EDC3

1.0 Introduction

In the last set of notes (EDC2), we saw how to use penalty factors in solving the EDC problem with losses. In this set of notes, we want to address two closely related issues.

- What are, exactly, penalty factors?
- How to obtain the penalty factors in practice?

2.0 What are penalty factors?

Recall the definition:

$$L_i = \frac{1}{\left[1 - \frac{\partial P_L(P_{G2}, \ldots, P_{Gm})}{\partial P_{Gi}}\right]} \tag{1}$$

In order to gain intuitive insight into what is a penalty factor, let's replace the numerator and denominator of the partial derivative in (1) with the approximation of $\Delta P_L / \Delta P_{Gi}$, so:

$$L_i = \frac{1}{\left[1 - \frac{\Delta P_L}{\Delta P_{Gi}}\right]} \qquad (2)$$

Multiplying top and bottom by ΔP_{Gi}, we get:

$$L_i = \frac{\Delta P_{Gi}}{\left[\Delta P_{Gi} - \Delta P_L\right]} \qquad (3)$$

What is ΔP_{Gi}? It is a small change in generation. But that cannot be all, because if you make a change in generation, then there must be a change in injection at, at least, one other bus. Let's assume that a compensating change is distributed throughout all other load buses according to a fixed percentage for each bus. By doing so, we are embracing the so-called "*conforming load*" assumption, which indicates that all loads change proportionally.

Therefore $\Delta P_{Gi}=\Delta P_D$. But this will also cause a change in losses of ΔP_L, which will be offset by a compensating change in swing bus generation ΔP_{G1}. So,

$$\Delta P_{Gi} + \Delta P_{G1} = \Delta P_D + \Delta P_L \qquad (4)$$

where we see generation changes are on the left and load & loss changes are on the right. Solving for ΔP_{Gi}-ΔP_L (because it is in the denominator of (3)), we get

$$\Delta P_{Gi} - \Delta P_L = \Delta P_D - \Delta P_{G1} \qquad (5)$$

Substituting (5) into (3), we obtain:

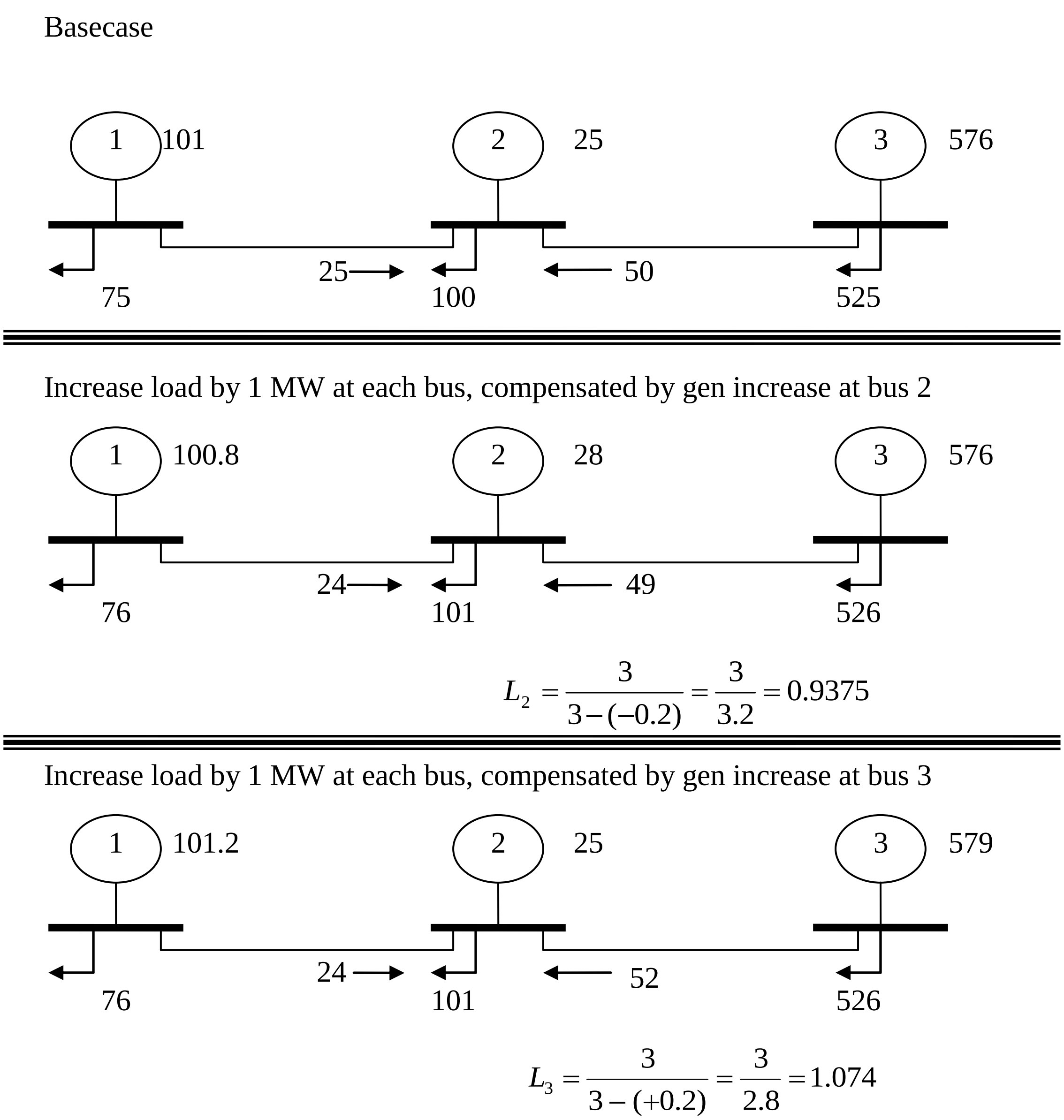
$$L_i = \frac{\Delta P_{Gi}}{\Delta P_D - \Delta P_{G1}} \qquad (6a)$$

Recognize that ΔP_{G1} in (6a) reflects the losses, we have

$$L_i = \frac{\Delta P_{Gi}}{\Delta P_D - \Delta P_L} \qquad (6b)$$

So from (6b), we extract the following interpretation of the penalty factor: It is the amount of generation at unit i necessary to supply ΔP_D, as a percentage of ΔP_D-ΔP_L. This depends on how the load is changed (which is why we use the *conforming load* assumption). If the change increases losses (ΔP_L>0), then L_i>1. If the change decreases losses(ΔP_L<0), then L_i<1.

An example will illustrate the significance of (6a) & (6b). Consider Fig. 1. Observe that the flows given on the circuits are into bus 2 (the flows along the line out of buses 1 and 3, respectively, are higher).



$$L_2 = \frac{3}{3-(-0.2)} = \frac{3}{3.2} = 0.9375$$

$$L_3 = \frac{3}{3-(+0.2)} = \frac{3}{2.8} = 1.074$$

Fig. 1

One observes that $L_2<1$. This is because a load change compensated by a gen change at bus 2 decreases the losses as indicated by the fact that the bus 1 generation decreased by 0.2 MW.
On the other hand, $L_3>1$. This is because a load change compensated by a gen change at bus 3 increases the losses as indicated by the fact that the bus 1 generation increases by 0.2. MW.

Why does the bus 2 generation reduce losses whereas the bus 3 generation increases losses?

Answer: Because increasing bus 2 tends to reduce line flows, whereas increasing bus 3 tends to increase line flows.

So we see that in general, generators on the receiving end of flows will tend to have lower penalty factors (below 1.0); generators on the sending end of flows will tend to have higher penalty factors (above 1.0).

Because transmission systems are in fact relatively efficient, with reasonably small losses in the circuits, the amount of generation necessary to supply a load change tends to be very close to that load change. Therefore penalty factors tend to be relatively close to 1.0.

A list of typical penalty factors for the power system in Northern California is illustrated in Fig. 2. Generators marked to the right are units in the San Francisco Bay Area, which is a relatively high import area for the Northern California system. Most of the penalty factors for these units are below 1.0. Units having penalty factors>1.1 are mainly units close to the Oregon border (a long way from the SF load center), such that they tend to add to the north-to-south flow that results from the northwest hydro being sold into the California load centers.

HALTY FACTORS FOR BASE-CASE GENERATION AND LOAD LEVEL

1985 SPRING PENALTY FACTORS   RANCHO SECO DOWN

IS	BUS	GENERATOR AXIS NAME	AXIS MW	PENALTY FACTOR
1	1	MALIN 500. INPUT FROM NORTHWEST.	2499.9995	1.142489
2	2	MIDWAY 500. INPUT FROM S. C. E.	-1199.9997	0.995580
3	3	SIERRA PACIFIC INTERTIE FROM SIERRA	0.8000	1.164796
4	4	SHASTA 230. SHT, KSW, CARR, SP CK, TRN	739.9998	1.117241
5	5	HUMBOLDT 115. HUM. P. P. 1-3.	30.0000	0.916989
6	6	ROUND MT. 230. PIT 3-7, BLACK	629.9999	1.125850
7	7	COTTONWD 230. PIT 1	60.0000	1.104298
8	8	CARIBOU 230. CARIBOU UNIT 4&5, BELDEN	120.0000	1.141188
9	9	MID/TID - INTERCHANGE FR PARKER & WALNUT	-36.4000	1.000940
10	10	POE 230: POE,CRESTA,BUCKS&ROK CRK,BELDEN	474.9999	1.185117
11	11	RANCHO SECO 230.	0.1000	1.021453
12	12	TABLE MT 230. INPUT FR STATE O/T AT TM	397.9999	1.094720
13	13	PALERMO 115. FORBSTOWN , WOODLEAF	80.0000	1.131894
14	14	DRUM 115. DRUM,DTCH FLT 1&2,CHICAGO PK	148.0000	1.163768
15	15	GOLD HILL 230. MID FK,FR MEADWS,RALSTON	197.0000	1.069882
16	16	CARIBOU 115. CARIBOU UNIT 1-3, BUTT VLY	65.0000	1.152115
17	17	FOLSOM 230. FOLSOM 1-3, NIMBUS	128.0000	1.043291
18	18	COLGATE 230. COLGATE, NARROWS 1&2	344.9999	1.116326
19	19	TRACY P. 230. INPUT FR TRACY PUMP & CCID	-75.0000	1.003675
20	20	WOKL. EQ 230. ELECTRA, SLT. SPG. AND TG	157.0000	1.045539
21	21	WRWRVILE 230. INTERCHANGE FROM CITY S.F.	23.9000	1.038416
22	22	NEWARK 115. INTERCHANGE FROM CITY S.F.	72.7000	0.977999
23	23	STANISLAS 115. STANISLAUS 0	55.0000	1.088110
24	24	MELONES 115. DONNELLS,BEARDSLEY,TULLOCH	76.0000	1.075023
25	25	CON. CSTA 230. CCPP 1-7	200.0000	1.006337
26	26	PITTSBRG 230. PTSB PP 3-7	654.7870	0.984111
27	27	PITTSBRG 115. PTSB PP 1 & 2	0.1000	0.981577
28	28	MARTINEZ 115. AVON AND MARTINEZ	7.0000	0.960786
29	29	OLEUM 115. OLEUM 1&2	10.0000	0.968778
30	30	HNTRS. PT. 115. HUNTERS POINT PP 1-4	185.0000	0.949433
31	31	POTRERO 115. POTRERO PP 1-6	200.0000	0.947466
32	32	MOSS LDG. 500. MOSS LANDING PP 6 & 7	699.9999	1.007338
33	33	MOSS LDG. 230. MOSS LANDING PP 1-5	0.1000	1.004706
34	34	AMES 115. INTERCHANGE FROM AMES	-80.0000	0.969340
35	35	SLAC 230. INTERCHANGE FROM SLAC	-55.0000	0.979686
36	36	MORRO BAY 230. MORRO BAY PP 1-4	499.9999	1.015732
37	37	PIEDRA SH. 115. KINGS RIVER	40.0000	1.009973
38	38	KERCKHOFF 115. KERCKHOFF GEN	120.0000	1.086544
39	39	EXCHEQUER 115. EXCHEQUER GEN	70.0000	1.119342
40	40	BALCH EQ. 230. BALCH 2, HAAS & PINE FLT	324.9999	1.045456
41	41	HELMS PP 230KV	0.1000	0.988100
42	42	UARP-SMUD HYDRO	439.9999	1.042280
43	43	OAKLAND STA C 115, STA C GAS TURB GEN	0.1000	0.966901
44	44	NEW MELONES 230. (LOOPED)	200.0000	1.046535
45	45	DELTA P. 230. INTERCHANGE FR DELTA PUMP	-57.0000	1.001290
46	46	DS AMIGS 230.INTERCHANGE FR DS AMIGOS P.	-17.0000	1.004139
47	47	LS BANOS 230.INTERCHANGE FR SAN LUIS GEN	0.1000	1.006157
48	48	GEYSERS 230. GEYSER UNITS ON 230KV	1089.9998	1.089819
49	49	GEYSERS 115. GEYSER UNITS ON 115KV	145.0000	1.091903
50	50	DIABLO 500. DIABLO 1&2	999.9998	1.003612

Note: A generation level of 0.1MW is equivalent to the unit(s) being shut down.

Figure 13-3

Fig. 2

But why do we actually call them penalty factors? Consider the criterion for optimality in the EDC with losses:

$$\lambda = L_i \frac{\partial C_i(P_{Gi})}{\partial P_{Gi}} \qquad \forall i = 1,\ldots m \qquad (7)$$

This says that all units (or all regulating units) must be at a generation level such that the product of their incremental cost and their penalty factor must be equal to the system incremental cost λ.

Let's do an experiment to see what this means. Consider that we have three identical units such that their incremental cost-rate curves are identical, given by $IC(P_G)=45+0.02P_G$.

Now consider the three units are so located such that unit 1 has penalty factor of 0.98, unit 2 has penalty factor of 1.0, and unit 3 has penalty factor of 1.02, and the demand is 300 MW.

Without accounting for losses, this problem would be very simple in that each unit would carry 100 MW.

But with losses, the problem is as follows:

λ=0.98(45+0.02P_{G1})=44.1+0.196P_{G1}
λ=1.0(45+0.02P_{G2})=45+0.02P_{G2}
λ=1.02(45+0.02P_{G3})=45.9+0.0204P_{G3}

Putting these three equations into matrix form results in:

$$\begin{bmatrix} 0.0196 & 0 & 0 & -1 \\ 0 & 0.02 & 0 & -1 \\ 0 & 0 & 0.0204 & -1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} P_{G1} \\ P_{G2} \\ P_{G3} \\ \lambda \end{bmatrix} = \begin{bmatrix} -44.1 \\ -45 \\ -45.9 \\ 300 \end{bmatrix}$$

Solving in Matlab yields:

$$\begin{bmatrix} P_{G1} \\ P_{G2} \\ P_{G3} \\ \lambda \end{bmatrix} = \begin{bmatrix} 147.32 \\ 99.37 \\ 53.31 \\ 46.9875 \end{bmatrix}$$

One notes that the unit with the lower penalty (unit 1) was “turned up” and the unit with the higher penalty (unit 3) was “turned down.” The reason for this is that unit 1 has a better effect on losses.

3.0 Penalty factor calculation

There are several methods for penalty factor calculation. We will review several of them in this section.

This method is described in [1]. Consider a power system with total of *n* buses of which bus 1 is the swing bus, buses 1…*m* are the PV buses, and buses *m*+1…n are the PQ buses.

Consider that losses must be equal to the difference between the total system generation and the total system demand:

$$P_L = P_G - P_D \qquad (8)$$

Recall the definition for bus injections, which is

$$P_i = P_{Gi} - P_{Di} \qquad (9)$$

Now sum the injections over all buses to get:

$$\sum_{i=1}^{n} P_i = \sum_{i=1}^{n} (P_{Gi} - P_{Di})$$

$$= \sum_{i=1}^{n} P_{Gi} - \sum_{i=1}^{n} P_{Di} = P_G - P_D \quad (10)$$

Therefore,

$$P_L = \sum_{i=1}^{n} P_i \quad (11)$$

Now differentiate with respect to a particular bus angle θ_k (where k is any bus number except 1) to obtain:

$$\frac{\partial P_L}{\partial \theta_k} = \frac{\partial P_1}{\partial \theta_k} + \frac{\partial P_2}{\partial \theta_k} \ldots + \frac{\partial P_m}{\partial \theta_k} + \frac{\partial P_{m+1}}{\partial \theta_k} + \ldots + \frac{\partial P_n}{\partial \theta_k}, k = 2,\ldots,n \quad (12)$$

Assumption to the above: All voltages are fixed at 1.0; this relieves us from accounting for variation in power with angle through the voltage magnitude term. Otherwise, each term in (12) would appear as

$$\frac{\partial \boldsymbol{P_i}}{\partial \boldsymbol{\theta_k}} + \frac{\partial \boldsymbol{P_i}}{\partial \mathbf{V}_{\boldsymbol{k}}} \frac{\partial \mathbf{V}_{\boldsymbol{k}}}{\partial \boldsymbol{\theta_k}}$$

Now let's assume that we have an expression for losses P_L as a function of generation $P_{G2}, P_{G3},\ldots,P_{Gm}$, i.e.,

$$P_L = P_L(P_{G2}, P_{G3}, \ldots, P_{Gm}) \tag{13}$$

Then we can use the chain rule of differentiation to express that

$$\frac{\partial P_L}{\partial \theta_k} = \frac{\partial P_L(\underline{P}_G)}{\partial P_{G2}}\frac{\partial P_2}{\partial \theta_k} + \ldots + \frac{\partial P_L(\underline{P}_G)}{\partial P_{Gm}}\frac{\partial P_m}{\partial \theta_k}, k = 2,\ldots,n \tag{14}$$

In (14), we assume that at generator buses, loads are constant, and $\partial P_{Gi}/\partial \theta_i = \partial P_i/\partial \theta_i$.

Subtracting (14) from (12), we obtain, for k=2,…,n:

$$\left.\frac{\partial \boldsymbol{P}_1}{\partial \boldsymbol{\theta}_k} + \frac{\partial \boldsymbol{P}_2}{\partial \boldsymbol{\theta}_k}\ldots + \frac{\partial \boldsymbol{P}_m}{\partial \boldsymbol{\theta}_k} + \frac{\partial \boldsymbol{P}_{m+1}}{\partial \boldsymbol{\theta}_k} + \ldots + \frac{\partial \boldsymbol{P}_n}{\partial \boldsymbol{\theta}_k}\right\} \leftarrow \frac{\partial \boldsymbol{P}_L}{\partial \boldsymbol{\theta}_k}\ \text{from (12)}$$

$$\left.-\left(\frac{\partial \boldsymbol{P}_L(\underline{\boldsymbol{P}}_G)}{\partial \boldsymbol{P}_{G2}}\frac{\partial \boldsymbol{P}_2}{\partial \boldsymbol{\theta}_k} + \ldots + \frac{\partial \boldsymbol{P}_L(\underline{\boldsymbol{P}}_G)}{\partial \boldsymbol{P}_{Gm}}\frac{\partial \boldsymbol{P}_m}{\partial \boldsymbol{\theta}_k}\right)\right\} \leftarrow \frac{\partial \boldsymbol{P}_L}{\partial \boldsymbol{\theta}_k}\ \text{from (14)}$$

$$0 = \frac{\partial \boldsymbol{P}_1}{\partial \boldsymbol{\theta}_k} + \frac{\partial \boldsymbol{P}_2}{\partial \boldsymbol{\theta}_k}\left(1 - \frac{\partial \boldsymbol{P}_L(\underline{\boldsymbol{P}}_G)}{\partial \boldsymbol{P}_{G2}}\right) + \ldots + \frac{\partial \boldsymbol{P}_m}{\partial \boldsymbol{\theta}_k}\left(1 - \frac{\partial \boldsymbol{P}_L(\underline{\boldsymbol{P}}_G)}{\partial \boldsymbol{P}_{Gm}}\right)$$

$$+ \frac{\partial \boldsymbol{P}_{m+1}}{\partial \boldsymbol{\theta}_k} + \ldots + \frac{\partial \boldsymbol{P}_n}{\partial \boldsymbol{\theta}_k}$$

Now bring the first term to the left-hand-side, for k=2,…,n

Writing the above

$$-\frac{\partial P_1}{\partial \theta_k} = \frac{\partial P_2}{\partial \theta_k}\left(1-\frac{\partial P_L(\underline{P}_G)}{\partial P_{G2}}\right)+\ldots+\frac{\partial P_m}{\partial \theta_k}\left(1-\frac{\partial P_L(\underline{P}_G)}{\partial P_{Gm}}\right)$$

$$+\frac{\partial P_{m+1}}{\partial \theta_k}+\ldots+\frac{\partial P_n}{\partial \theta_k}$$

The above equation, when written for k=2,…,n, can be expressed in matrix form as

$$\begin{bmatrix} \frac{\partial P_2}{\partial \theta_2} & \cdots & \frac{\partial P_m}{\partial \theta_2} & \cdots & \frac{\partial P_n}{\partial \theta_2} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ \frac{\partial P_2}{\partial \theta_n} & \cdots & \frac{\partial P_m}{\partial \theta_n} & \cdots & \frac{\partial P_n}{\partial \theta_n} \end{bmatrix} \begin{bmatrix} 1-\frac{\partial P_L(\underline{P}_G)}{\partial P_{G2}} \\ \vdots \\ 1-\frac{\partial P_L(\underline{P}_G)}{\partial P_{Gm}} \\ 1 \\ \vdots \\ 1 \end{bmatrix} = -\begin{bmatrix} \frac{\partial P_1}{\partial \theta_2} \\ \vdots \\ \frac{\partial P_1}{\partial \theta_2} \end{bmatrix} \qquad (15)$$

The matrix on the left-hand side is the transpose of the upper left-hand submatrix of the power flow Jacobian (we called it $\underline{J}^{P\theta}$), and so codes are readily available to compute it. The elements of the right-hand-side vector may be found by differentiating the real power equation for bus 1, which is:

$$\boldsymbol{P}_1 = \sum_{\boldsymbol{i}=1}^{\boldsymbol{N}} |\boldsymbol{V}_1||\boldsymbol{V}_{\boldsymbol{i}}|\left(\boldsymbol{G}_{1\boldsymbol{i}}\cos(\boldsymbol{\theta}_1-\boldsymbol{\theta}_{\boldsymbol{i}})+\boldsymbol{B}_{1\boldsymbol{i}}\sin(\boldsymbol{\theta}_1-\boldsymbol{\theta}_{\boldsymbol{i}})\right) \qquad (16)$$

with respect to each angle, resulting in

$$\frac{\partial P_1}{\partial \theta_i} = |V_1||V_i|\left[G_{1i}\sin(\theta_1-\theta_i) - B_{1i}\cos(\theta_1-\theta_i)\right]$$

The solution vector contains the inverse of the penalty factors in the first m-1 terms.

4.0 Using loss formula

The method of *loss formula* results in an approximate expression given by

$$P_L = \underline{P}_G^T \underline{B}\underline{P}_G + \underline{B}_0^T \underline{P}_G + B_{00} \qquad (17)$$

where $\underline{P}_G$ is the vector of generation

$$\underline{P}_G^T = \begin{bmatrix} P_{G1} \\ \vdots \\ P_{Gm} \end{bmatrix} \qquad (18)$$

Development of the coefficient matrices in (17) has been done in several ways. The first edition of the W&W text (1986) presented a method developed by Meyer [2] in Appendix B of chapter 4; it was removed from the second edition.

I developed another method based on the work of Kron, which is partially articulated in the book by El-Harawry and Christenson, and attached to the end of these notes.

Some important similarities in the methods:

1. Both are dependent on the following assumptions:
   - Each bus can be clearly distinguished as either a load bus or a generation bus.
   - Reactive generation varies linearly with generation, i.e., $Q_{gk}=Q_{go}+f_kP_{gk}$.
2. Both end up with expressions for P_L of the same form.
3. Both expressions for P_L are dependent on the elements of the Z_{bus} matrix.

But there is one major difference between the formulations in that Kron's approach makes no assumption regarding conforming loads. However, the method of W&W (Meyers) does, i.e., in Meyer's approach, all loads must increase or decrease uniformly.

We assume that we have the so-called B-coefficients in the example which follows.

Example of ED solution using Loss Formula

$F_1(P_{g_1}) = 100 + 4.1 P_{g_1} + .0035 P_{g_1}^2$

$F_2(P_{g_2}) = 200 + 4.1 P_{g_2} + .0035 P_{g_2}^2$

$20 \le P_{g_i} \le 550$

$$\underline{B} = \begin{bmatrix} .0001 & -.000005 \\ -.000005 & .00013 \end{bmatrix} \qquad P_d = 400 \text{ MW}$$

We desire to find the optimal dispatch.

Let $\epsilon = 1$ MW (tolerance on Lambda Iteration)
$\delta = 1.5$ MW (overall solution tolerance)

First, let's solve w/o losses to get a starting solution. Let's use closed form.

$$\frac{\partial F_1}{\partial P_{g_1}} = 4.1 + .007 P_{g_1} = \lambda$$

$$\frac{\partial F_2}{\partial P_{g_2}} = 4.1 + .007 P_{g_2} = \lambda$$

$$P_{g_1} + P_{g_2} = P_d$$

$$\begin{bmatrix} .007 & 0 & -1 \\ 0 & .007 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} P_{g_1} \\ P_{g_2} \\ \lambda \end{bmatrix} = \begin{bmatrix} -4.1 \\ -4.1 \\ 400 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} P_{g_1}^{(0)} \\ P_{g_2}^{(0)} \\ \lambda^{(0)} \end{bmatrix} = \begin{bmatrix} 200 \\ 200 \\ 5.5 \end{bmatrix} \Rightarrow \text{This is our starting solution.}$$

Compute Losses

$$P_{loss} = \underline{P_G^T} \, \underline{B} \, \underline{P_G} = \begin{bmatrix} P_{g_1} & P_{g_2} \end{bmatrix} \begin{pmatrix} .0001 & -.000005 \\ -.000005 & .00013 \end{pmatrix} \begin{pmatrix} P_{g_1} \\ P_{g_2} \end{pmatrix}$$

$$= .0001 P_{g_1}^2 - .00001 P_{g_1} P_{g_2} + .00013 P_{g_2}^2$$

$$P_L^{(0)} = (.0001)(200)^2 - .00001(200)(200) + .00013(200)^2 = 8.8$$

$$\Rightarrow P_R = 400 + 8.8 = 408.8 \text{ MW}$$

Compute penalty factors

$$pf_1 = \frac{1}{1 - \frac{\partial P_L}{\partial P_{g_1}}} = \frac{1}{1 - 2\sum_{j=1}^{2} B_{1j} P_{g_j} - B_{10}^{\,0}}$$

$$= \frac{1}{1 - 2\left[B_{11} P_{g_1} + B_{12} P_{g_2}\right]} = \frac{1}{1 - 2\left[(.0001)(200) - (.000005)(200)\right]} = 1.0395$$

$$pf_2 = \frac{1}{1 - \frac{\partial P_L}{\partial P_{g_2}}} = \frac{1}{1 - 2\sum_{j=1}^{2} B_{2j} P_{g_j} - B_{20}^{\,0}}$$

$$= \frac{1}{1 - 2\left[B_{21} P_1 + B_{22} P_2\right]} = \frac{1}{1 - 2\left[-.000005(200) + (.00013)(200)\right]} = 1.053$$

~~= 2.0~~

So the coordination eqts are

$$1.0395 \frac{\partial F_1}{\partial P_{g_1}} = \lambda$$

$$1.053 \frac{\partial F_2}{\partial P_{g_2}} = \lambda$$

$$P_{g_1} + P_{g_2} - P_L(P_g) = P_R$$

OR

$$1.0345\left[4.1+.007P_{g_1}\right]=\lambda \Rightarrow P_{g_1}=\frac{.962\lambda-4.1}{.007}$$

$$1.053\left[4.1+.007P_{g_2}\right]=\lambda \Rightarrow P_{g_2}=\frac{0.95\lambda-4.1}{.007}$$

Note $\frac{\partial P_T}{\partial \lambda}=\frac{.962}{.007}+\frac{.95}{.007}=273.1$

$$\Rightarrow \Delta\lambda=\frac{\Delta P_T}{273.1}$$

Try $\lambda = 5.5$

$$\Rightarrow P_{g_1}=170.1$$
$$P_{g_2}=160.7$$
$$330.81 \Rightarrow \Delta P_T = 408.8-330.8 = 78.0$$

$$\Rightarrow \Delta\lambda = \frac{78.0}{273.1}=0.286 \Rightarrow \lambda = 5.5+.286=5.79$$

$\lambda = 5.79$

$$\Rightarrow P_{g_1}=209.4$$
$$P_{g_2}=199.5$$
$$408.9 \Rightarrow \Delta P_T = 408.8-408.9=-.1 < \varepsilon \text{ OK!}$$

Now check stopping criterion:

$$\Delta = \max\left\{\left|P_{g_i}^{new}-P_{g_i}^{old}\right| \forall i\right\} = \max\left\{|[illegible]-209.4|, |199.5-[illegible]|\right\}$$

$$= \max\{9.4, 0.5\} = 9.4 > \varepsilon \quad \therefore \text{ we must repeat!}$$

Compute new losses:

$$P_L^{(1)} = (.0001)(208.4)^2 - .00001(208.4)(199.5) + .00013(199.5)^2$$
$$= 9.14$$

$\Rightarrow P_R = 400 + 9.14 = 409.14$ MW

Compute new penalty factors:

$$pf_1 = \frac{1}{1 - \frac{\partial P_L}{\partial P_{g1}}} = \frac{1}{1 - 2[.0001(208.4) - .000005(199.5)]}$$

$$= 1.042$$

$$pf_2 = \frac{1}{1 - \frac{\partial P_L}{\partial P_{g2}}} = \frac{1}{1 - 2[-.000005(208.4) + .00013(199.5)]}$$

$$= 1.052$$

Coordination Eqts:

$$1.042[4.1 + .007 P_{g1}] = \lambda \Rightarrow P_{g1} = \frac{.96\lambda - 4.1}{.007}$$

$$1.052[4.1 + .007 P_{g2}] = \lambda \Rightarrow P_{g2} = \frac{.951\lambda - 4.1}{.007}$$

$$\frac{\partial P_T}{\partial \lambda} = \frac{.96}{.007} + \frac{.951}{.007} = 273.0$$

$$\Rightarrow \Delta\lambda = \frac{\Delta P_T}{273}$$

Try $\lambda = 5.79$

$$\Rightarrow P_{g_1} = 208.3$$
$$P_{g_2} = 200.9$$
$$409.2 \Rightarrow \Delta P_L = 409.14 - 409.2 = -.06 < \epsilon, \ \text{OK!}$$

Stopping Criterion?

$$\Delta = \max\{|208.3 - 209.4|, |200.9 - 199.5|\}$$
$$= \max\{1.1, 1.4\} = 1.4 < 5$$

STOP!

[1] A. Bergen and V. Vittal, “Power System Analysis,” Prentice-Hall, 2000.
[2] W. Meyer, “Efficient computer solution for Kron and Kron-Early Loss Formulas,” Proc of the 1973 PICA conference, IEEE 73 CHO 740-1, PWR, pp. 428-432.

Derivation of Loss Formula

(From Optimal Economic Operation of Electric Power Systems
by M. El-Hawary and G. Christensen)

If you add all MVA injections ($S_{LOAD} + S_{gen}$) at every bus, you get the losses:

$$\bar{S}_L = \sum_{i=1}^{N} \bar{S}_i$$

Define $\underline{\bar{V}_B} = \begin{bmatrix} \bar{V}_1 \\ \vdots \\ \bar{V}_N \end{bmatrix}$ $\quad \bar{I}_B = \begin{bmatrix} \bar{I}_1 \\ \vdots \\ \bar{I}_N \end{bmatrix}$

$$\Rightarrow \bar{S}_L = \underline{\bar{V}_B}^T \bar{I}_B^*$$

Define $\quad \underline{\bar{Z}_B} = \begin{bmatrix} \bar{Z}_{11} & \cdots & \bar{Z}_{1N} \\ \vdots & & \vdots \\ \bar{Z}_{N1} & \cdots & \bar{Z}_{NN} \end{bmatrix}$

Then $\underline{\bar{V}_B} = \underline{\bar{Z}_B}\,\underline{\bar{I}_B}$

Express $\underline{\bar{Z}_B}$ and $\underline{\bar{I}_B}$ in rectangular form

$$\underline{\bar{Z}_B} = \underline{R} + j\underline{X} = \begin{bmatrix} R_{11} & \cdots & R_{1N} \\ \vdots & & \vdots \\ R_{N1} & \cdots & R_{NN} \end{bmatrix} + j \begin{bmatrix} X_{11} & \cdots & X_{1N} \\ \vdots & & \vdots \\ X_{N1} & \cdots & X_{NN} \end{bmatrix}$$

$$\underline{\bar{I}_B} = \underline{\hat{I}_p} + j\underline{\hat{I}_q} = \begin{bmatrix} \hat{I}_{p1} \\ \vdots \\ \hat{I}_{pN} \end{bmatrix} + j \begin{bmatrix} \hat{I}_{q1} \\ \vdots \\ \hat{I}_{q2} \end{bmatrix}$$

Step 2: Derive P_L, Q_L

Then

$$\bar{S}_L = \underline{\bar{V}}_B^T \underline{\bar{I}}_B^* = (\underline{\bar{Z}}_B \underline{\bar{I}}_B)^T \underline{\bar{I}}_B^* = \underline{\bar{I}}_B^T \underline{\bar{Z}}_B^T \underline{\bar{I}}_B^*$$

But $= (\underline{\bar{I}}_p^T + j\underline{\bar{I}}_q^T)(\underline{R}^T + j\underline{X}^T)(\underline{\bar{I}}_p - j\underline{\bar{I}}_q)$

$$= \left[(\underline{\bar{I}}_p^T \underline{R}^T - \underline{\bar{I}}_q^T \underline{X}^T) + j(\underline{\bar{I}}_q^T \underline{R}^T + \underline{\bar{I}}_p^T \underline{X}^T)\right]\left[\underline{\bar{I}}_p - j\underline{\bar{I}}_q\right]$$

$$= \left[(\underline{\bar{I}}_p^T \underline{R}^T - \underline{\bar{I}}_q^T \underline{X}^T)\underline{\bar{I}}_p + (\underline{\bar{I}}_q^T \underline{R}^T + \underline{\bar{I}}_p^T \underline{X}^T)\underline{\bar{I}}_q\right]$$

$$+ j\left[(\underline{\bar{I}}_q^T \underline{R}^T + \underline{\bar{I}}_p^T \underline{X}^T)\underline{\bar{I}}_p - (\underline{\bar{I}}_p^T \underline{R}^T - \underline{\bar{I}}_q^T \underline{X}^T)\underline{\bar{I}}_q\right]$$

$$\Rightarrow P_L = \underline{\bar{I}}_p^T \underline{R}^T \underline{\bar{I}}_p - \underline{\bar{I}}_q^T \underline{X}^T \underline{\bar{I}}_p + \underline{\bar{I}}_q^T \underline{R}^T \underline{\bar{I}}_q + \underline{\bar{I}}_p^T \underline{X}^T \underline{\bar{I}}_q$$

$$P_L = \underline{\bar{I}}_p^T \underline{R}^T \underline{\bar{I}}_p + \underline{\bar{I}}_q^T \underline{R}^T \underline{\bar{I}}_q \qquad \text{(Eqt 1)}$$

$$Q_L = \underline{\bar{I}}_q^T \underline{R}^T \underline{\bar{I}}_p + \underline{\bar{I}}_p^T \underline{X}^T \underline{\bar{I}}_p - \underline{\bar{I}}_p^T \underline{R}^T \underline{\bar{I}}_q + \underline{\bar{I}}_q^T \underline{X}^T \underline{\bar{I}}_q$$

$$Q_L = \underline{\bar{I}}_p^T \underline{X}^T \underline{\bar{I}}_p + \underline{\bar{I}}_q^T \underline{X}^T \underline{\bar{I}}_q \qquad \text{(Eqt 2)}$$

Step 3: Eliminate Current Variables

Recall $\bar{V}_i = V_i(\cos\theta_i + j\sin\theta_i)$

$\bar{I}_i = \bar{I}_{pi} + j\bar{I}_{qi}$

$$\Rightarrow \bar{S}_i = P_i + jQ_i = \bar{V}_i \bar{I}_i^* = V_i(\cos\theta_i + j\sin\theta_i)(\bar{I}_{pi} - jI_{qi})$$

$$= V_i\left[I_{pi}\cos\theta_i + I_{qi}\sin\theta_i + j(I_{qi}\cos\theta_i + I_{pi}\sin\theta_i)\right]$$

$$\Rightarrow P_i = V_i(I_{pi}\cos\theta_i + I_{qi}\sin\theta_i)$$

$$Q_i = V_i(I_{pi}\sin\theta_i - I_{qi}\cos\theta_i)$$

Laborius manipulation of these eqts yield explicit expressions for I_{pi} and I_{qi} as follows

$$I_{pi} = \frac{1}{V_i}\left[P_i \cos\theta_i + Q_i \sin\theta_i\right] \qquad \text{Eqt 3}$$

$$I_{qi} = \frac{1}{V_i}\left[P_i \sin\theta_i - Q_i \cos\theta_i\right] \qquad \text{Eqt 4}$$

Define

$$\underline{C} = \begin{bmatrix} \frac{\cos\theta_1}{V_1} & & & 0 \\ & \frac{\cos\theta_2}{V_2} & & \\ & & \ddots & \\ 0 & & & \frac{\cos\theta_N}{V_N} \end{bmatrix}$$

$$\underline{D} = \begin{bmatrix} \frac{\sin\theta_1}{V_1} & & & \\ & \frac{\sin\theta_2}{V_2} & & \\ & & \ddots & \\ & & & \frac{\sin\theta_N}{V_N} \end{bmatrix}$$

Then Eqts 3 and 4 may be written in vector form

$$\underline{I_p} = \underline{C}\,\underline{P} + \underline{D}\,\underline{Q} \qquad \text{Eqt 5}$$

$$\underline{I_q} = \underline{D}\,\underline{P} - \underline{C}\,\underline{Q} \qquad \text{Eqt 6}$$

where

$$\underline{P} = \begin{bmatrix} P_1 \\ \vdots \\ P_N \end{bmatrix} \qquad \underline{Q} = \begin{bmatrix} Q_1 \\ \vdots \\ Q_N \end{bmatrix}$$

are the bus injections into the system.

Now substitute Eqts 5 and 6 into Eqt 1 and 2

$$\Rightarrow P_L = \left[\underline{C}\,\underline{P} + \underline{D}\,\underline{Q}\right]^T \underline{R}^T \left[\underline{C}\,\underline{P} + \underline{D}\,\underline{Q}\right]$$

$$+ \left[\underline{D}\,\underline{P} - \underline{C}\,\underline{Q}\right]^T \underline{R}^T \left[\underline{D}\,\underline{P} - \underline{C}\,\underline{Q}\right]$$

Taking transposes,

$$P_L = [\mathbf{P}^T\mathbf{C}^T + \mathbf{Q}^T\mathbf{D}^T][\mathbf{R}^T\mathbf{C}\mathbf{P} + \mathbf{R}^T\mathbf{D}\mathbf{Q}]$$

$$+ [\mathbf{P}^T\mathbf{D}^T - \mathbf{Q}^T\mathbf{C}^T][\mathbf{R}^T\mathbf{D}\mathbf{P} - \mathbf{R}^T\mathbf{C}\mathbf{Q}]$$

$$= \mathbf{P}^T\mathbf{C}^T\mathbf{R}^T\mathbf{C}\mathbf{P} + \mathbf{P}^T\mathbf{C}^T\mathbf{R}^T\mathbf{D}\mathbf{Q} + \mathbf{Q}^T\mathbf{D}^T\mathbf{R}^T\mathbf{C}\mathbf{P} + \mathbf{Q}^T\mathbf{D}^T\mathbf{R}^T\mathbf{D}\mathbf{Q}$$

$$+ \mathbf{P}^T\mathbf{D}^T\mathbf{R}^T\mathbf{D}\mathbf{P} - \mathbf{P}^T\mathbf{D}^T\mathbf{R}^T\mathbf{C}\mathbf{Q} - \mathbf{Q}^T\mathbf{C}^T\mathbf{R}^T\mathbf{D}\mathbf{P} + \mathbf{Q}^T\mathbf{C}^T\mathbf{R}^T\mathbf{C}\mathbf{Q}$$

$$= [\mathbf{P}^T \;\; \mathbf{Q}^T]\begin{bmatrix} \mathbf{C}^T\mathbf{R}^T\mathbf{C}\mathbf{P} + \mathbf{C}^T\mathbf{R}^T\mathbf{D}\mathbf{Q} + \mathbf{D}^T\mathbf{R}^T\mathbf{D}\mathbf{P} - \mathbf{D}^T\mathbf{R}^T\mathbf{C}\mathbf{Q} \\ \mathbf{D}^T\mathbf{R}^T\mathbf{C}\mathbf{P} + \mathbf{D}^T\mathbf{R}^T\mathbf{D}\mathbf{Q} - \mathbf{C}^T\mathbf{R}^T\mathbf{D}\mathbf{P} + \mathbf{C}^T\mathbf{R}^T\mathbf{C}\mathbf{Q} \end{bmatrix} \left\| \begin{matrix} \mathbf{P} \\ \mathbf{Q} \end{matrix} \right\|$$

$$= [\mathbf{P}^T \;\; \mathbf{Q}^T]\begin{bmatrix} \mathbf{C}^T\mathbf{R}^T\mathbf{C} + \mathbf{D}^T\mathbf{R}^T\mathbf{D} & \mathbf{C}^T\mathbf{R}^T\mathbf{D} - \mathbf{D}^T\mathbf{R}^T\mathbf{C} \\ \mathbf{D}^T\mathbf{R}^T\mathbf{C} - \mathbf{C}^T\mathbf{R}^T\mathbf{D} & \mathbf{D}^T\mathbf{R}^T\mathbf{D} + \mathbf{C}^T\mathbf{R}^T\mathbf{C} \end{bmatrix}\begin{bmatrix} \mathbf{P} \\ \mathbf{Q} \end{bmatrix}$$

$$P_L = [\mathbf{P}^T \;\; \mathbf{Q}^T]\begin{bmatrix} \mathbf{A}_P & -\mathbf{B}_P \\ \mathbf{B}_P & \mathbf{A}_P \end{bmatrix}\begin{bmatrix} \mathbf{P} \\ \mathbf{Q} \end{bmatrix} \qquad \text{Eqt 7}$$

Where
$$\left.\begin{aligned} \mathbf{A}_P &= \mathbf{C}^T\mathbf{R}^T\mathbf{C} + \mathbf{D}^T\mathbf{R}^T\mathbf{D} \\ \mathbf{B}_P &= \mathbf{D}^T\mathbf{R}^T\mathbf{C} - \mathbf{C}^T\mathbf{R}^T\mathbf{D} \end{aligned}\right\} \text{Eqts 7-A}$$

$$\left.\begin{aligned} a_{ij_P} &= r_{ij}\cos(\theta_i - \theta_j) / V_1 V_2 \\ b_{ij_P} &= r_{ij}\sin(\theta_i - \theta_j) / V_1 V_2 \end{aligned}\right\} \text{Eqts 7-B}$$

We now require the following assumption:

ASSUMPT. #1 **All dispatchable generators have no load**

With this assumption, we may order the generator buses as $1, \ldots, ng$ and the load buses as the remainder. The assumption implies, for gen buses $P_i = P_{gi}$ and for load buses $P_i = -P_{di}$. Then

$$\underline{P} = \begin{bmatrix} P_{g_1} \\ \vdots \\ P_{g_{ng}} \\ -P_{d_{ng+1}} \\ \vdots \\ -P_{d_N} \end{bmatrix} = \begin{bmatrix} \underline{P_g} \\ -\underline{P_d} \end{bmatrix}$$

Similarly, we have

$$\underline{Q} = \begin{bmatrix} Q_{g_1} \\ \vdots \\ Q_{g_{ng}} \\ -Q_{d_{ng+1}} \\ \vdots \\ -Q_{d_N} \end{bmatrix} = \begin{bmatrix} \underline{Q_g} \\ -\underline{Q_d} \end{bmatrix}$$

We can partition $\underline{A_p}$ and $\underline{B_p}$ appropriately so that

$$\underline{A_p} = \begin{bmatrix} \underline{A_{pgg}} & \underline{A_{pgd}} \\ \underline{A_{pdg}} & \underline{A_{pdd}} \end{bmatrix}$$

$$\underline{B_p} = \begin{bmatrix} \underline{B_{pgg}} & \underline{B_{pgd}} \\ \underline{B_{pdg}} & \underline{B_{pdd}} \end{bmatrix}$$

Then Eqt 17 becomes

$$P_L = \left[\left(\underline{P}_g^T \;\vdots\; -\underline{P}_d^T \right) \;\vdots\; \left(\underline{Q}_g^T \;\vdots\; -\underline{Q}_d^T \right) \right] \begin{bmatrix} \underline{A}_{pgg} & \underline{A}_{pgd} & -\underline{B}_{pgg} & -\underline{B}_{pgd} \\ \underline{A}_{pdg} & \underline{A}_{pdd} & -\underline{B}_{pdg} & -\underline{B}_{pdd} \\ \underline{B}_{pgg} & \underline{B}_{pgd} & \underline{A}_{pgg} & \underline{A}_{pgd} \\ \underline{B}_{pdg} & \underline{B}_{pdd} & \underline{A}_{pdg} & \underline{A}_{pdd} \end{bmatrix} \begin{bmatrix} \underline{P}_g \\ -\underline{P}_d \\ \underline{Q}_g \\ -\underline{Q}_d \end{bmatrix}$$

$$= \left[\begin{matrix} \underline{P}_g^T \underline{A}_{pgg} - \underline{P}_d^T \underline{A}_{pdg} \\ + \underline{Q}_g^T \underline{B}_{pgg} - \underline{Q}_d^T \underline{B}_{pdg} \end{matrix} \;\middle|\; \begin{matrix} \underline{P}_g^T \underline{A}_{pgd} - \underline{P}_d^T \underline{A}_{pdd} \\ + \underline{Q}_g^T \underline{B}_{pgd} - \underline{Q}_d^T \underline{B}_{pdd} \end{matrix} \;\middle|\; \begin{matrix} - \underline{P}_g^T \underline{B}_{pgg} + \underline{P}_d^T \underline{B}_{pdg} \\ + \underline{Q}_g^T \underline{A}_{pgg} - \underline{Q}_d^T \underline{A}_{pdg} \end{matrix} \;\middle|\; \begin{matrix} - \underline{P}_g^T \underline{B}_{pgd} + \underline{P}_d^T \underline{B}_{pdd} \\ + \underline{Q}_g^T \underline{A}_{pgd} - \underline{Q}_d^T \underline{A}_{pdd} \end{matrix} \right] \begin{bmatrix} \underline{P}_g \\ -\underline{P}_d \\ \underline{Q}_g \\ -\underline{Q}_d \end{bmatrix}$$

$$= \underline{P}_g^T \underline{A}_{pgg} \underline{P}_g - \underline{P}_d^T \underline{A}_{pdg} \underline{P}_g + \underline{Q}_g^T \underline{B}_{pgg} \underline{P}_g - \underline{Q}_d^T \underline{B}_{pdg} \underline{P}_g$$

$$- \underline{P}_g^T \underline{A}_{pgd} \underline{P}_d + \underline{P}_d^T \underline{A}_{pdd} \underline{P}_d - \underline{Q}_g^T \underline{B}_{pgd} \underline{P}_d + \underline{Q}_d^T \underline{B}_{pdd} \underline{P}_d$$

$$- \underline{P}_g^T \underline{B}_{pgg} \underline{Q}_g + \underline{P}_d^T \underline{B}_{pdg} \underline{Q}_g + \underline{Q}_g^T \underline{A}_{pgg} \underline{Q}_g - \underline{Q}_d^T \underline{A}_{pdg} \underline{Q}_g$$

$$+ \underline{P}_g^T \underline{B}_{pgd} \underline{Q}_d - \underline{P}_d^T \underline{B}_{pdd} \underline{Q}_d - \underline{Q}_g^T \underline{A}_{pgd} \underline{Q}_d + \underline{Q}_d^T \underline{A}_{pdd} \underline{Q}_d$$

Now our goal is to separate the terms dependent on P_g, Q_g from the terms dependent on P_d, Q_d and also from the terms dependent on both.

$$\Rightarrow P_L = \underline{P}_g^T \underline{A}_{pgg} \underline{P}_g + \underline{Q}_g^T \underline{B}_{pgg} \underline{P}_g - \underline{P}_g^T \underline{B}_{pgg} \underline{Q}_g + \underline{Q}_g^T \underline{A}_{pgg} \underline{Q}_g \Big\} G_1$$

$$+ \underline{P}_d^T \underline{A}_{pdd} \underline{P}_d + \underline{Q}_d^T \underline{B}_{pdd} \underline{P}_d - \underline{P}_d^T \underline{B}_{pdd} \underline{Q}_d + \underline{Q}_d^T \underline{A}_{pdd} \underline{Q}_d \Big\} G_2$$

$$\left. \begin{array}{l} - \underline{P}_d^T \underline{A}_{pdg} \underline{P}_g - \underline{Q}_d^T \underline{B}_{pdg} \underline{P}_g - \underline{P}_g^T \underline{A}_{pgd} \underline{P}_d - \underline{Q}_g^T \underline{B}_{pgd} \underline{P}_d \\ + \underline{P}_d^T \underline{B}_{pdg} \underline{Q}_g - \underline{Q}_d^T \underline{A}_{pdg} \underline{Q}_g + \underline{P}_g^T \underline{B}_{pgd} \underline{Q}_d - \underline{Q}_g^T \underline{A}_{pgd} \underline{Q}_d \end{array} \right\} G_3$$

$$\Rightarrow P_L = G_1 + G_3 + G_2 \quad \text{Eqt 8} \quad \text{where}$$

$$G_1 = \begin{bmatrix} \underline{P}_g^T & \underline{Q}_g^T \end{bmatrix} \begin{bmatrix} \underline{A}_{pgg} & -\underline{B}_{pgg} \\ \underline{B}_{pgg} & \underline{A}_{pgg} \end{bmatrix} \begin{bmatrix} \underline{P}_g \\ \underline{Q}_g \end{bmatrix}$$

$$G_2 = \begin{bmatrix} \underline{P}_d^T & \underline{Q}_d^T \end{bmatrix} \begin{bmatrix} \underline{A}_{pdd} & -\underline{B}_{pdd} \\ \underline{B}_{pdd} & \underline{A}_{pdd} \end{bmatrix} \begin{bmatrix} \underline{P}_d \\ \underline{Q}_d \end{bmatrix}$$

To write G_3 compactly, note from eqts 7-B (pg LF-4) that

$$\underline{A}_p = \underline{A}_p^T \qquad (\underline{A}_p \text{ is symmetric})$$

$$\underline{B}_p = -\underline{B}_p^T \qquad (\underline{B}_p \text{ is symmetric with a sign change})$$

Therefore we see that several terms in $\underline{G}_3$ are the same as follows:

$$-\underline{P}_d^T \underline{A}_{pdg} \underline{P}_g = -\underline{P}_g^T \underline{A}_{pgd} \underline{P}_d$$

$$-\underline{Q}_d^T \underline{B}_{pdg} \underline{P}_g = \underline{P}_g^T \underline{B}_{pgd} \underline{Q}_d$$

$$-\underline{Q}_g^T \underline{B}_{pgd} \underline{P}_d = \underline{P}_d^T \underline{B}_{pdg} \underline{Q}_g$$

$$-\underline{Q}_d^T \underline{A}_{pdg} \underline{Q}_g = -\underline{Q}_g^T \underline{A}_{pgd} \underline{Q}_d$$

Therefore

$$G_3 = 2\left[-\underline{P}_d^T \underline{A}_{pdg} \underline{P}_g - \underline{Q}_d^T \underline{B}_{pdg} \underline{P}_g + \underline{P}_d^T \underline{B}_{pdg} \underline{Q}_g - \underline{Q}_d^T \underline{A}_{pdg} \underline{Q}_g\right]$$

$$= 2\left[-\underline{P}_d^T \underline{A}_{pdg} - \underline{Q}_d^T \underline{B}_{pdg} \;\vdots\; \underline{P}_d^T \underline{B}_{pdg} - \underline{Q}_d^T \underline{A}_{pdg}\right] \begin{bmatrix} \underline{P}_g \\ \underline{Q}_g \end{bmatrix}$$

$$= \left[2(\underline{Q}_d^T \underline{B}_{pgd} - \underline{P}_d^T \underline{A}_{pdg}) \;\vdots\; 2(\underline{P}_d^T \underline{B}_{pdg} - \underline{Q}_d^T \underline{A}_{pdg})\right] \begin{bmatrix} \underline{P}_g \\ \underline{Q}_g \end{bmatrix}$$

Define: $\underline{E}_{pp} = 2(\underline{B}_{pgd}^T \underline{Q}_d^T - \underline{A}_{pgd} \underline{P}_d)$

$$\underline{E}_{pq} = 2(\underline{B}_{pdg}^T \underline{P}_d - \underline{A}_{pdg}^T \underline{Q}_d)$$

and we have that

$$G_3 = \left[\underline{E}_{pp}^T \quad \underline{E}_{pq}^T\right] \begin{bmatrix} \underline{P}_G \\ \underline{Q}_g \end{bmatrix}$$

Eqt 8 can be written as

$$P_L = \begin{bmatrix} \underline{P}_g^T & \underline{Q}_g^T \end{bmatrix} \begin{bmatrix} \underline{A}_{pgg} & -\underline{B}_{pgg} \\ \underline{B}_{pgg} & \underline{A}_{pgg} \end{bmatrix} \begin{bmatrix} \underline{P}_g \\ \underline{Q}_g \end{bmatrix} \leftarrow G_1$$

Eqt 9

$$+ \begin{bmatrix} \underline{E}_{pp}^T & \underline{E}_{pq}^T \end{bmatrix} \begin{bmatrix} \underline{P}_g \\ \underline{Q}_g \end{bmatrix} \leftarrow G_3$$

$$+ \begin{bmatrix} \underline{P}_d^T & \underline{Q}_d^T \end{bmatrix} \begin{bmatrix} \underline{A}_{pdd} & -\underline{B}_{pdd} \\ \underline{B}_{pdd} & \underline{A}_{pdd} \end{bmatrix} \leftarrow G_2$$

And this is an *exact* expression for the real power losses, assuming Assumpt. #1 holds.

Now we still need to get rid of $\underline{Q}_g$

ASSUMPTION #2

> Reactive generation varies linearly with real generation.

$\Rightarrow \quad Q_{gi} = Q_{gio} + f_i P_{gi}$; f_i a constant for each gen

If $\quad \underline{Q}_{go} = \begin{bmatrix} Q_{g1o} \\ \vdots \\ Q_{gn_go} \end{bmatrix}$

$$\underline{F} = \begin{bmatrix} f_1 & & & 0 \\ & f_2 & & \\ & & \ddots & \\ 0 & & & f_{ng} \end{bmatrix}$$

These are the same because $\underline{F}$ is diagonal.

Then $\quad \underline{Q}_g = \underline{Q}_{go} + \underline{F}\,\underline{P}_g = \underline{Q}_{go} + \underline{P}_g^T \underline{F}$

Then $\underline{G}_1$ and $\underline{G}_2$ are modified as follows

$$G_1 = \left[\underline{P}_g^T \; \vdots \; \underline{Q}_{go}^T + \underline{P}_g^T \underline{F} \right] \begin{bmatrix} \underline{A}_{pgg} & -\underline{B}_{pgg} \\ \underline{B}_{pgg} & \underline{A}_{pgg} \end{bmatrix} \begin{bmatrix} \underline{P}_g \\ \underline{Q}_{go} + \underline{F}\,\underline{P}_g \end{bmatrix}$$

$$= \left(\underline{P}_g^T \underline{A}_{pgg} + \underline{Q}_{go}^T \underline{B}_{pgg} + \underline{P}_g^T \underline{F}\, \underline{B}_{pgg} \right) \underline{P}_g$$

$$+ \left(-\underline{P}_g^T \underline{B}_{pgg} + \underline{Q}_{go}^T \underline{A}_{pgg} + \underline{P}_g^T \underline{F}\, \underline{A}_{pgg} \right) \left(\underline{Q}_{go} + \underline{F}\, \underline{P}_g \right)$$

$$\Rightarrow G_1 = \underline{P}_g^T \underline{A}_{pgg} \underline{P}_g + \underline{Q}_{go}^T \underline{B}_{pgg} \underline{P}_g + \underline{P}_g^T \underline{F} \underline{B}_{pgg} \underline{P}_g$$

$$- \underline{P}_g^T \underline{B}_{pgg} \underline{Q}_{go} + \underline{Q}_{go}^T \underline{A}_{pgg} \underline{Q}_{go} + \underline{P}_g^T \underline{F} \underline{A}_{pgg} \underline{Q}_{go}$$

$$- \underline{P}_g^T \underline{B}_{pgg} \underline{F} \underline{P}_g + \underline{Q}_{go}^T \underline{A}_{pgg} \underline{F} \underline{P}_g + \underline{P}_g^T \underline{F} \underline{A}_{pgg} \underline{F} \underline{P}_g$$

$$= \underline{P}_g^T \left[\underline{A}_{pgg} + \underline{F} \underline{B}_{pgg} - \underline{B}_{pgg} \underline{F} + \underline{F} \underline{A}_{pgg} \underline{F} \right] \underline{P}_g$$ term 1

note: I disagree with text here

$$+ \left[\underline{Q}_{go}^T \underline{B}_{pgg} + \underline{Q}_{go}^T \underline{A}_{pgg} \underline{F} \right] \underline{P}_g$$ term 2

$$+ \underline{P}_g^T \left[-\underline{B}_{pgg} \underline{Q}_{go} + \underline{F} \underline{A}_{pgg} \underline{Q}_{go} \right]$$ term 3

Note in term 1 that

$$\underline{F} \underline{B}_{pgg} = \underline{B}_{pgg} \underline{F}$$

and in term 3 that

$$\underline{P}_g^T \left[-\underline{B}_{pgg} \underline{Q}_{go} + \underline{F} \underline{A}_{pgg} \underline{Q}_{go} \right] = \underline{P}_g^T \left[\underline{F} \underline{A}_{pgg} - \underline{B}_{pgg} \right] \underline{Q}_{go}$$

$$= \underline{Q}_{go}^T \left[\underline{A}_{pgg} \underline{F} + \underline{B}_{pgg} \right] \underline{P}_g$$

$$\therefore G_1 = \underline{P}_g^T \left[\underline{A}_{pgg} + \underline{F} \underline{A}_{pgg} \underline{F} \right] \underline{P}_g + 2 \underline{Q}_{go}^T \left[\underline{A}_{pgg} \underline{F} + \underline{B}_{pgg} \right] \underline{P}_g$$

Also: $G_3 = \underline{E}_{pp}^T \underline{P}_g + \underline{E}_{pq}^T \underline{Q}_g$ (from eq 9)

$$= \underline{E}_{pp}^T \underline{P}_g + \underline{E}_{pq}^T (\underline{Q}_{go} + \underline{F} \underline{P}_g)$$

$$= (\underline{E}_{pp}^T + \underline{E}_{pq}^T \underline{F}) \underline{P}_g + \underline{E}_{pq}^T \underline{Q}_{go}$$

we therefore can write Eqt. 9 as

$$\underline{P_L} = \underline{G_1} + \underline{G_3} + \underline{G_2}$$

$$= \underline{P_g}^T \left[\underline{A_{pgg}} + \underline{F}\,\underline{A_{pgg}}\,\underline{F} \right] \underline{P_g} + 2\,\underline{Q_{go}}^T \left[\underline{A_{pgg}}\,\underline{E} + \underline{B_{pgg}} \right] \underline{P_g}$$

$$+ \left(\underline{E_{pp}}^T + \underline{E_{pq}}^T \underline{F} \right) \underline{P_g} + \underline{E_{pq}}^T \underline{Q_{go}}$$

$$+ \left[\underline{P_d}^T \;\; \underline{Q_d}^T \right] \begin{bmatrix} \underline{A_{pdd}} & -\underline{B_{pdd}} \\ \underline{B_{pdd}} & \underline{A_{pdd}} \end{bmatrix}$$

note: El-Haway's text includes a term $2\underline{E}\,\underline{B_{pgg}}$ here. I'm not sure who is right.

$$\Rightarrow P_L = \underline{P_g}^T \left[\underline{A_{pgg}} + \underline{F}\,\underline{A_{pgg}}\,\underline{F} \right] \underline{P_g}$$

$$+ \left\{ 2\,\underline{Q_{go}}^T \left[\underline{A_{pgg}}\,\underline{F} + \underline{B_{pgg}} \right] + \underline{E_{pp}}^T + \underline{E_{pq}}^T \underline{F} \right\} \underline{P_g}$$

Eqt 10

$$+ \left[P_d^T \;\; Q_d^T \right] \begin{bmatrix} \underline{A_{pdd}} & -\underline{B_{pdd}} \\ \underline{B_{pdd}} & \underline{A_{pdd}} \end{bmatrix} + \underline{E_{pq}}^T \underline{Q_{go}}$$

which can be written as

$$P_L = \underline{P_g}^T \underline{B}\,\underline{P_g} + \underline{B_o}^T \underline{P_G} + B_{oo} \qquad \text{Eqt 11}$$

where

$$\underline{B} = \underline{A_{pgg}} + \underline{F}\,\underline{A_{pgg}}\,\underline{F}$$

$$\underline{B_o} = 2\,\underline{Q_{go}}^T \left[\underline{A_{pgg}}\,\underline{F} + \underline{B_{pgg}} \right] + \underline{E_{pp}}^T + \underline{E_{pq}}^T F$$

$$B_{oo} = \left[P_d^T \; Q_d^T \right] \begin{pmatrix} \underline{A_{pdd}} & -\underline{B_{pdd}} \\ \underline{B_{pdd}} & \underline{A_{pdd}} \end{pmatrix} + \underline{E_{pq}}^T \underline{Q_{go}}$$