Chapter 9
Multimachine Systems with Constant Impedance Loads

9.1 If the \( \mathbf{Y} \) matrix of the network, reduced to the generator nodes, is such that \( \theta_{ij} = 90^\circ, i \neq j \), derive the general form of the matrix \( \mathbf{M} \).

**Solution**

For \( \theta_{ij} = 90^\circ \), derive the general expression for \( \mathbf{M} \)

\[
\mathbf{M} = \begin{bmatrix}
Y_{11}e^{j\theta_{11}} & Y_{12}e^{j(\theta_{12} - \delta_{12})} & \cdots & Y_{1n}e^{j(\theta_{1n} - \delta_{1n})} \\
Y_{21}e^{j(\theta_{21} - \delta_{21})} & Y_{22}e^{j\theta_{22}} & \cdots & Y_{2n}e^{j(\theta_{2n} - \delta_{2n})} \\
\vdots & \vdots & \ddots & \vdots \\
Y_{n1}e^{j(\theta_{n1} - \delta_{n1})} & Y_{n2}e^{j(\theta_{n2} - \delta_{n2})} & \cdots & Y_{nn}e^{j\theta_{nn}}
\end{bmatrix}
\]

For \( \theta_{ij} = 90^\circ \),
\[e^{j(\theta_{ij} - \delta_{ij})} = \cos(\theta_{ij} - \delta_{ij}) + j\sin(\theta_{ij} - \delta_{ij})\]
\[= \sin\delta_{ij} + j\cos\delta_{ij}\]

The diagonal terms for diagonal element \( i \) are:
\[Y_{ii}e^{j\theta_{ii}} = Y_{ii}(\cos\theta_{ii} + j\sin\theta_{ii}) = G_{ii} + jB_{ii}\]

Then
\[
\mathbf{M} = \begin{bmatrix}
G_{11} & B_{12} \sin\delta_{12} & B_{13} \sin\delta_{13} & \cdots & B_{1n} \sin\delta_{1n} \\
B_{21} \sin\delta_{21} & G_{22} & B_{23} \sin\delta_{23} & \cdots & B_{2n} \sin\delta_{2n} \\
B_{31} \sin\delta_{31} & B_{32} \sin\delta_{32} & G_{33} & \cdots & B_{3n} \sin\delta_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
B_{n1} \sin\delta_{n1} & B_{n2} \sin\delta_{n2} & B_{n3} \sin\delta_{n3} & \cdots & G_{nn}
\end{bmatrix}
\]

\[+ j \begin{bmatrix}
B_{11} \cos\delta_{11} & B_{12} \cos\delta_{12} & B_{13} \cos\delta_{13} & \cdots & B_{1n} \cos\delta_{1n} \\
B_{21} \cos\delta_{21} & B_{22} \cos\delta_{22} & B_{23} \cos\delta_{23} & \cdots & B_{2n} \cos\delta_{2n} \\
B_{31} \cos\delta_{31} & B_{32} \cos\delta_{32} & B_{33} & \cdots & B_{3n} \cos\delta_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
B_{n1} \cos\delta_{n1} & B_{n2} \cos\delta_{n2} & B_{n3} \cos\delta_{n3} & \cdots & B_{nn}
\end{bmatrix}
\]

9.2 For the conditions of Problem 9.1, obtain the real matrices for \( \mathbf{I}_q \) and \( \mathbf{I}_d \) in terms of \( \mathbf{V}_q \) and \( \mathbf{V}_d \). Compare with (9.40) for a two-machine system with \( G_{12} = G_{21} = 0 \).

Solution

For a two-machine system

\[
H = \begin{bmatrix}
G_{11} & B_{12} \sin \delta_{12} \\
B_{21} \sin \delta_{21} & G_{22}
\end{bmatrix}
\]

\[
S = \begin{bmatrix}
B_{11} & B_{12} \cos \delta_{12} \\
B_{21} \cos \delta_{21} & B_{22}
\end{bmatrix}
\]

\[
I = \left[HV_q - SV_d\right] + j\left[SV_q + HV_d\right]
\]

\[
\begin{bmatrix}
I_{q1} \\
I_{q2} \\
I_{d1} \\
I_{d2}
\end{bmatrix} = \begin{bmatrix}
G_{11} & B_{12} \sin \delta_{12} & V_{q1} & B_{11} & B_{12} \cos \delta_{12} & V_{d1} \\
B_{21} \sin \delta_{21} & G_{22} & V_{q2} & B_{21} \cos \delta_{21} & B_{22} & V_{d2}
\end{bmatrix}
\]

9.3 Repeat Example 9.3, using the synchronous machine model called the one-axis model (see Section 4.15.4).

Solution

The one-axis model is obtained from Section 4.15.4, p. 141:

The first equation for the one-axis model is developed starting from (4.288) as follows:

\[
\tau_{q0}' \dot{E}'_d = -E'_d - (x'_q - x'_q)I_q
\]

(4.288)

In the one-axis model, we have that \( \dot{E}'_d = 0 \) and so (4.288) becomes:

\[
0 = -E'_d - (x'_q - x'_q)I_q
\]

so that

\[
E'_d = -(x'_q - x'_q)I_q
\]

(4.288')

And using the assumption \( x'_d \approx x'_q \approx x' \) (see Example 9.3, p. 376), (4.288') becomes

\[
E'_d = -(x'_q - x'_q)I_q
\]

(a)

The second equation for the one-axis model uses (4.294)

\[
\tau_{d0}' \dot{E}'_q = E_{FD} - E; \quad E = E'_q - (x'_d - x'_d)I_d
\]

Substitution yields

\[
\tau_{d0}' \dot{E}'_q = E_{FD} - E'_q + (x'_d - x'_d)I_d
\]

And again using the assumption \( x'_d \approx x'_q \approx x' \) we have:

\[
\tau_{d0}' \dot{E}'_q = E_{FD} - E'_q + (x'_d - x'_d)I_d
\]

(b)

The third equation for the one-axis model uses (4.297)
\[ \tau_j \dot{\omega} = T_m - T_e \]

where \( T_e \) is expressed by (4.296)
\[ T_e = E'_q I_q - \left(x_q - x'_d\right) I_d I_q \]

so that
\[ \tau_j \dot{\omega} = T_m - E'_q I_q + \left(x_q - x'_d\right) I_d I_q \]

And again using the assumption \( x'_d \approx x'_q \equiv x' \) we have
\[ \tau_j \dot{\omega} = T_m - E'_q I_q + \left(x_q - x'\right) I_d I_q \]

Recalling equation (a) above, repeated here for convenience:
\[ E'_d = -(x_q - x') I_q \]

we recognize \( E'_d \) so that
\[ \tau_j \dot{\omega} = T_m - E'_q I_q - E'_d I_d \] (i)

Or
\[ \tau_j \dot{\omega} = T_m - (E'_q I_q + E'_d I_d) \] (c)

The fourth equation for the one-axis model is also from (4.297),
\[ \dot{\delta} = \omega - 1 \] (d)

Collecting (a), (b), (c), and (d), and subscripting with "i" (since we have a two-generator system, we will need to write these equations for \( i=1 \) and \( i=2 \)):
\[
\begin{align*}
E'_{d1} &= -(x_{q1} - x'_{1}) I_{q1} \\
\tau'_{d0} \dot{E}'_{q1} &= E_{FD} - E'_{q1} + (x_{d1} - x'_{1}) I_{d1} \\
\tau_{ji} \dot{E}'_{i} &= T_{mi} - (I_{di} E'_{di1} + I_{qi} E'_{qi}) \\
\dot{\delta}_{i} &= \omega_{i} - 1
\end{align*}
\]

Then, in (9.42), for the top two equations, the derivatives on \( E'_d \) go to zero (per the one-axis model). This gives us two algebraic equations in \( E'_{d1} \), \( E'_{d2} \), \( E'_{q1} \), and \( E'_{q2} \).
\[
\begin{align*}
&\begin{bmatrix} 1 - (x_{q1} - x'_{1})B_{11} \ - (x_{q1} - x'_{1})F_{B-G}(\delta_{12}) E'_{d2} \ - (x_{q2} - x'_{2})F_{B-G}(\delta_{21}) E'_{d1} \ + \begin{bmatrix} 1 - (x_{q2} - x'_{2})B_{22} \ - (x_{q2} - x'_{2})F_{G-B}(\delta_{21}) E'_{q1} \ - (x_{q2} - x'_{2})G_{22} E'_{q2} \end{bmatrix} \end{bmatrix} \cr
&= \begin{bmatrix} -(x_{q1} - x'_{1})G_{11} E'_{q1} \ - (x_{q1} - x'_{1})F_{G+B}(\delta_{12}) E'_{q2} \ - (x_{q2} - x'_{2})F_{B-G}(\delta_{21}) E'_{q1} \ - (x_{q2} - x'_{2})G_{22} E'_{q2} \end{bmatrix}
\end{align*}
\]

Then, \( E'_{d1} \) and \( E'_{d2} \) can be expressed in terms of \( E'_{q1} \) and \( E'_{q2} \). The rest of the equations in (9.42) remain the same.