \frac{L_{MQ}}{l_Q}\lambda_Q + \frac{L_{MQ}}{l_G}\lambda_G$$

and

$$\frac{1}{L_{MD}} = \left[\frac{1}{L_{AD}} + \frac{1}{l_d} + \frac{1}{l_F} + \frac{1}{l_D}\right]$$

$$\frac{1}{L_{MQ}} = \left[\frac{1}{L_{AQ}} + \frac{1}{l_q} + \frac{1}{l_Q} + \frac{1}{l_G}\right]$$

The state variables are λ_d, λ_F, λ_D, λ_q, λ_Q, λ_G, ω, and δ.
The forcing functions are v_d, v_q, v_F, and T_m (v_d and v_q depend on the currents, which in turn depend on the network loading).

Recall that λ_{AD}, λ_{AQ} represents the per-unit d-axis and q-axis mutual flux linkages, respectively, i.e.,

$$\lambda_{AD}=\lambda_d-l_d i_d=\lambda_F-l_F i_F=\lambda_D-l_D i_D \tag{4.110}$$

$$\lambda_{AQ}=\lambda_q-l_q i_q=\lambda_Q-l_Q i_Q=\lambda_G-l_G i_G \tag{4.110-Q}$$

and that it is only in them, the mutual flux, where saturation occurs. That is, of the total flux seen by each winding, saturation does not affect the leakage, only the mutual. Since these mutual fluxes occur in the same material, it is reasonable to assume the saturation characteristics are the same for all. This allows us to address saturation in a very simple way, through λ_{AD}, λ_{AQ}.

If we were to eliminate these two terms by making the appropriate substitution, then the effects of saturation would be distributed throughout our model, and its treatment would be very complex.

➔ So how do we deal with saturation?