Solving Linear Programs #3

1.0 Tableau method

The tableau (French for "picture") method is a tabular method of solving linear programs by hand. For our purposes, it is just a good way of learning and remembering the steps of the simplex method. I call it a labeled matrix approach.

Recall the LP problem we have been working on, as given below.

$$F - 3x_1 - 5x_2 = 0 \quad (1)$$

$$x_1 + x_3 = 4 \quad (2)$$

$$2x_2 + x_4 = 12 \quad (3)$$

$$3x_1 + 2x_2 + x_5 = 18 \quad (4)$$

We will write this into a tableau as follows:

Tableau 1a

Basic variable	Eq. #	Coefficients of						Right side
		F	x_1	x_2	x_3	x_4	x_5	
F	0	1	-3	-5	0	0	0	0
x_3	1	0	1	0	1	0	0	4
x_4	2	0	0	2	0	1	0	12
x_5	3	0	3	2	0	0	1	18

We can see that the above solution is not yet optimal because there are still coefficients in the first row, (the row corresponding to the objective function), that are negative.

Our **first step in iteration** is to determine the entering variable. Remember,

> Select the variable that improves the objective
> at the highest rate (i.e., the largest amount
> of objective per unit change in variable).

This variable is the one in the first row that is most negative. This would be x_2, with the coefficient of -5.

We will call the column below this coefficient, and below the entering variable, the **pivot column**. We have drawn a box around the corresponding column in the tableau below.

Tableau 1b

Basic variable	Eq. #	Coefficients of						Right side
		F	x_1	x_2	x_3	x_4	x_5	
F	0	1	-3	-5	0	0	0	0
x_3	1	0	1	0	1	0	0	4
x_4	2	0	0	2	0	1	0	12
x_5	3	0	3	2	0	0	1	18

Our **second step in iteration** is to determine the leaving variable. Remember:

> Choose the leaving variable to be the one that
> hits 0 first as the entering variable is increased,
> as dictated by one of the m constraint equations.

To understand procedurally what this means, recall our Table 3 in the notes “LPSimple2,” which is repeated below for convenience:

Table 3: Determination of leaving variable for first step of example, when x_2 is the entering variable

Basic variable	Equation	Upper bound for x_2
x_3	$x_1+x_3=4$	No limit imposed
x_4	$2x_2+x_4=12$	x_2=(12-0)/2=6
x_5	$3x_1+2x_2+x_5=18$	x_2=(18-3(0)-0)/2=9

Inspection of Table 3 will convince yourself that we found the leaving variable in the following way:

1. Identify each equation that contains the entering variable (x_2) and therefore imposes a constraint on how much it can be increased. In Table 3, this is the last two equations (the ones for x_4 and x_5).
2. For each identified equation, we solved for the entering variable (x_2). Notice in Table 3 that in both cases, this turned out to be

$$x_2 = \frac{\text{Right Hand Side - 0 - 0 ....}}{\text{Coefficient of } x_2}$$

   The numerator subtracts zero(s) because, except for the entering variable and the right-hand-side, all other terms in each equation are zero! This is because each equation has only one basic (non-zero) term in it, and we are pushing this term to zero in order to see how much we can increase the entering variable (x_2).
3. The leaving variable is the one that hits zero for the least value of the entering variable.

These three steps, relative to our Tableau 1b, are:

1. Identify each equation that contains the entering variable (x_2) and therefore imposes a constraint on how much it can be increased. In a Tableau, this will be the rows that have non-zero values for the entering variable, i.e., the rows that have non-zero values in the pivot column. In Tableau 1b, this includes the last two rows.
2. For each identified row in the Tableau, solve for the entering variable (x_2) by dividing the right-hand-side by the coefficient of the entering variable, i.e.,

$$x_2 = \frac{\text{Right Hand Side}}{\text{Coefficient of } x_2}$$

3. The leaving variable is identified by the equation having minimum ratio given in step 2 as the previously basic (nonzero) variable of this equation.

Tableau 2a illustrates, with the calculation corresponding to the chosen variable circled.

Tableau 2a

Basic variable	Eq. #	Coefficients of						Right side	
		F	x_1	x_2	x_3	x_4	x_5		
F	0	1	-3	-5	0	0	0	0	
x_3	1	0	1	0	1	0	0	4	
x_4	2	0	0	2	0	1	0	12	$\frac{12}{2} = 6$
x_5	3	0	3	2	0	0	1	18	$\frac{18}{2} = 9$

To indicate the leaving variable, we place a box around its row, as shown in Tableau 2b.

Tableau 2b

Basic variable	Eq. #	Coefficients of						Right side
		F	x_1	x_2	x_3	x_4	x_5	
F	0	1	-3	-5	0	0	0	0
x_3	1	0	1	0	1	0	0	4
x_4	2	0	0	2	0	1	0	12
x_5	3	0	3	2	0	0	1	18

The pivot element is the intersection of the two boxes.

Our **third step in iteration** is to reconstruct the equations so that the entering variable becomes basic and the leaving variable becomes nonbasic. To do this, we re-write Tableau 2b so that

- x_2 replaces x_4 in the left-hand-column of basic variables, and
- the pivot row is divided by the pivot element

This is shown in Tableau 3a below.

Tableau 3a

Basic variable	Eq. #	Coefficients of						Right side	
		F	x_1	x_2	x_3	x_4	x_5		
F	0	1	-3	-5	0	0	0	0	
x_3	1	0	1	0	1	0	0	4	
x_2	2	0	0	1	0	0.5	0	6	← Divided by 2
x_5	3	0	3	2	0	0	1	18	

Now in order to eliminate x_2 from all other equations (including the objective function), we add an appropriate multiple of it to each row. The result is shown in Tableau 3b.

Tableau 3b

Basic variable	Eq. #	Coefficients of						Right side	
		F	x_1	x_2	x_3	x_4	x_5		
F	0	1	-3	0	0	2.5	0	30	Add 5 × pivot row
x_3	1	0	1	0	1	0	0	4	
x_2	2	0	0	1	0	0.5	0	6	
x_5	3	0	3	0	0	-1	1	6	Add -2 × pivot row

Because each basic variable always equals its right-hand-side, we can immediately read off the solution as $(x_1,x_2,x_3,x_4,x_5)=(0,6,4,0,6)$, with $F=30$.

And so now we test for optimality. Here, we want to see if the objective function can improve any more. The test for this is to see whether there are any variables in the objective function having positive coefficients. In the Tableau, because we have expressed all variables on the left-hand side (with *F*), we look to see whether there are any variables in the objective function row having negative coefficients. In this case, there is one (for x_1) and so this solution is not optimal. We must do another iteration.

2.0 Exceptions

We have established rules for making certain decisions in the simplex method. What happens if these rules do not lead to a clear-cut decision? Let's consider several situations.

2.1 Tie for the entering variable

Recall that we select the entering variable as the one with the largest positive coefficient in the objective function. But what happens if there are two variables with the same coefficient?

Recall that in our example, the objective function was

$$F = 3x_1 + 5x_2$$

and we choose x_2 as the entering variable on the first iteration.

But what if, in our example, our objective function would have been

$$F = 3x_1 + 3x_2$$

In this case, the rule is to choose one of them arbitrarily as the entering variable.

This means you either move to one corner point or another. Either way, the simplex will arrive at the optimal answer eventually. Choosing one over the other may get you there faster (with fewer iterations), but there is, in general, no way to know at this point.

2.2 Tie for the leaving variable

Recall that we selected the leaving variable be the one that hits 0 first as the entering variable is increased, as dictated by one of the *m* constraint equations. But what happens if we have two variables hitting zero for the same value of the entering variable?

Recall in our example that we used Table 3 to make this choice.

Table 3: Determination of leaving variable for first step of example, when x_2 is the entering variable

Basic variable	Equation	Upper bound for x_2
x_3	$x_1+x_3=4$	No limit imposed
x_4	$2x_2+x_4=12$	x_2=(12-0)/2=6
x_5	$3x_1+2x_2+x_5=18$	x_2=(18-3(0)-0)/2=9

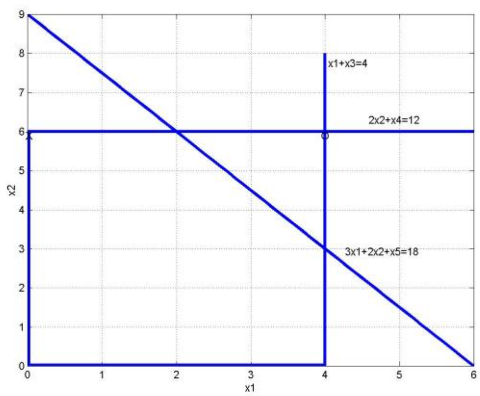
In this case, the second equation was more limiting that the third, and so there was no problem choosing.

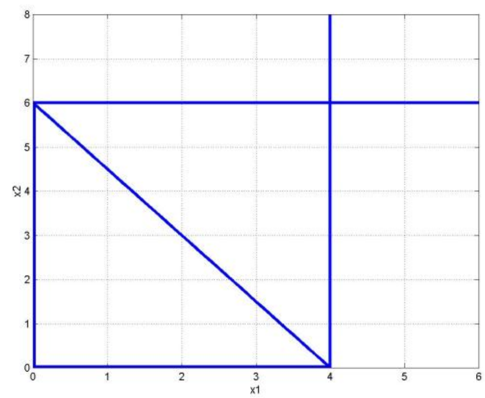
But what if the situation would have been as below?

Table 3: Determination of leaving variable for first step of example, when x_2 is the entering variable

Basic variable	Equation	Upper bound for x_2
x_3	$x_1+x_3=4$	No limit imposed
x_4	$2x_2+x_4=12$	x_2=(12-0)/2=6
x_5	$3x_1+2x_2+x_5=12$	x_2=(12-3(0)-0)/2=6

Let's compare this situation to the original one in terms of the x_1 -x_2 Cartesian plane.





Original situation New situation

In both cases, the first iteration moves us from the origin to the corner point (0,6), but in the new situation, the corner point (0,6) is defined by the intersection of 3 different constraints rather than 2.

And as a result, we end up seeing that both the $2x_2=12$ constraint and the $3x_1+2x_2=12$ constraint are equally limiting in regards to how much we can increase the entering variable x_2.

This seems problematic because in the new situation, we do NOT want to move along the $x_2=6$ boundary (meaning that x_4, the slack variable for this constraint, should not be the leaving variable) since this will carry us into an infeasible region. Clearly, we need to move along the $3x_1+2x_2=12$ boundary in order to remain feasible (meaning that x_5, the slack variable for this constraint, should be the leaving variable).

There are some complex rules for making this judgment; however, we may also make it arbitrarily.

What actually happens if you select x_4 as the leaving variable is that you will cycle on the point (0,6) for an extra iteration, and in the second iteration, you will choose the leaving variable to be x_5.

2.3 No leaving basic variable – unbounded *F*

Recall we selected the leaving variable to be the one that hits 0 first as the entering variable is increased, as dictated by one of the *m* constraint equations. But what happens if we have NO variables hitting zero as the entering variable is increased?

Recall in our example that we used Table 3 to make this choice.

Table 3: Determination of leaving variable for first step of example, when x_2 is the entering variable

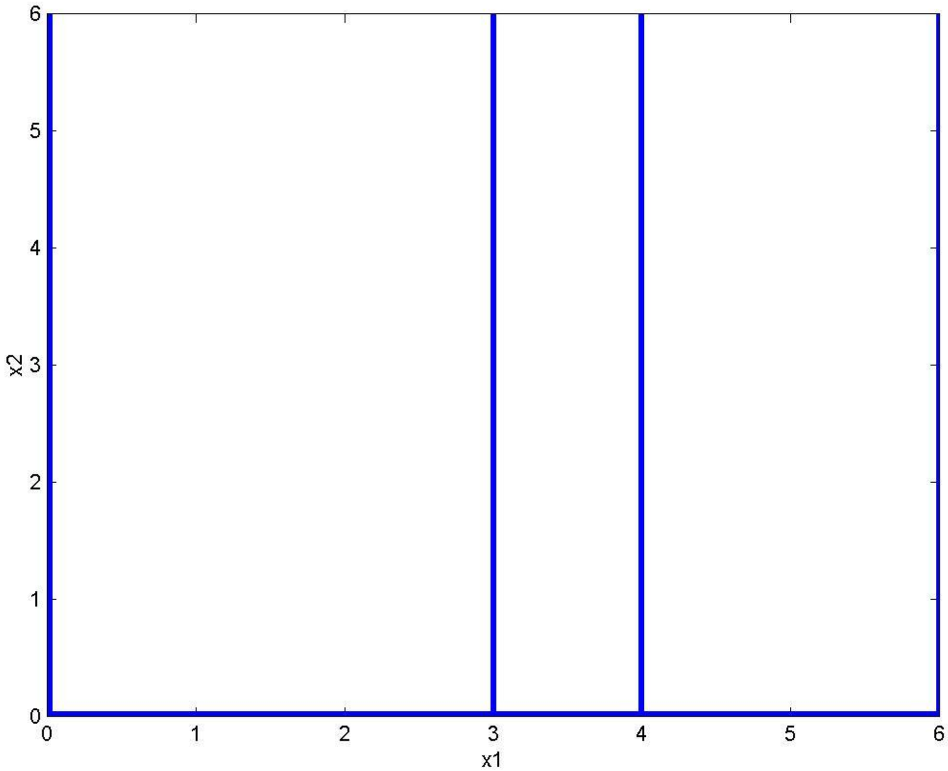
Basic variable	Equation	Upper bound for x_2
x_3	$x_1+x_3=4$	No limit imposed
x_4	$2x_2+x_4=12$	x_2=(12-0)/2=6
x_5	$3x_1+2x_2+x_5=18$	x_2=(18-3(0)-0)/2=9

Notice that the first constraint does not limit x_2. What if our second and third constraints also did not limit x_2? For example:

Table 3: Determination of leaving variable for first step of example, when x_2 is the entering variable

Basic variable	Equation	Upper bound for x_2
x_3	$x_1+x_3=4$	No limit imposed
x_4	$4x_1+x_4=12$	No limit imposed
x_5	$3x_1+x_5=18$	No limit imposed

In this case, we find that there is no constraint that limits what x_2 can be. What does this mean? The figure below illustrates.



Clearly, x_2 is unbounded, and there is no feasible solution to this problem. We recognize this situation when we cannot choose the leaving variable due to no limits imposed on the increase in the entering variable. In such case, we stop the iterations and report that the solution is unbounded.

2.4 Multiple optimal solutions

It is possible to see multiple optimal solutions. This happens, for example, when the slope of the objective function in the decision variable space is exactly the same as the slope of some constraint. We will look at detection of this situation later.