(1)

The LaGrangian of the problem is given by —

$$\mathcal{L} = x_1^2 + 4x_2^2 - \lambda(x_1 + x_2 - 4).$$

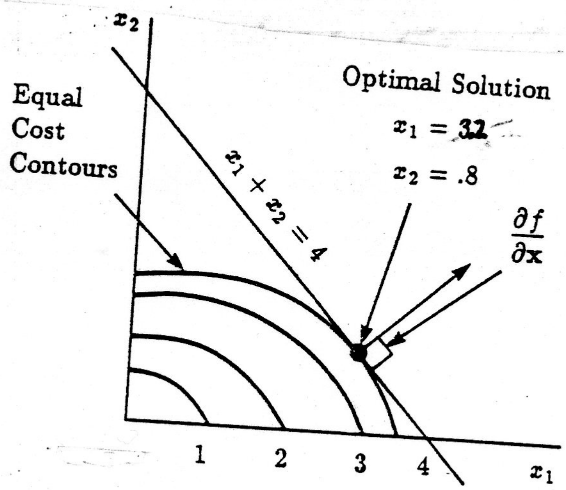
Hence, the necessary conditions of optimality are given by —

$$\frac{\partial \mathcal{L}}{\partial x_1} = 0 = 2x_1 - \lambda$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = 0 = 8x_2 - \lambda$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 = -x_1 - x_2 + 4.$$

$$\Rightarrow \begin{bmatrix} 2 & 0 & -1 \\ 0 & 8 & -1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -4 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ \lambda \end{bmatrix} = \begin{bmatrix} 3.2 \\ 0.8 \\ 6.4 \end{bmatrix}$$



The optimal value is:

$f(\underline{x}) = x_1^2 + 4x_2^2$

$= 3.2^2 + 4(.8)^2 = 12.8$

(2)

First, we restate the inequality constraints to be in the form: $g_i \leq 0$ —

$$\begin{aligned} g_1(\mathbf{x}) &= 4 - x_1 - x_2 \leq 0 \\ g_2(\mathbf{x}) &= x_1 - 3 \leq 0 \\ g_3(\mathbf{x}) &= x_2 - 5 \leq 0. \end{aligned}$$

The Langrangian of the problem is —

$$\begin{aligned} \mathcal{L} = & \; x_1^2 + x_2^2 + \beta_1(4 - x_1 - x_2) \\ & + \beta_2(x_1 - 3) + \beta_3(x_2 - 5). \end{aligned}$$

Our guess is that the solution will be on the line defined by —

$$x_1 + x_2 - 4 = 0.$$

Problem 2 (Continued)

In other words, we will guess that g_1 is binding (which means $\beta_1 > 0$) and g_2 and g_3 are non-binding (which mean $\beta_2 = \beta_3 = 0$). The necessary optimality cdts are

$$\frac{\partial \mathcal{L}}{\partial x_1} = 2x_1 - \beta_1 = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = 2x_2 - \beta_1 = 0$$

$$\frac{\partial \mathcal{L}}{\partial \beta_1} = x_1 + x_2 - 4 = 0$$

$$\Rightarrow \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 2.0 \\ 2.0 \\ 4.0 \end{pmatrix}$$

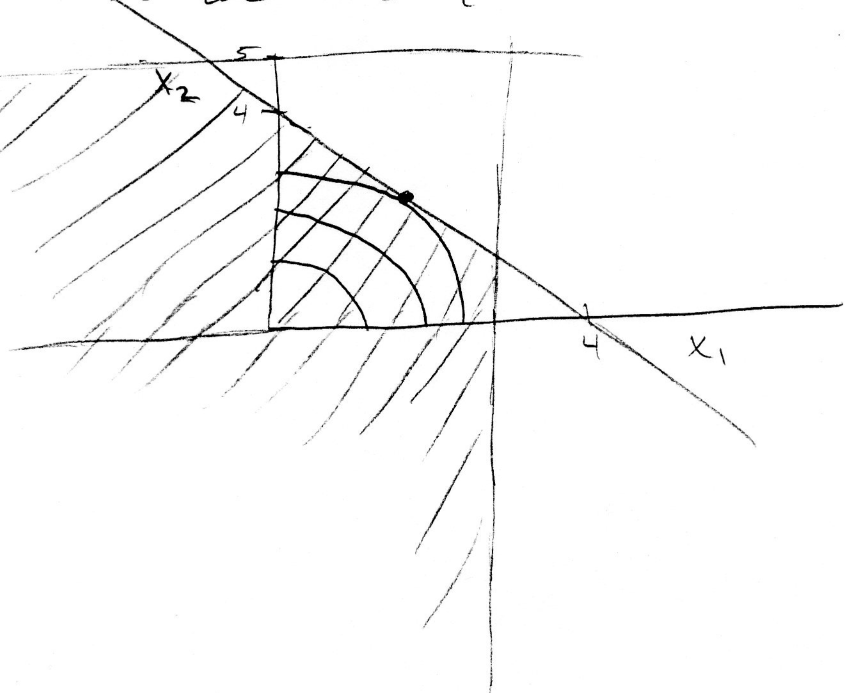
note $\beta_1 > 0$ which is required.

what about g_2 and g_3?

$g_2(\underline{x}) = x_1 - 3 = 2 - 3 = -1 \le 0$ <u>OK!</u>

$g_3(\underline{x}) = x_2 - 5 = 2 - 5 = -3 \le 0$ <u>OK!</u>

So we have found the SOLUTION!



$f(\underline{x}) = x_1^2 + x_2^2$

$= 2^2 + 2^2 = 8$

The LaGrangian is given by —

$$\begin{aligned}\mathcal{L} &= x_1^2 + 3x_2^2 + 4x_3^2 + \lambda(x_1 + x_2 + x_3 - 5) \\ &\quad +\beta_1(x_1 - 3) + \beta_2(x_2 - 2) + \beta_3(x_2 + x_3 - 5).\end{aligned}$$

Hence, the necessary conditions of optimality are given by

$$\begin{aligned}0 &= \frac{\partial \mathcal{L}}{\partial x_1} &= 2x_1 - \lambda + \beta_1 \\ 0 &= \frac{\partial \mathcal{L}}{\partial x_2} &= 6x_2 - \lambda + \beta_2 + \beta_3 \\ 0 &= \frac{\partial \mathcal{L}}{\partial x_3} &= 8x_2 - \lambda + \beta_3 \\ 0 &= \frac{\partial \mathcal{L}}{\partial \lambda} &= x_1 + x_2 + x_3 - 5\end{aligned}$$

$$\begin{aligned}0 &= \beta_1(x_1 - 3),\ \beta_1 &\geq 0 \\ 0 &= \beta_2(x_2 - 2),\ \beta_2 &\geq 0 \\ 0 &= \beta_3(x_2 + x_3 - 5),\ \beta_3 &\geq 0.\end{aligned}$$

We shall start by ignoring the inequality constraints, i.e., set $\beta_1 = \beta_2 = \beta_3 = 0$. The resulting equations are —

$$\begin{aligned}2x_1 - \lambda &= 0, \Rightarrow x_1 = \lambda/2 \\ 6x_2 - \lambda &= 0, \Rightarrow x_2 = \lambda/6 \\ 8x_3 - \lambda &= 0, \Rightarrow x_3 = \lambda/8 \\ x_1 + x_2 + x_3 &= 5.\end{aligned}$$

Substituting into the last equation, we get —

$$\frac{\lambda}{2} + \frac{\lambda}{6} + \frac{\lambda}{8} = 5$$

which implies that $\lambda = 6.3158$. In turn, this implies that $x_1 = 3.158$, $x_2 = 1.053$, $x_3 = .79$.

From this solution, it is clear that the first inequality constraint is violated. As a result we set $x_1 = 3$, while keeping $\beta_2 = \beta_3 = 0$. The resulting equations are —

$$\begin{aligned}2x_1 - \lambda + \beta_1 &= 6 - \lambda + \beta_1 &= 0 \\ x_2 &= \lambda/6 \\ x_3 &= \lambda/8 \\ x_2 + x_3 &= 5 - x_1 &= 2.\end{aligned}$$

This implies that $\frac{\lambda}{6} + \frac{\lambda}{8} = 2$. Consequently, $\lambda = 6.87$. Substituting this last value of λ, one obtains the solution —

$$x_2 = 1.14,\ x_3 = .88,\ \beta_1 = .87 > 0.$$

This solution meets all the necessary conditions of optimality.

In the above example we started by ignoring the inequality constraints. This led to a solution in which one of the constraints is violated. As a result we *guessed* that the optimal solutions are on the boundary of the violated constraint. Effectively this converted that into a new *equality constraint*. Our guess was correct, in the sense that the resulting solution met the necessary conditions of optimality.