

# EDC1

## 1.0 EDC Problem Formulation

Each plant  $i$  has a cost-rate curve that gives the cost  $C_i$  in \$/hour as a function of its generation level  $P_{Gi}$  (the 3 phase power). So we denote the cost-rate functions as  $C_i(P_{Gi})$ . These functions are often assumed to be quadratic (and therefore convex). For example, two such functions are given as

$$C_1(P_{G1}) = 900 + 45P_{G1} + 0.01P_{G1}^2 \quad (1)$$

$$C_2(P_{G2}) = 2500 + 43P_{G2} + 0.003P_{G2}^2 \quad (2)$$

If we have  $m$  generating units, then the total system cost will be given by

$$C_T = \sum_{i=1}^m C_i(P_{Gi}) \quad (3)$$

Equation (3), represents our objective function, and we desire to minimize it. The generation values  $P_{Gi}$  are the decision variables.

There are two basic kinds of constraints for our problem.

1. Power balance
2. Generation limits

### 1.1 Power balance constraint

In regards to power balance, it must be the case that the total generation equals the total demand  $P_D$  plus the total losses  $P_L$ .

$$\sum_{i=1}^m P_{Gi} = P_D + P_L \quad (4a)$$

The demand  $P_D$  is assumed to be a fixed value. However, the losses  $P_L$  depend on the solution (given by the  $P_{Gi}$ ) which we do not know until we solve the problem. This dependency is due to the fact that the losses depend on the flows in the circuits, and the flows in the circuits depend on the generation dispatch. Therefore we represent this dependency according to eq. (4b).

$$\sum_{i=1}^m P_{Gi} = P_D + P_L(P_{G1}, P_{G2}, \dots, P_{Gm}) \quad (4b)$$

Note that only  $m-1$  of the  $P_{Gi}$  are independent variables. Given demand and losses, one of the generation values is determined once the other  $m-1$  of them are set. We will assume this generator is unit 1.

Therefore we need to remove  $P_{G1}$  from the arguments of  $P_L$  so that eq. (4b) becomes

$$\sum_{i=1}^m P_{Gi} = P_D + P_L(P_{G2}, \dots, P_{Gm}) \quad (4c)$$

We rearrange eq. (4c) so that all terms dependent of the decision variables are on the left-hand-side, according to:

$$\sum_{i=1}^m P_{Gi} - P_L(P_{G2}, \dots, P_{Gm}) = P_D \quad (4d)$$

## 1.2 Generation limits

There are physical constraints on the generation levels. The generators cannot exceed their maximum capabilities, represented by  $P_{Gi}^{\max}$ . And clearly, they cannot operate below 0 (otherwise they are

operating as a motor, attempting to drive the turbine – not a good operational state!). Most units actually cannot operate at 0; as a result, we will denote the minimum as  $P_{Gi}^{\min}$ . Therefore, the generation limits are represented by

$$P_{Gi}^{\min} \leq P_{Gi} \leq P_{Gi}^{\max} \quad (5)$$

### 1.3 Problem statement

This leads us to the statement of the problem, i.e., the articulation of the mathematical program, which is, from eqs. (3), (4d), and (5), as follows.

$$\text{Min } C_T = \sum_{i=1}^m C_i(P_{Gi})$$

Subject to

$$\sum_{i=1}^m P_{Gi} - P_L(P_{G2}, \dots, P_{Gm}) = P_D$$

$$P_{Gi}^{\min} \leq P_{Gi} \leq P_{Gi}^{\max} \quad \forall \quad i = 1, \dots, m$$

$$P_{Gi} \geq 0 \quad \forall \quad i = 1, \dots, m$$

## 2.0 Application of KKT conditions

In formulating the KKT conditions, we recall that the complementary condition is handled procedurally as below.

- Solve the problem without any inequality constraint
- Check solution against inequality constraints. For those that are violated, bring them in as equality constraints and re-solve the problem. Repeat this step until you obtain a solution for which no inequality constraints are violated.

This is an iterative solution procedure, and represents a procedural equivalent to the complementary condition. Thus, for any given iteration, we can assume there are no inequality constraints.

Therefore we may state the required conditions for the solution more simply as

$$\frac{\partial F}{\partial x_i} = 0 \quad \forall i = 1, n \quad (6)$$

$$\frac{\partial F}{\partial \lambda_j} = 0 \quad \forall j = 1, J \quad (7)$$

where it is assumed that any binding inequality constraints are included in eq. (7) as equality constraints.

Let's apply these conditions to the problem statement of Section 1.3 above, assuming that no inequality constraints are binding so that there is only one equality constraint to consider (the power balance constraint).

First, we form the Lagrangian function:

$$F = \sum_{i=1}^m C_i(P_{Gi}) - \lambda \left[ \sum_{i=1}^m P_{Gi} - P_L(P_{G2}, \dots, P_{Gm}) - P_D \right] \quad (8)$$

Now applying the KKT conditions of (6) and (7), we get:

$$\frac{\partial F}{\partial P_{Gi}} = \frac{\partial C_i(P_{Gi})}{\partial P_{Gi}} - \lambda \left[ 1 - \frac{\partial P_L(P_{G2}, \dots, P_{Gm})}{\partial P_{Gi}} \right] = 0, \quad \forall i = 1, \dots, m \quad (9)$$

$$\frac{\partial F}{\partial \lambda} = \sum_{i=1}^m P_{Gi} - P_L(P_{G2}, \dots, P_{Gm}) - P_D = 0 \quad (10)$$

Observe that we have  $m$  equations of the form given in (9). However, the one

corresponding to  $i=1$  will not have a loss term and therefore will be:

$$\frac{\partial F}{\partial P_{G1}} = \frac{\partial C_1(P_{G1})}{\partial P_{G1}} - \lambda = 0 \quad (11)$$

Because  $\frac{\partial P_L}{\partial P_1} = 0$ , eq. (9) appropriately captures eq. (11).

The term  $\frac{\partial C_i}{\partial P_{Gi}}$  is the incremental cost of unit  $i$  and is denoted by  $IC_i$ .

Let's consider eq. (9) more closely. In particular, let's solve it for  $\lambda$ .

$$\lambda = \frac{1}{\left[1 - \frac{\partial P_L(P_{G2}, \dots, P_{Gm})}{\partial P_{Gi}}\right]} \frac{\partial C_i(P_{Gi})}{\partial P_{Gi}} \quad \forall i = 1, \dots, m \quad (12)$$

Define the fraction out front as  $L_i$ , that is

$$L_i = \frac{1}{\left[1 - \frac{\partial P_L(P_{G2}, \dots, P_{Gm})}{\partial P_{Gi}}\right]} \quad (13)$$

We call  $L_i$  the *penalty factor* for the  $i^{th}$  unit. Note that  $L_1 = 1$ .

Substituting eq. (13) into (12) results in

$$\lambda = L_i \frac{\partial C_i(P_{Gi})}{\partial P_{Gi}} \quad \forall i = 1, \dots, m \quad (14)$$

What eq. (14) says is that, at the optimum dispatch, for each unit not at a binding inequality constraint, the product of the penalty factor and the incremental cost of unit is the same and is equal to  $\lambda$ .

**Example:**

Consider the cost-rate functions:

$$C_1(P_{G1}) = 900 + 45P_{G1} + 0.01P_{G1}^2 \quad (1)$$

$$C_2(P_{G2}) = 2500 + 43P_{G2} + 0.003P_{G2}^2 \quad (2)$$

The load is specified as  $P_D = 600$  MW.

Generator limits are given as

$$50MW \leq P_{G1} \leq 200MW$$

$$50MW \leq P_{G2} \leq 600MW$$

In this example, we assume that there are no losses. This means that all penalty factors are 1.0. Assuming there are no binding inequality constraints, eq. (14) is

$$\lambda = \frac{\partial C_1(P_{G1})}{\partial P_{G1}} \quad \lambda = \frac{\partial C_2(P_{G2})}{\partial P_{G2}}$$



Writing out these equations, we have:

$$\lambda = 45 + 0.02P_{G1}$$

$$\lambda = 43 + 0.006P_{G2}$$

We also have our equality constraint eq. (10)

$$P_{G1} + P_{G2} = 600$$

We can solve these equations in matrix representation, as a set of linear equations, as given below.

$$\begin{bmatrix} 0.02 & 0 & -1 \\ 0 & 0.006 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} P_{G1} \\ P_{G2} \\ \lambda \end{bmatrix} = \begin{bmatrix} -45 \\ -43 \\ 600 \end{bmatrix}$$

Solution to this equation yields:

$$\begin{bmatrix} P_{G1} \\ P_{G2} \\ \lambda \end{bmatrix} = \begin{bmatrix} 61.64 \\ 538.46 \\ 46.23 \end{bmatrix}$$

Checking the inequality limits, we see that we have found the solution.

Let's explore another solution method. The previous one is fine, but it requires that all equations be linear. This may not always be the case, e.g., when we include losses, the power balance equation can be nonlinear.

The method is known as Lambda-iteration and is best understood via Fig. 1 which shows incremental cost curves  $IC_1$ ,  $IC_2$ , given by

$$IC_1 = 45 + 0.02P_{G1}$$

$$IC_2 = 43 + 0.006P_{G2}$$

The incremental cost curves are just the derivatives of the cost-rate curves. Observe that the expressions derived for  $\lambda$  under the KKT conditions specify a certain relation among the incremental cost curves. An implication here is that the incremental cost curves express the derivatives (or the incremental costs) under any condition, not necessarily at the optimum.

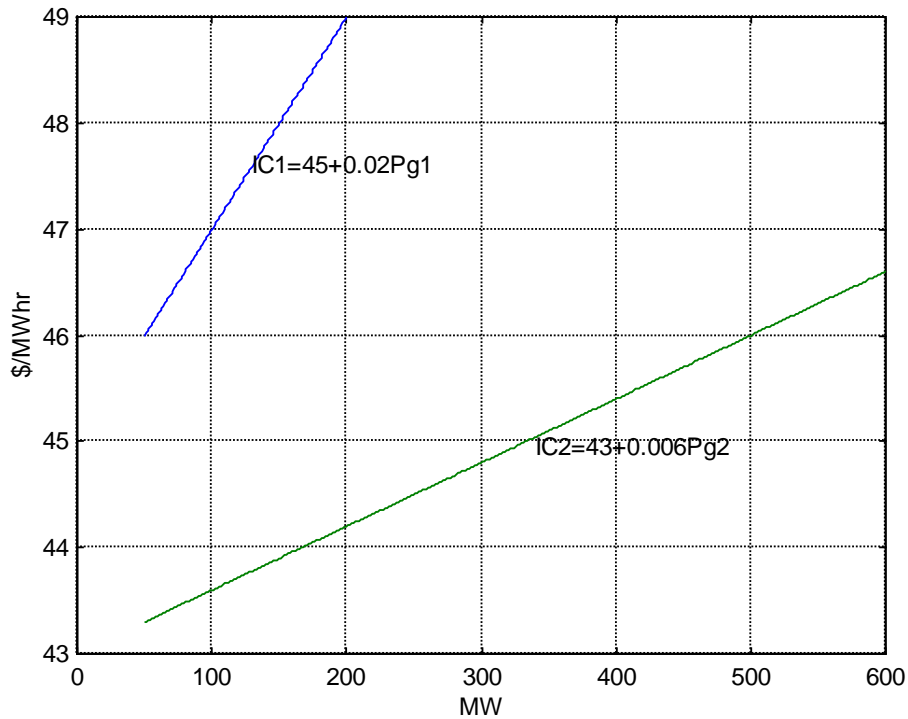


Fig. 1

The lambda iteration method begins with a guess in regards to a value of  $\lambda$  which satisfies the KKT conditions (such that all incremental costs are equal), and the total demand equals the load.

The lambda iteration may be performed graphically. Let's guess that  $\lambda=46$ . To determine what the corresponding generation levels are at the optimum, draw a

horizontal line across our IC curves, as shown by the dark horizontal line in Fig. 2.

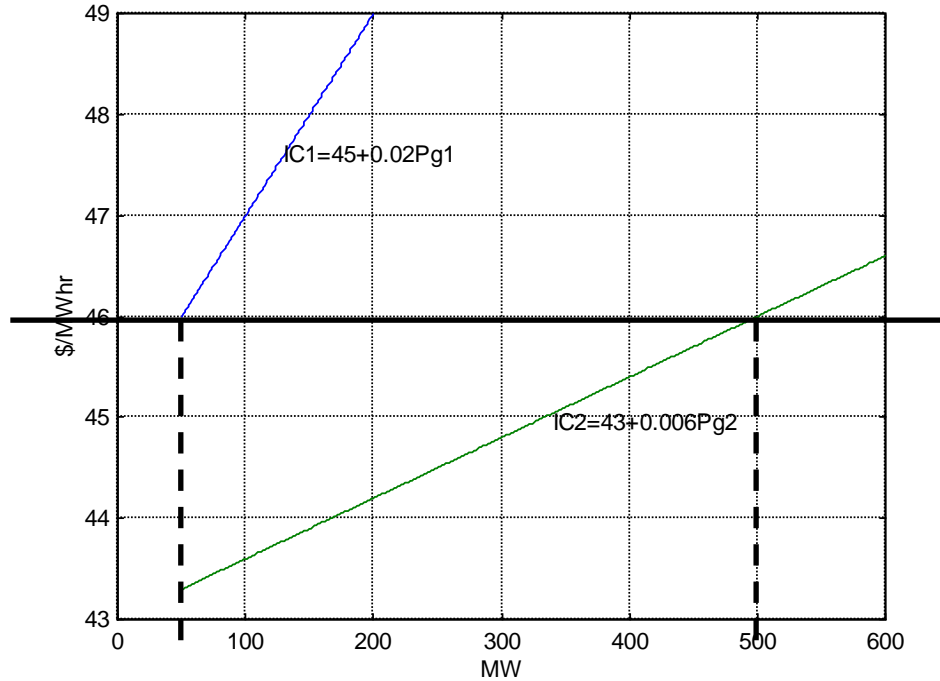


Fig. 2

The corresponding generation values are the dark vertical dashed lines, so we can see that  $P_{G1}=50$  and  $P_{G2}=500$ , for a total generation of 550 MW. This is less than the desired 600 MW so let's increase our guess. Let's try about 46.4, as shown in Fig. 3.

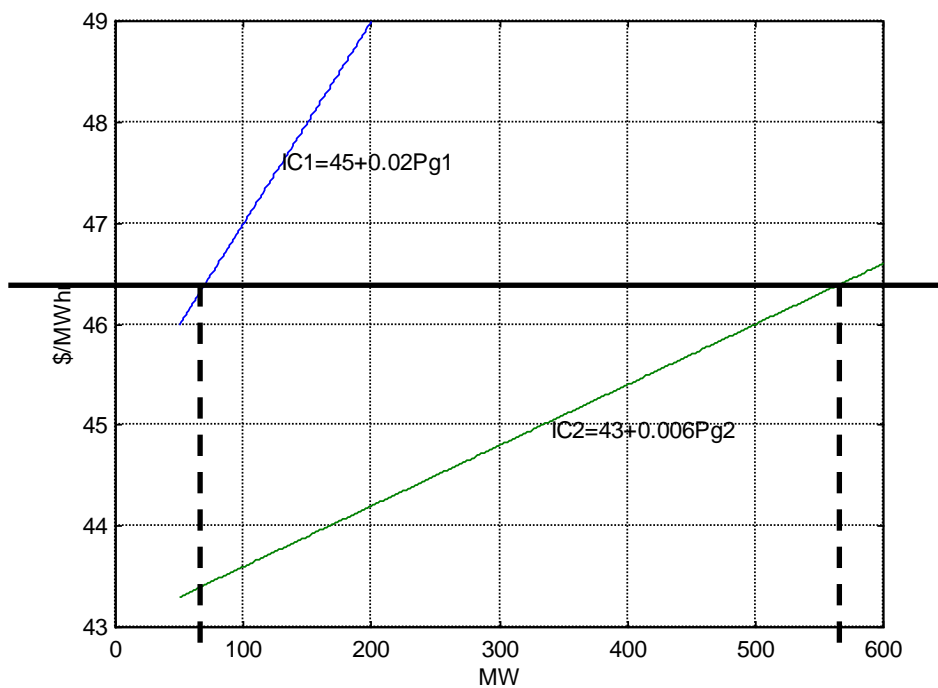


Fig. 3

The corresponding generation levels are about  $P_{G1}=65$  MW and  $P_{G2}=565$  MW, for a total of 630 MW, and so this is a little too high. Let's try  $\lambda=46.2$  as shown in Fig. 4.

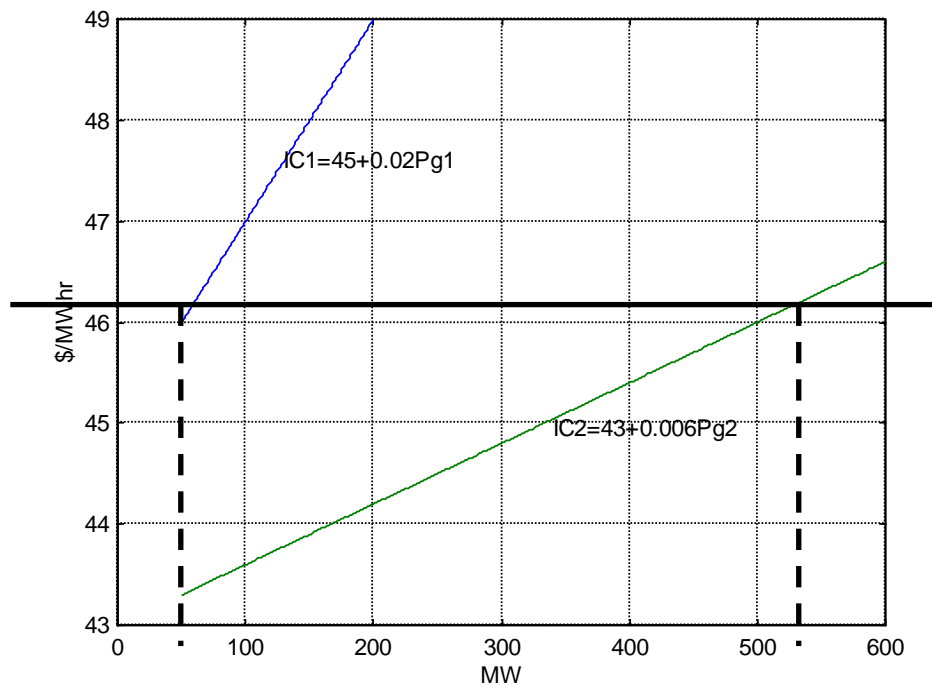


Fig. 4

The corresponding generation levels are about  $P_{G1}=55$  MW and  $P_{G2}=540$  MW for a total of 595 MW, so this is just a small bit too low. It is probably not possible to do better than this unless we use a more granular axis in our plots.

This method can be stated analytically as well. Notice what we are doing: we choose  $\lambda$  and then obtain the generation levels from the plots.

The plots are really analytical relations between  $\lambda$  and the generation levels, and we can easily manipulate them so that they give the generation levels as a function of  $\lambda$ , as shown below.

$$0.02P_{G1} = \lambda - 45 \Rightarrow P_{G1} = 50\lambda - 2250$$

$$0.006P_{G2} = \lambda - 43 \Rightarrow P_{G2} = 166.67\lambda - 7166.7$$

Now we can proceed analytically.

As before, guess 46 and calculate:

$$P_{G1} = 50(46) - 2250 = 50$$

$$P_{G2} = 166.67(46) - 7166.7 = 500$$

Total is 550 MW which is too low so let's try 46.4 (we could try anything we like, as long as it is higher, since the generation is too low in our first guess):

$$P_{G1} = 50(46.4) - 2250 = 70$$

$$P_{G2} = 166.67(46.4) - 7166.7 = 566.78$$

Total is 636.78, so now we need to try a lower  $\lambda$ . But let's use linear interpolation to guide our next value of  $\lambda$ :

$$\frac{46.4 - 46}{636.78 - 550} = \frac{\lambda - 46}{600 - 550} \Rightarrow \lambda = 46.2305$$

Because our equations for  $P_{G1}$  and  $P_{G2}$  are linear with  $\lambda$ , the linear interpolation will provide an exact answer. We can check to see:

$$P_{G1} = 50(46.2305) - 2250 = 61.525$$

$$P_{G2} = 166.67(46.2305) - 7166.7 = 538.5374$$

And the sum is 600.06 MW, as desired.

This method will also work when the IC curves and/or the power balance equation are nonlinear. For nonlinear relations, however, linear interpolation will not find the solution in one shot, and so it is necessary to iterate. On each iteration, one may employ a stopping criterion by checking to see whether the total generation is within some tolerance of the load.



## Example (extended):

Now let's reconsider our example, but with a load of 700 MW instead of 600 MW. Using our graphical method again, and with the knowledge gained from our previous example, we know that  $\lambda$  will exceed 46.4. But it cannot go too much higher without causing Unit 2 to exceed its upper limit. Let's try it at the value of  $\lambda$  that causes unit 2 to be at its upper limit. This is shown in Fig. 5.

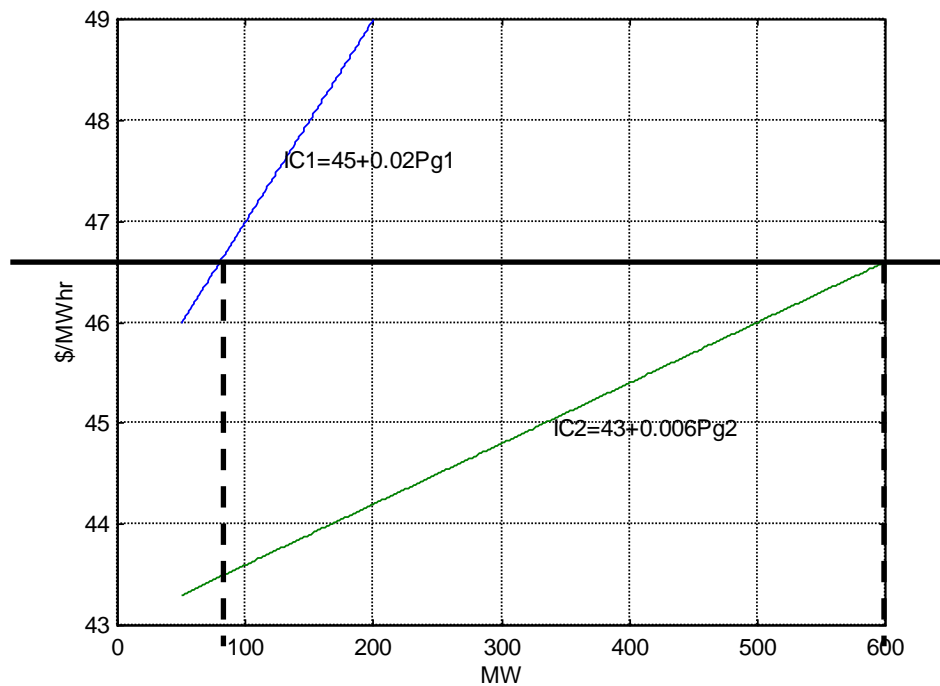


Fig. 5

It appears that  $\lambda$  is about 46.6, and  $P_{G1} \approx 85$  MW,  $P_{G2} \approx 600$  MW, for a total of 685 MW. So this is not enough, and we must therefore raise  $\lambda$ . However, we cannot raise  $\lambda$  on unit 2 because it is already at its upper limit. So we have to clamp  $P_{G2}$  at 600 MW. In other words, we will no longer use  $P_{G2}$  in our  $\lambda$ -iteration, although we will need to account for its generation of 600 MW.

So we will now perform  $\lambda$ -iteration on only the remaining units. In this case, the “remaining units” is just unit 1. In addition, our stopping criteria will now be that the total generation of the remaining units be equal to  $P_D - P_{g2} = 700 - 600 = 100$ .

The upshot of this is that we need to perform  $\lambda$ -iteration on unit 1’s ability to supply 100 MW. The horizontal solid-dark line of Fig. 6 illustrates.

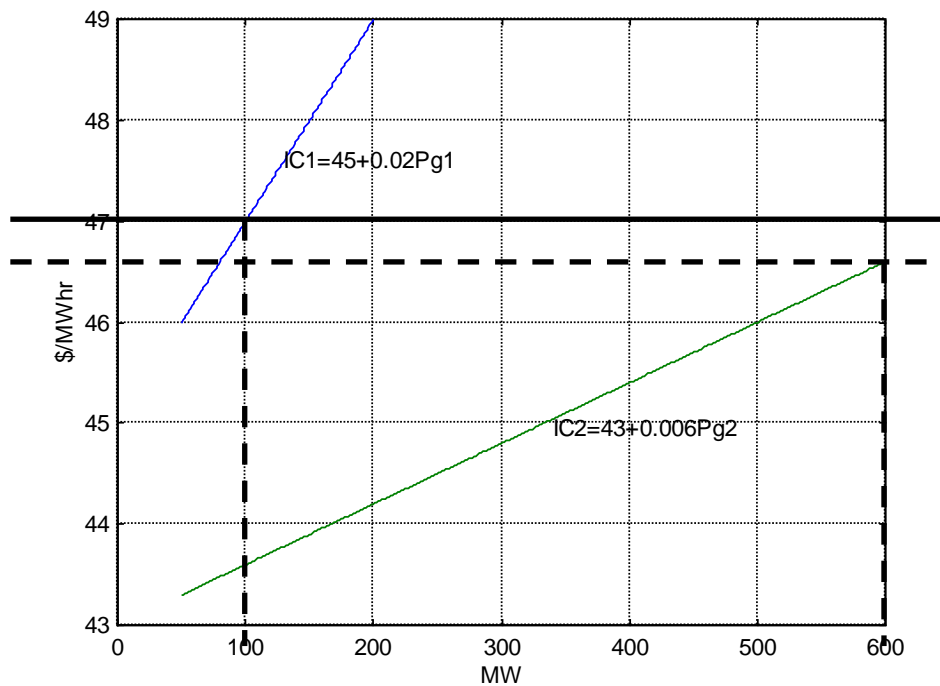


Fig. 5

Observe, however that there are now two horizontal lines,

- the solid one for unit 1 at 47;
- the dashed one for unit 2 at about 46.6

So which one is  $\lambda$ ?

➔  $\lambda$  is the SYSTEM incremental cost and indicates the cost of *optimally* supplying another MW from the system for the next hour. If the system has to supply another MW for the next hour, in this case (because there is only 1 regulating unit), it would have no choice but to do it with unit 1.

Therefore  $\lambda=47$ .

Then what is 46.6? It is the incremental cost of unit 2 (but not the system incremental cost). The unit incremental cost is normally understood as the cost for the unit to supply another MW for one hour. It can equivalently be understood as the savings if the unit was off-loaded by 1 MW for one hour, and in this case, that is a better interpretation since the unit cannot supply more power.

Let's look at this example in terms of a real-time market. In this case, the generator offer curves might appear as in Fig. 6.

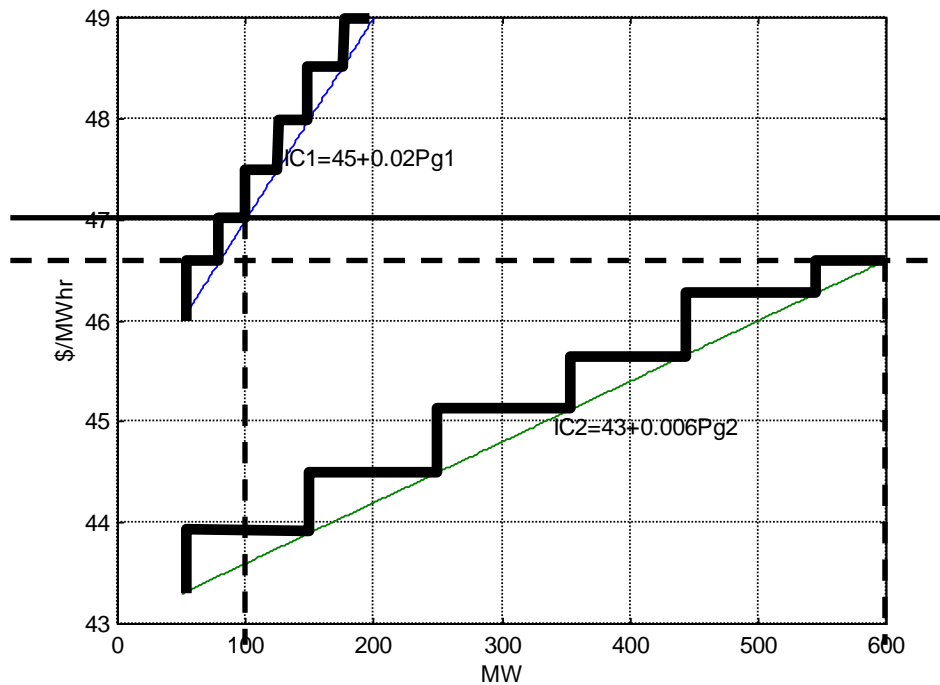


Fig. 6

The stacked offers appear as in Table 1.

Offer/bid order	Offers to sell 1 MWhr		Price (\$/MWhr)
	Seller	Quantity (MW)	
1	S2	100	43.9
2	S2	100	44.5
3	S2	100	45.1
4	S2	100	45.6
5	S2	100	46.3
6	S2	100	46.6
7	S1	25	46.6
8	S1	25	47.0
9	S1	25	47.5
10	S1	25	48.0
11	S1	25	48.5
12	S1	25	49.0

Observe that with this representation, the 700 MW solution would not change from that which we obtained from the linear incremental cost curves.

However, the 600 MW solution might differ, depending on “market rules” which would need to decide whether to award the last 25 MW to S1 or to S2. If it awarded it to S1, then the solution would be  $P_{g1}=75\text{MW}$ ,  $P_{g2}=515\text{ MW}$ . This is about what we got in the solution with linear incremental costs ( $P_{g1}=61.5\text{MW}$ ,  $P_{g2}=538.5\text{MW}$ ), the

difference caused by the approximation made by the step functions associated with the market offers. This approximation can be reduced by taking smaller “steps” in the incremental cost curve.

It is necessary to represent the offers in this stepwise approach in order to maintain linearity in the objective function and consequently enable solution by linear programming. If one utilized the linear incremental cost functions, then the objective functions would have quadratics, as we have seen in this example.