# Linearized optimal power flow

## 1.0 Some introductory comments

We looked at the economic dispatch problem under the "Cost Curves" notes at the beginning of the course. This problem was

Minimize:

$$F_T = \sum_{i=1}^{n} F_i(P_i) \tag{1}$$

Subject to:

$$\sum_{i=1}^{n} P_i = P_{load} ==> \phi(P_i) = P_{load} - \sum_{i=1}^{N} P_i = 0 \tag{2}$$

$$
\begin{aligned}
P_i &\geq P_{i,\min} \Rightarrow -P_i \leq -P_{i,\min} \\
P_i &\leq P_{i,\max} \\
P_i &\geq 0
\end{aligned} \tag{3}
$$

The advantage of the economic dispatch formulation to obtain minimum cost allocation of demand to the generation units is that it is computationally very fast and reasonably easy to solve. However, it suffers from two main drawbacks:

1. It is somewhat inaccurate due to loss approximation and due to implicit assumption that transmission has infinite capacity;
2. It provides no information about line flows.

The optimal power flow (OPF) compensates for these drawbacks by replacing the one power balance equation with the nodal power flow equations. The generator limits are retained as inequality constraints, but we add to the inequality constraints the constraints on line flows.

We desire to bring network constraints into the problem in order to study the influence of transportation (transmission) constraints on the most economic distribution of generation. There are a number of ways to solve the problem, including the following.

- Linearized
- Nonlinear, generalized reduced gradient
- Nonlinear, penalty function approach.

In these notes, we will focus on the linearized approach. It is very attractive to do so because

1. Linear optimization problems may be solved using linear programming (LP). LP has been under development now for over 50 years, extremely efficient algorithms have been developed so that linear programs, even very large ones, may be solved quickly;
2. There are available software packages which make LP solutions relatively easy to develop;
3. This is what industry does; and
4. We have learned the basics of how it works.

The OPF has been a tool for many years in the power engineering community, and has been effective in solving the following kinds of problems:

- Minimizing dispatch cost while meeting transmission constraints (this is the "classical" OPF)
- Security-constrained OPF (corrective and preventative)
- Voltage-var optimization
- Maximum power transfer
- Loss minimization

But it is the advent of electricity markets that has made OPF a central, essential, and everyday part of running power systems.

There are two comments to be made here:

- Most electricity markets utilize the linear programmed security-constrained OPF (LP-SCOPF). We will focus only on the LP-OPF in these notes. However, the main difference between LP-SCOPF and LP-OPF is that additional constraints ("security" constraints) associated with branch overload under contingency conditions are present in the former and not in the latter.
- Most electricity markets allow both offers (to sell) and bids (to buy). We will allow only the former in these notes, but introduction of the latter provides nothing conceptually new.

In linearizing the OPF, to maintain simplicity, we will assume that each generator makes only a single-price offer. Thus, each unit is represented in the objective function by a constant times the generation level for that unit. This approach ($s_i$), and an alternative ($s_{i1}$, $s_{i2}$, $s_{i3}$), are illustrated in Fig. 0 below.
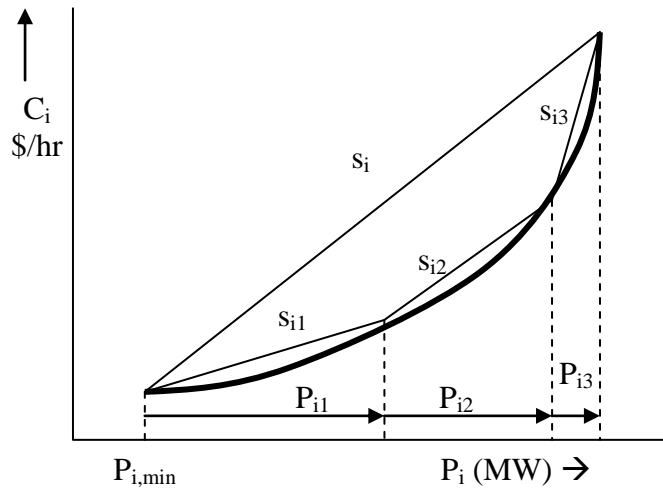


Fig. 0

So here is the formal statement of our problem:

$$\min \sum_{k \in \{generator\_buses\}} s_k P_{gk} \qquad 1$$

Subject to:

$$\underline{P} = \underline{B}' \underline{\theta} \qquad 2$$

$$\underline{P}_B = (\underline{D} \times \underline{A}) \times \underline{\theta} \qquad 3$$

$$-\underline{P}_{B,\max} \leq \underline{P}_B \leq \underline{P}_{B,\max} \qquad 4$$

$$0 \leq P_{gk} \leq P_{gk,\max}, \forall k \in \{generator\_buses\} \qquad 5$$

where

$$P_k = P_{gk} - P_{dk}, k = 1,...N \qquad 6$$

Before we proceed with formulating this problem, we must make one important observation. In the DC power flow problem, we found that if we included an equation for all buses in the network, using all angles in the network, that there was a dependency in our set of equations, and the matrix was not invertable.

Now, however, we will include all N equations for our network because we want a Lagrange multiplier (dual variable) for each bus, in order to know the *locational marginal price* (LMP) of energy. This implies that we must also include all of the angles as our unknowns. This will not create the same problem of matrix singularity that we had before because our overall system of equations will include (2), (3), and (6). In fact, this system will be under-constrained (there will be more unknowns than equations).

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[1] k is index on the units.

[2] These are DC power flow equations to represent the network. However, we must include all nodal injections $P_1,...P_N$ and all angles $\theta_1 ...\theta_N$ in this set of equations.

[3] These are equation to get line flows. Again, we need to include all angles $\theta_1 ...\theta_N$ in this set of equations.

[4] These are the limits on the line flows. Notice that there is only one circuit rating, but it must be enforced as a limit if the flow is in one direction or in the other.

[5] These are the limits on the linear cost curve variables.

[6] This relates variables used in the cost curves ($P_{kj}$) to variables used in DC power flow equations ($P_k$).

This is acceptable because
• we are not trying to solve them (because they are underconstrained, there are many solutions);
• rather, we are trying to minimize a function that is subject to them (with appropriate convexity requirements, there is only one solution that will do this).

Therefore we will include in $\underline{P}=\underline{B}'\underline{\theta}$ all DC power flow equations, one for each node, as a function of all angles in the network.

So be aware that in all of what follows, the vector $\underline{P}$ includes the reference bus injection, the vector $\underline{\theta}$ includes the reference bus angle, the matrix $\underline{B}'$ is N×N (i.e., it does not have a row and column eliminated corresponding to the reference bus), and adjacency (node-arc) matrix $\underline{D}$ is M×N (i.e, we do not eliminate the column corresponding to the reference bus).

To solve this problem as a linear program, we need to be able to write it in a form that a standard LP solver will handle. This means equality and inequality constraints must be written as a function of a single vector of unknowns (the "solution vector").

Special note on Lagrange multipliers and dual variables:
We concluded in our notes on "Basics of Optimization" that:
"It is important to understand the significance of $\mu$ and $\lambda$. The optimal values of the LaGrangian Multipliers are in fact the rates of change of the optimum attainable objective value $f(\underline{x})$ with respect to changes in the right-hand-side elements of the constraints. Economists know these variables as shadow prices or marginal values. This information can be used not only to investigate changes to the original problem but also to accelerate repeat solutions. The marginal values $\lambda_j$ or $\mu_k$ indicate how much the objective $f(\underline{x})$ would improve if a constraint $b_j$ or $c_k$, respectively, were changed."

We also stated in our notes on Duality (for LPs) that

> *The coefficients of the slack variables in the objective function expression of the final tableau (dual variables) give the improvement in the objective for a unit increase in the right-hand-sides of the corresponding constraints.*

Thus we see that the dual variables we were discussing for LPs are the same as the Lagrange multipliers.

Special note on Lagrange multipliers for equality constraints:
We saw in our notes called "LPSimplex1" that equality constraints may be converted into two inequality constraints via the following approach:

$$\underline{h}(\underline{x}) = \underline{c} \Leftrightarrow \begin{array}{c} \underline{h}(\underline{x}) \leq \underline{c} \\ \underline{h}(\underline{x}) \geq \underline{c} \end{array}$$

and we may then reverse the sign of the second inequality, resulting in:

$$\underline{h}(\underline{x}) = \underline{c} \Leftrightarrow \begin{array}{c} \underline{h}(\underline{x}) \leq \underline{c} \\ -\underline{h}(\underline{x}) \leq -\underline{c} \end{array}$$

This means we may include all of our equality constraints $\underline{h}(\underline{x}) = \underline{c}$ from our general form (1) in our inequality constraints $\underline{g}(\underline{x}) \leq \underline{b}$. However, we should also comment about Lagrange multipliers for equality constraints. As stated in [1, pg. 116-117],

> "Because an equality constraint is represented as two inequality constraints, we will get two dual variables, both nonnegative, having the same coefficients but opposite signs. Instead of treating these two variables separately, the difference can be treated as a variable unrestricted in sign. Hence the dual of an equality constraint is a variable unrestricted in sign."

Reference [1] gives a nice example illustrating this concept. Our main "take-away" is that equality constraints also contribute dual variables.

Special note on LP solver: The below formulation was developed for use in Matlab. However, this formulation may also be implemented in CPLEX. To do so, you need to realize that CPLEX is quite flexible in handling input constraints. The below example illustrates this flexibility:

```
\Problem name: alldiet.lp

Minimize
 DOLLARS: 10.89 pizza + 0.79 FRFries + 2.89 M.cherry + 2.59 MILK + 2.69 C.Milk
 + 0.47 P.towels + 1.49 !SUGARS! + 1.27 cereal + 3.29 SixPackX + 3.29 SixPackY
Subject To
 aluminum: 137 pizza + 71.4 SixPackX + 71.4 SixPackY >= 27230
 Vacuous:  >= 0
 RED#2:    1.9 pizza + 18 M.cherry + 1.9 P.towels + 1.6 !SUGARS! + 1.6 cereal
           + 1.7 SixPackX + 1.7 SixPackY - RgRED#2  = 18390
 salt:     67.2 pizza + 36 FRFries + 2.6 M.cherry + 3.4 P.towels + 4.3 !SUGARS!
           + 4.3 cereal + 31.4 SixPackX + 31.4 SixPackY + 0.000067 Air >= 21740
 fat:      41.8 pizza + 210.6 FRFries + 243 MILK + 223 C.Milk + 1.3 SixPackX
           + 1.2 SixPackY - Rgfat  = 30270
 Fiber:    3.4 pizza + 9.3 FRFries + 0.084 M.cherry + 0.45 P.towels
           + 1.78 !SUGARS! + 1.78 cereal - RgFiber  = 4789
 CALCIUM:  45.2 MILK + 43.2 C.Milk - RgCALCIUM  = 11460
 Sparkle:  - pizza - FRFries <= 10000
 Dirt:     pizza + FRFries >= -10000
Bounds
       FRFries >= 20
 500 <= M.cherry <= 800
 20 <= MILK <= 5000
 0 <= C.Milk <= 8888

       P.towels >= -27000
       !SUGARS! >= 144
 0 <= SixPackX <= 99999999
       SixPackY = 24
       Air Free
 -10 <= RgRED#2 <= 0
 0 <= Rgfat <= 99999
 -34789 <= RgFiber <= 0
 0 <= RgCALCIUM <= 240
End
```

*Equality constraints*

*Less-than inequality constraint*

*Greater-than inequality constraint*

*Less-than, greater-than inequality constraints*

You will find a great deal of additional information on how to use CPLEX at www.iro.umontreal.ca/~gendron/IFT6551/CPLEX/HTML.

## 2.0 Formulation of the linearized OPF

We define the following solution vector for our problem as:

$$\underline{x} = \begin{bmatrix} \underline{P}_g \\ \underline{P}_B \\ \underline{\theta} \end{bmatrix} \qquad (1)$$

where

- $\underline{P}_g$ is the vector of generation increments $[P_{gk} \ \ldots]^T$ for all bus k that has generation.
- $\underline{P}_B$ is the vector of line flows $[P_{b1} \ P_{b2} \ldots P_{bM}]^T$, M is # of branches.
- $\underline{\theta}$ is the vector of bus angles, in radians $[\theta_1 \ \ \theta_2 \ldots \theta_N]$, N is # of buses.

We want to capture all of our equality constraints in a single matrix relation.

There are two kinds of equality constraints: one due to line flows

$$\underline{P}_B = (\underline{D} \times \underline{A}) \times \underline{\theta} \qquad (2)$$

and the other due to injections:

$$\underline{P} = \underline{B}' \underline{\theta} \qquad (3)$$

We rewrite these slightly modified to make them more convenient to embed in a single matrix equation.

$$-\underline{P}_B + (\underline{D} \times \underline{A}) \times \underline{\theta} = \underline{0} \qquad (4)$$

$$-\underline{P} + \underline{B}' \underline{\theta} = \underline{0} \qquad (5)$$

The inequality constraints are straightforward given our definition of the solution vector. The only issue here is what to use as constraints on the angles? Clearly, all angles must reside between $-\pi$ radians and $+\pi$ radians. Therefore, inequality constraints will be:

$$\begin{bmatrix} \underline{P}_{g,\min} \\ -\underline{P}_{B,\max} \\ -\underline{\pi} \end{bmatrix} \leq \begin{bmatrix} \underline{P}_g \\ \underline{P}_B \\ \underline{\theta} \end{bmatrix} \leq \begin{bmatrix} \underline{P}_g \\ \underline{P}_{B,\max} \\ \underline{\pi} \end{bmatrix} \qquad (6)$$

**Example**:

We illustrate using an example that has 3 units connected to 3 different buses in a 4 bus network supplying load at 2 different buses.

The one-line diagram for the example system is given in Fig. 1. We will modify the load so that it has a total of 2.1787 per unit (or 217.87 MW), with 1 per unit load at bus 2 ($P_{d2}$=1.0) and 1.1787 per unit load at bus 3 ($P_{d3}$=1.1787).
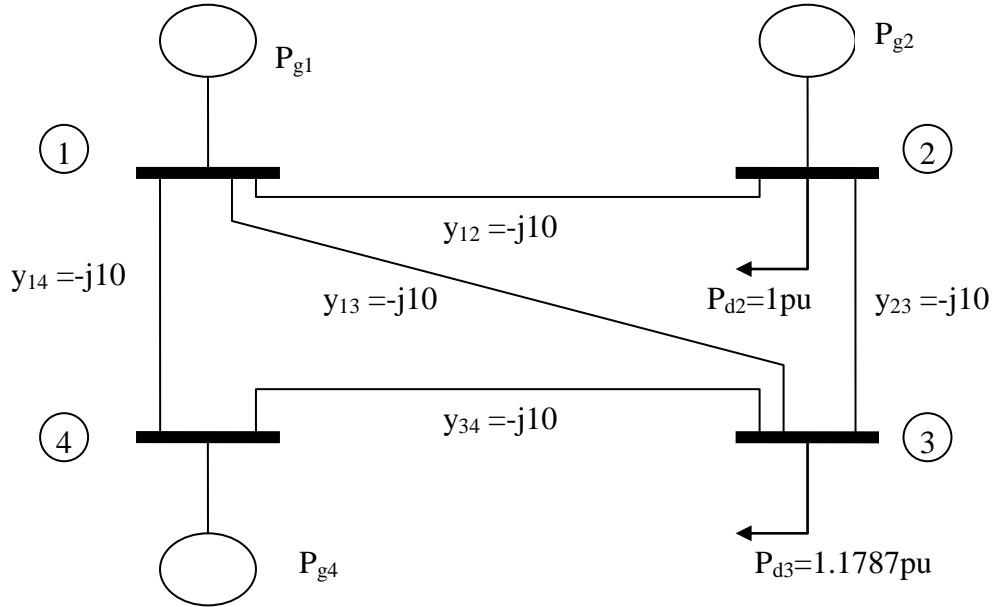


Fig. 1: One line diagram for example system

The offers and corresponding min and max generation limits are as follows:

$$K_1(P_{g1}) = s_1 P_{g1}, \ 50 \le P_{g1} \le 200$$

$$K_2(P_{g2}) = s_2 P_{g2}, \ 37.5 \le P_{g2} \le 150$$

$$K_4(P_{g4}) = s_4 P_{g4}, \ 45 \le P_{g4} \le 180$$

with

$s_1$=13.07 \$/MWhr
$s_2$=12.11 \$/MWhr
$s_4$=12.54 \$/MWhr

A modification is necessary at this point in that we need to **change the generation variables to per-unit.** This is necessary in order to bring in the transmission equations, which are in per-unit.

Thus, the decision variables $P_{g1}$, $P_{g2}$, and $P_{g3}$, are all divided by 100. We can compensate for this in terms of obtaining the correct evaluation for the cost by multiplying the offers by 100. Thus, we will use:

$s_1$=1307 \$/puMWhr
$s_2$=1211 \$/puMWhr
$s_4$=1254 \$/puMWhr

*Objective function*: We explicitly write the solution vector; then we will be able to write the objective function immediately.

$$\underline{x} = \begin{bmatrix} \underline{P}_g \\ \underline{P}_B \\ \underline{\theta} \end{bmatrix} = \begin{bmatrix} P_{g1} \\ P_{g2} \\ P_{g4} \\ P_{B1} \\ P_{B2} \\ P_{B3} \\ P_{B4} \\ P_{B5} \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix}$$

Now we see that the objective function is given by:

$$Z(\underline{x}) = \begin{bmatrix} 1307 & 1211 & 1254 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_{g1} \\ P_{g2} \\ P_{g4} \\ P_{B1} \\ P_{B2} \\ P_{B3} \\ P_{B4} \\ P_{B5} \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix}$$

*Equality constraints*: The equality constraints are given in eqs. (4) and (5), repeated here for convenience:

$$-\underline{P}_B + (\underline{D} \times \underline{A}) \times \underline{\theta} = \underline{0} \qquad (4)$$

$$-\underline{P} + \underline{B}'\underline{\theta} = \underline{0} \qquad (5)$$

We will build all of these equality codisplnstraints into a matrix form of $\underline{A}_{eq}\underline{x} = \underline{b}_{eq}$. We begin by noting dimensions.

- Columns: Since the solution vector $\underline{x}$ is 12×1, $\underline{A}_{eq}$ must have 12 columns in order to pre-multiply $\underline{x}$.
- Rows: Since there are 5 branches, eq. (4) will contribute 5 rows to $\underline{A}_{eq}$. Since there are 4 buses, eq. (5) will contribute 4 rows to $\underline{A}_{eq}$. So $\underline{A}_{eq}$ will have total of 9 rows.

Therefore, the dimensions of $\underline{A}_{eq}$ will be 9×12.

We begin with the line flow equations, eq. (4). The $\underline{D}$ matrix is:

$$\underline{D} = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 10 \end{bmatrix}$$

The node-arc incidence matrix, $\underline{A}$, is:

$$\underline{A} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$$

The $\underline{D} \times \underline{A}$ product required by eq. (4) is then given by:

$$\underline{D} \times \underline{A} = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 10 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 10 & 0 & 0 & -10 \\ 10 & -10 & 0 & 0 \\ 0 & 10 & -10 & 0 \\ 0 & 0 & -10 & 10 \\ 10 & 0 & -10 & 0 \end{bmatrix}$$

So based on eq. (4) and the solution vector, we can see that these elements will occupy the upper right hand corner of $\underline{A}_{eq}$. So that will take care of the last 4 columns in the first 5 rows.

But what about the first 8 columns? These are the elements in the line flow equations that multiply the variables $P_{g1}$, $P_{g2}$, $P_{g4}$, $P_{B1}$, $P_{B2}$, $P_{B3}$, $P_{B4}$, $P_{B5}$. Since we do not use the generation variables within the line flow equations, the first 3 columns of these top 5 rows will be zeros. The next 5 columns in these top 5 rows (columns 4-8) will also be zeros, except the one element in each of

these rows that multiplies the corresponding line flow variable, and that element will be -1.

Finally, with respect to these top 5 equations, eq. (4) indicates that the right-hand-side will be 0 for each of them.

Thus, we can now write down all elements in the first 5 rows of our matrix, as follows:

$$
\underline{A}_{eq}\,\underline{x} =
\begin{bmatrix}
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & -10 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 10 & -10 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 10 & -10 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -10 & 10 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 10 & 0 & -10 & 0 \\
- & - & - & - & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - & - & - & - & - \\
\end{bmatrix}
\begin{bmatrix}
P_{g1} \\ P_{g2} \\ P_{g4} \\ P_{B1} \\ P_{B2} \\ P_{B3} \\ P_{B4} \\ P_{B5} \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4
\end{bmatrix}
=
\begin{bmatrix}
0 \\ 0 \\ 0 \\ 0 \\ 0 \\ - \\ - \\ - \\ -
\end{bmatrix}
$$

Now we need to write the last 4 equations. These are the DC power flow equations corresponding to eq. (5).

Again, we must remember that the solution vector contains all 4 angles, and therefore the DC power flow matrix needs to be a 4×4.

This augmented DC power flow matrix is given below:

$$\underline{B}' = \begin{bmatrix} 30 & -10 & -10 & -10 \\ -10 & 20 & -10 & 0 \\ -10 & -10 & 30 & -10 \\ -10 & 0 & -10 & 20 \end{bmatrix}$$

So based on eq. (5) and the solution vector, we can see that this matrix will occupy the lower right hand side of the $\underline{A}_{eq}$ matrix. So that will take care of the last 4 columns in the bottom 4 rows. The resulting matrix appears as:

$$\underline{A}_{eq}\,\underline{x} = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & -10 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 10 & -10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 10 & -10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -10 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 10 & 0 & -10 & 0 \\ - & - & - & - & - & - & - & - & 30 & -10 & -10 & -10 \\ - & - & - & - & - & - & - & - & -10 & 20 & -10 & 0 \\ - & - & - & - & - & - & - & - & -10 & -10 & 30 & -10 \\ - & - & - & - & - & - & - & - & -10 & 0 & -10 & 20 \end{bmatrix} \begin{bmatrix} P_{g1} \\ P_{g2} \\ P_{g4} \\ P_{B1} \\ P_{B2} \\ P_{B3} \\ P_{B4} \\ P_{B5} \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ - \\ - \\ - \\ - \end{bmatrix}$$

Once again, we need to consider the first eight columns. Columns 4-8 correspond to the line flow variables, which do not appear in the DC power flow equations, so these will be zero.

$$
\underline{A}_{eq}\,\underline{x} = 
\begin{bmatrix}
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & -10 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 10 & -10 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 10 & -10 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -10 & 10 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 10 & 0 & -10 & 0 \\
- & - & - & 0 & 0 & 0 & 0 & 0 & 30 & -10 & -10 & -10 \\
- & - & - & 0 & 0 & 0 & 0 & 0 & -10 & 20 & -10 & 0 \\
- & - & - & 0 & 0 & 0 & 0 & 0 & -10 & -10 & 30 & -10 \\
- & - & - & 0 & 0 & 0 & 0 & 0 & -10 & 0 & -10 & 20
\end{bmatrix}
\begin{bmatrix}
P_{g1} \\ P_{g2} \\ P_{g4} \\ P_{B1} \\ P_{B2} \\ P_{B3} \\ P_{B4} \\ P_{B5} \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4
\end{bmatrix}
=
\begin{bmatrix}
0 \\ 0 \\ 0 \\ 0 \\ 0 \\ - \\ - \\ - \\ -
\end{bmatrix}
$$

The first three columns multiply the generation variables $P_{g1}$, $P_{g2}$, and $P_{g4}$. However, the DC power flow equations, eq. (5), require the negative of the *injections for all buses*, and the injections are the generation minus the load, i.e., $P_{gk}$-$P_{dk}$.

We do not have load variables $P_{dk}$ included in the solution vector. In addition, we do not have generation for bus 3 (it is just a load bus) included in the solution vector. So what do we do?

The answer to this lies in recognizing that the "variables" we do not have in the solution vector, $P_{d1}$, $P_{d2}$, $P_{d3}$, $P_{d4}$, and $P_{g3}$, are not (when the electricity market does not allow demand bids) "variables" at all! In fact, they are known quantities, constants, given by:

$P_{d1}$=0, $P_{d2}$=1.0, $P_{d3}$=1.1787, $P_{d4}$=0, $P_{g3}$=0

Since these are constants, they can go to the right-hand side. But with what sign? Eq. (5)

$$ -\underline{P} + \underline{B}'\underline{\theta} = \underline{0} \tag{5} $$

indicates that the injection, if modeled on the left-hand-side, should be negative, i.e., on the left-hand side, $P_{gk}$-$P_{dk}$ should be negative. So we should see on the left-hand-side $-P_{gk}$+$P_{dk}$. But now we will take the load term onto the right-hand-side by subtracting it from both sides.

Thus, we see that the load term should show up on the right-hand-side as a negative number. This same logic also shows us that the elements multiplying the generation terms in the $\underline{A}_{eq}$ matrix should be -1.

So we are now prepared to complete the matrix relation for the equality constraints.

$$\underline{A}_{eq}\,\underline{x} = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & -10 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 10 & -10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 10 & -10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -10 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 10 & 0 & -10 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 30 & -10 & -10 & -10 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -10 & 20 & -10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -10 & -10 & 30 & -10 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -10 & 0 & -10 & 20 \end{bmatrix} \begin{bmatrix} P_{g1} \\ P_{g2} \\ P_{g4} \\ P_{B1} \\ P_{B2} \\ P_{B3} \\ P_{B4} \\ P_{B5} \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ -1.1787 \\ 0 \end{bmatrix}$$

*Inequality constraints*:

The inequality constraints are simple, as given in what follows:

$$\begin{bmatrix} 0.5 \\ 0.375 \\ 0.45 \\ -500 \\ -500 \\ -500 \\ -500 \\ -500 \\ -\pi \\ -\pi \\ -\pi \\ -\pi \end{bmatrix} \leq \begin{bmatrix} P_{g1} \\ P_{g2} \\ P_{g4} \\ P_{B1} \\ P_{B2} \\ P_{B3} \\ P_{B4} \\ P_{B5} \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} \leq \begin{bmatrix} 2 \\ 1.5 \\ 1.8 \\ 500 \\ 500 \\ 500 \\ 500 \\ 500 \\ \pi \\ \pi \\ \pi \\ \pi \end{bmatrix}$$

*Solution by Matlab*: The code for solving this linear program using Matlab is given below:

%Load is system load plus losses
Load=2.1787;

%Build objective function vector.
c=[1307 1211 1254 0 0 0 0 0 0 0 0 0]';

%Build Aeq matrix for equality constraints.
Aeq=[0  0  0  -1 0 0  0  0   10   0   0 -10;
     0  0  0   0 -1 0  0  0   10 -10   0   0;
     0  0  0   0 0 -1  0  0    0  10  -10  0;
     0  0  0   0 0 0  -1  0    0   0  -10 10;
     0  0  0   0 0 0   0 -1   10   0  -10  0;
    -1  0  0   0 0 0   0  0   30 -10  -10 -10
     0 -1  0   0 0 0   0  0  -10  20  -10  0;
     0  0  0   0 0 0   0  0  -10 -10   30 -10;
     0  0 -1   0 0 0   0  0  -10   0  -10 20;];

%Build right-hand side of equality constraint.
beq=zeros(9,1);
beq(7)=-1;
beq(8)=-1.1787;

%Build upper and lower bounds on decision variables.
LB=[.50  .375  .45    -500 -500 -500 -500 -500  -pi -pi -pi -pi]';
UB=[2.00 1.50  1.80    500  500  500  500  500   pi  pi  pi  pi]';
[X,FVAL,EXITFLAG,OUTPUT,LAMBDA]=LINPROG(c,A,b,Aeq,beq,LB,UB);

The solution vector $\underline{x}$ is given by:

$$\underline{x} = \begin{bmatrix} P_{g1} \\ P_{g2} \\ P_{g4} \\ P_{B1} \\ P_{B2} \\ P_{B3} \\ P_{B4} \\ P_{B5} \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1.2287 \\ 0.45 \\ -0.0152 \\ 0.0955 \\ 0.3242 \\ 0.4348 \\ 0.4197 \\ 0.0125 \\ 0.003 \\ -0.0295 \\ 0.0140 \end{bmatrix} , \quad Z=2705.8$$

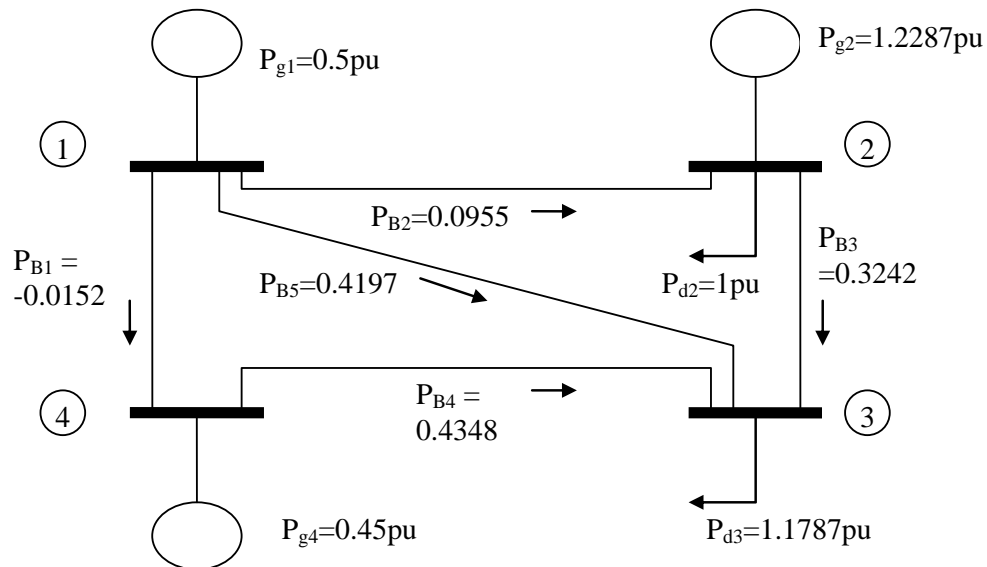The solution is provided on the one-line diagram of Fig. 4.



Fig. 4: Result in terms of generation levels and flows

One can easily check to see that the power is conserved at the buses.

The objective function that Matlab provides, is given above as Z=2705.8 $/hr.

It is of interest to compare this solution with a solution obtained from an economic dispatch (implying no representation of transmission). This problem will be

$$\min \ 1307P_{g1} + 1211P_{g2} + 1254P_{g4}$$

Subject to:

$$P_{g1} + P_{g2} + P_{g4} = 2.1787$$

$$0.5 \le P_{g1} \le 2.0$$

$$0.375 \le P_{g2} \le 1.5$$

$$0.45 \le P_{g4} \le 1.8$$

This is a LP that we can use Matlab or CPLEX to solve; however, it is actually a rather trivial solution. You can see that $P_{g1}$ and $P_{g4}$ are the more expensive units, and so we will take as little of those units as possible. Therefore we will assume

   $P_{g1}$=0.5
   $P_{g4}$=0.45

This means that $P_{g2}$= 2.1787-0.95=1.2287. The fact that this value of $P_{g2}$ is feasible (between its limits) means that this must be the answer to the problem, since this is the maximum amount of the cheapest unit that we can have.

Compare this to the solution we obtained in our LPOPF, and you see it is the same!!!

Should we expect the solutions to be the same, given the network representation in the LOPF approach? The answer is NO, if the flows are at the branch capacities.

But if the flows are all within the branch capacities (and we are not representing losses), then the transmission system has no impact on the dispatch, and in this case, the solution will be identical to the solution that we obtained in the economic dispatch scenario.

The LPOPF solution is for the case where all flows are within their branch capacities, since we have modeled all branch capacities to be 500 pu. In per-unit, this is effectively infinite branch capacity.

*BASE CASE*:
Now we will investigate the Lagrange multipliers for this loading level, assuming infinite capacity lines. These Lagrange multipliers, which are the same as the dual variables, are given in Table 2.

Table 2: Lagrange multipliers for $P_{d2}=1.0$, $P_{d3}=1.1787$
and infinite transmission capacity ($/per unit-hr)

| Equality constraints | | Lower bounds | | Upper bounds | |
|---|---|---|---|---|---|
| Equation | Value*$10^3$ | Variable | value | variable | value |
| $P_{B1}$ | 0.0000 | $P_{g1}$ | 96.0000 | $P_{g1}$ | 0.0000 |
| $P_{B2}$ | 0.0000 | $P_{g2}$ | 0 | $P_{g2}$ | 0.0000 |
| $P_{B3}$ | 0.0000 | $P_{g4}$ | 43.0000 | $P_{g4}$ | 0.0000 |
| $P_{B4}$ | 0.0000 | $P_{B1}$ | 0 | $P_{B1}$ | 0.0000 |
| $P_{B5}$ | 0.0000 | $P_{B2}$ | 0 | $P_{B2}$ | 0.0000 |
| $P_1$ | 1.2110 | $P_{B3}$ | 0 | $P_{B3}$ | 0.0000 |
| $P_2$ | 1.2110 | $P_{B4}$ | 0 | $P_{B4}$ | 0.0000 |
| $P_3$ | 1.2110 | $P_{B5}$ | 0 | $P_{B5}$ | 0.0000 |
| $P_4$ | 1.2110 | $\theta_1$ | 0 | $\theta_1$ | 0.0000 |
| | | $\theta_2$ | 0 | $\theta_2$ | 0.0000 |
| | | $\theta_3$ | 0 | $\theta_3$ | 0.0000 |
| | | $\theta_4$ | 0 | $\theta_4$ | 0.0000 |

Lagrange multipliers on the last four equality constraints are very interesting to us, since they give the improvement in the objective function if we increase the right-hand-side of the corresponding equation by 1 unit. These are the so-called nodal prices, i.e., the LMPs, given in $/per unit-hr. We see that the numbers are all $1211/per unit-hr, and if we divide this by the power base of 100 MVA, we get $12.11/MW-hr.

We also see that all Lagrange multipliers on the lower bounds are all zero, with the exception of the ones on $P_{g1}$ and $P_{g4}$, which are 96 and 43, respectively, with units of $/per unit-hr. Converting to $/MW-hr, these values are 0.96 and 0.43, respectively, indicating the amount of improvement we can expect if we increase the corresponding right-hand-side of these inequalities by 1 unit. Since these equations look like, for example, $-P_{g1} \leq -0.5$, an increase in the right-hand-side corresponds to a decrease in the lower limit. Thus, for $P_{g1}$, if we move the lower limit from 50 MW to 49 MW, we can expect to improve the objective function by 96 cents per hour.

Lagrange multipliers for all other lower bounds, and for all upper bounds, are zero, since none of the other inequality constraints are binding.

*CASE 1*: $P_{d2}=1.01$, $P_{d3}=1.1787$, and infinite transmission capacity

We now make a slight modification, by changing the loading of bus 2 from 1.0 to 1.01, an increase of .01 per unit or 1 MW. The changes necessary to the Matlab code are:

```
%Build right-hand side of equality constraint. It will be vector of zeros
%except for element in first row, which is load-sum of minimum
generation
beq=zeros(9,1);
beq(7)=-1.01;
beq(8)=-1.1787;
```

The solution is almost the same as in the base case, except for the objective function now evaluates to Z=2717.9. The previous

objective function was Z=2705.8, an increase of 12.1 $/hr, confirming our understanding of the meaning of the nodal prices.

We find that a 1 MW increase in the loading anywhere in this network has the same effect on the objective function, to increase it by 12.1$/hr. Thus, nodal prices have no locational variation in this network. Any network with infinite transmission capacity will have this attribute (assuming no losses), since the transmission system effectively makes the entire system look like a single bus.

_Case 2_: $P_{d2}$=1.0, $P_{d3}$=1.1787, and 0.3 capacity constraint on $P_{B3}$
In this case, we maintain loading at the same levels as the base case, where we saw the flows as in Fig. 5:
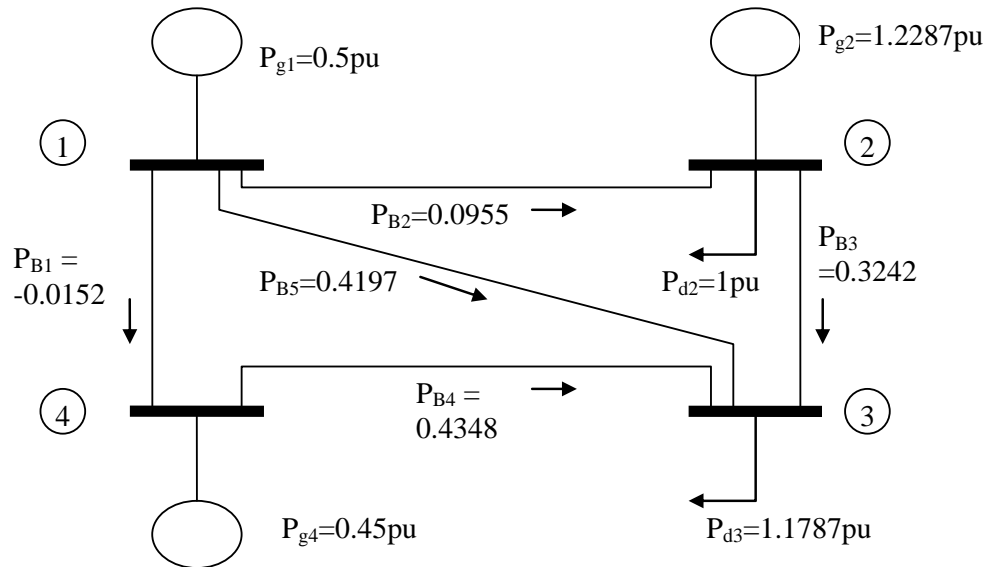


Fig. 5: Base case flows

We observe the flow on branch 3, $P_{B3}$=0.342. So let's consider that this branch has capacity of 0.3. This means that upper and lower bounds on $P_{B3}$ should be changed from -500,500 to -0.3, 0.3. The resulting solution is given in Fig. 6:
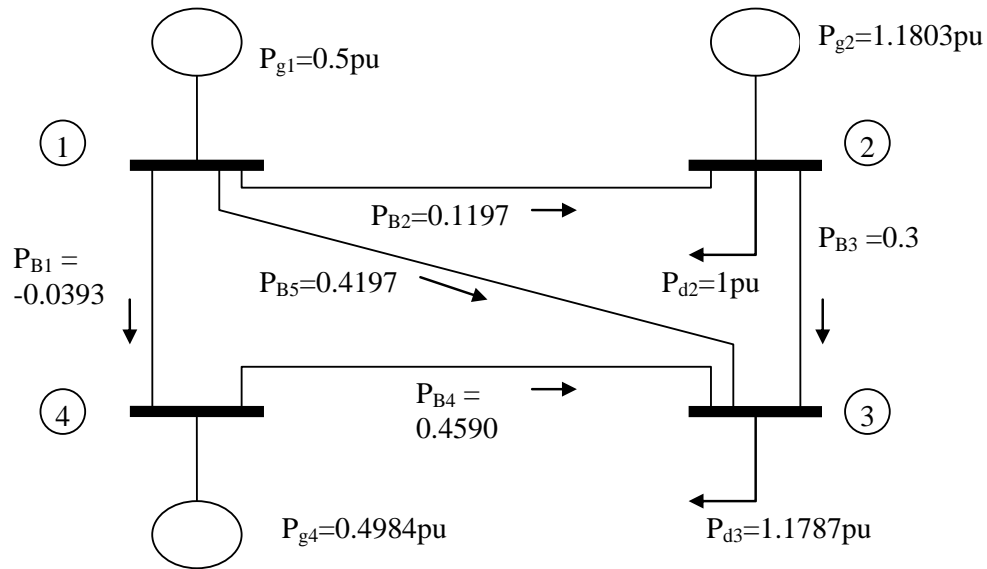
Fig. 6: Cases 2 flows

In comparing the previous two diagrams, we observe that
- The flow on branch 3 is constrained to 0.3 as desired.
- The flows all over the network have changed.
- The generation levels at buses 2 and 4 have changed.

Thus, the activation of a transmission constraint has changed the dispatch. This will affect the energy prices! This conclusion can be verified by looking at the Lagrange multipliers, given in the table below.

Table 3: Lagrange multipliers for $P_{d2}=1.0$, $P_{d3}=1.1787$
and infinite transmission capacity except for and
0.3 capacity constraint on $P_{B3}$ ($/per unit-hr)

| Equality constraints | | Lower bounds | | Upper bounds | |
|---|---|---|---|---|---|
| Equation | Value*$10^3$ | Variable | value | variable | value |
| $P_{B1}$ | 0.0000 | $P_{g1}$ | 63.7500 | $P_{g1}$ | 0.0000 |
| $P_{B2}$ | 0.0000 | $P_{g2}$ | 0.0000 | $P_{g2}$ | 0.0000 |
| $P_{B3}$ | 0.0860 | $P_{g4}$ | 0.0000 | $P_{g4}$ | 0.0000 |
| $P_{B4}$ | 0.0000 | $P_{B1}$ | 0.0000 | $P_{B1}$ | 0.0000 |
| $P_{B5}$ | 0.0000 | $P_{B2}$ | 0.0000 | $P_{B2}$ | 0.0000 |
| $P_1$ | 1.2432 | $P_{B3}$ | 0.0000 | $P_{B3}$ | 86.0000 |
| $P_2$ | 1.2110 | $P_{B4}$ | 0.0000 | $P_{B4}$ | 0.0000 |
| $P_3$ | 1.2647 | $P_{B5}$ | 0.0000 | $P_{B5}$ | 0.0000 |
| $P_4$ | 1.2540 | $\theta_1$ | 0.0000 | $\theta_1$ | 0.0000 |
| | | $\theta_2$ | 0.0000 | $\theta_2$ | 0.0000 |
| | | $\theta_3$ | 0.0000 | $\theta_3$ | 0.0000 |
| | | $\theta_4$ | 0.0000 | $\theta_4$ | 0.0000 |

Some comments about Lagrange multipliers in Table 3:

1. <u>Generation limits</u>: We still see a non-zero Lagrange multiplier on the $P_{g1}$ lower limit, as before, but the Lagrange multiplier on the $P_{g4}$ lower limit has become zero, reflecting that $P_{g4}$ had to increase and come off of its lower limit to compensate for the decrease in $P_{g2}$ necessary to redispatch around the $P_{B3}$ constraint.

2. <u>Branch limits</u>: The Lagrange multiplier on the $P_{B3}$ upper bound is 86, and after dividing by 100 to change from per-unit to MW, it is 0.86 $/MW-hr, reflecting the improvement in objective function that can be obtained from increasing the $P_{B3}$ branch limit by 1 MW (from 0.30 per-unit to 0.31 per-unit).

3. <u>Branch flows</u>: The Lagrange multiplier on the $P_{B3}$ flow (the $P_{B3}$ equality constraint) is $0.086*10^3=86$, and after dividing by 100 to change from per-unit to MW, it is 0.86 $/MW-hr, reflecting the improvement in objective function that can be obtained from

increasing the flow in this branch by 1 MW (from 0.30 per-unit to 0.31 per-unit). It makes sense that the Lagrange multiplier on the flow would be the same as the Lagrange multiplier on the branch limit.

4. <u>Nodal prices</u>: The Lagrange multipliers on the equality constraints corresponding to the 4 nodes are the nodal prices. Without transmission constraints, these prices were all the same, at 12.11 $/MW-hr, a price set entirely by the generator at bus 2 since it was the bus 2 generator that responded to any load change. But now they are all different, with only bus 2 price at 12.11 $/MW-hr. This difference reflects that, because of the transmission constraint, a load increase at one bus will incur a different cost than a load increase at another bus.

Comment #4 is worth investigating further. Let's increase the load at the highest price bus, bus #3, from 1.1787 to 1.1887 per unit, an increase of 1 MW. The resulting dispatch and flows are shown in Fig. 7. In order to gain intuition into what has happened, we have repeated Fig. 6 just below it so as to provide a convenient comparison.
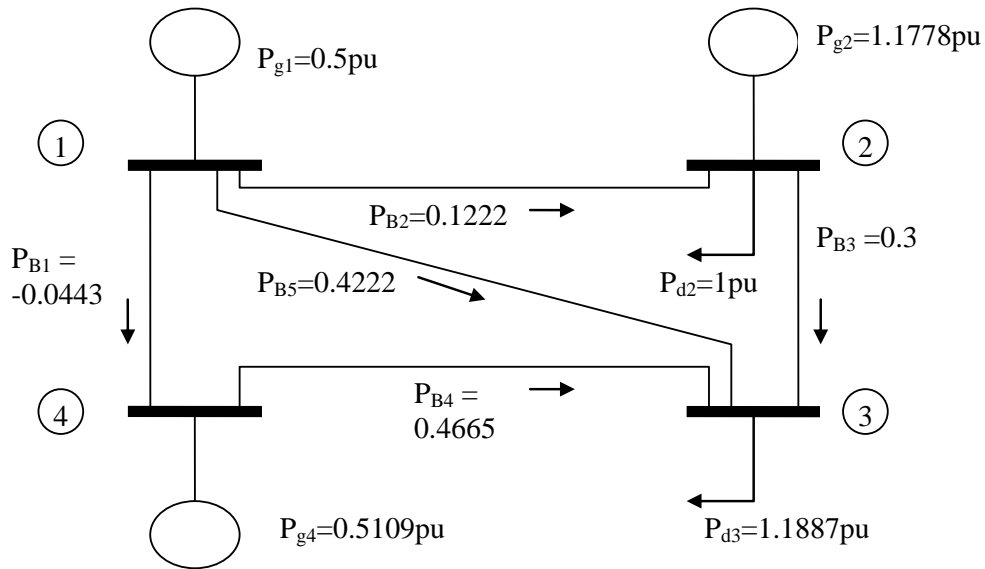
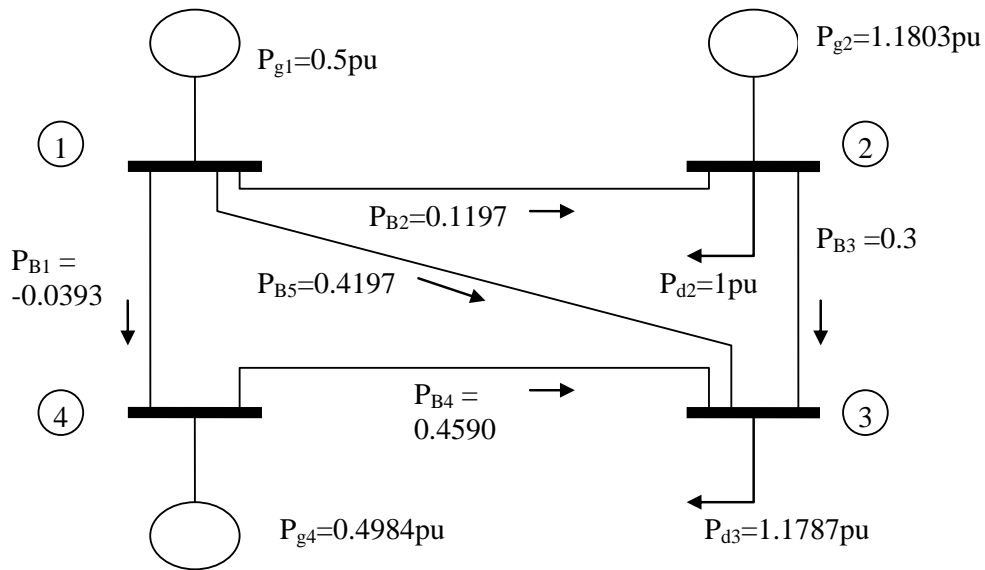Fig. 7: Flows for case 2 with 1 MW increase in $P_{d3}$



Fig. 6: Original case 2 flows

The comparison shows that in order to supply an additional MW at bus 3, *the generation levels of 2 different units had to be modified*. Specifically, Unit 2 was decreased from 1.1803 to 1.1778, a decrease of 0.0025 per unit (0.25 MW) and Unit 4 was increased from 0.4984 per unit to 0.5109 per unit, an increase of 0.0125 (1.25 MW). Thus, Unit 4 was increased enough to supply the increased load at bus 3 and the decreased generation at bus 2.

Question: Why did we not just increase Unit 2 or increase Unit 4 by 1 MW?

Answer: Because the resulting flow on branch 3 would exceed its capacity!!!

In fact, it is not possible to supply additional load at bus 3 with only a single unit increase. We will always have to compensate for the load AND redispatch to compensate for the additional flow on the branch 3. As a result, the nodal price at bus 3 is a function of the generation costs at those buses that are used in the particular redispatch that achieves the minimum cost. Although we only used two different units here, there could be situations where more than two units must be redispatched to supply an additional MW at a bus when congestion is present. How many units must be redispatched depends on

- the bids of the units;
- the location of the units relative to the congested line (or lines).

It will find the minimum cost redispatch, and so it will tend to utilize the less expensive units. However, it will also tend to use the units that have to move the least amount relative to the solution before the additional MW was added. Thus, it could be the case that a unit bid at $13 gets moved up instead of a unit bid at $12 if the first unit has location which required that it only move 1.1 MW and the second unit has a location which requires that it move 1.5 MW.

[1] S.Zionts, "Linear and Integer Programming," Prentice-Hall, 1974.