

## Homework #4 (Due Wednesday October 5, 2010)

1. A two generator system has cost curves (\$/hr) of  $C_1(P_1)=0.006P_1^2 + 5P_1+3$  , and  $C_2(P_2)=0.01P_2^2+4P_2+2$ , where  $P_1$  and  $P_2$  are given in MW. The total demand is  $P_T=500$  MW. The limits on these generators are  $0 \leq P_1 \leq 300$  and  $0 \leq P_2 \leq 300$ .
  - a. Determine the unconstrained values of  $P_1$  and  $P_2$  that minimize the cost of supplying the 500 MW, and indicate whether this solution is feasible or not.
  - b. For the solution found in (a), how much would the total cost of supply change if the total demand increased to 501 MW for one hour ?
  - c. Use the complementary condition (the third condition in the KKT conditions), to identify the values of each Lagrange multiplier associated with the inequality constraints.

### Solution:

- a. The unconstrained solution is found from applying the first and second KKT conditions. This results in:

$$\begin{bmatrix} 0.012 & 0 & -1 \\ 0 & 0.02 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ \lambda \end{bmatrix} = \begin{bmatrix} -5 \\ -4 \\ 500 \end{bmatrix} \Rightarrow \begin{bmatrix} P_1 \\ P_2 \\ \lambda \end{bmatrix} = \begin{bmatrix} 281.25 \\ 218.75 \\ 8.375 \end{bmatrix}$$

Both generation values are within their constraints; therefore the solution is feasible.

- b. The LaGrange multiplier  $\lambda$  provides the change in the objective function (which is the total cost of supply) for a 1 MW increase in demand over the next hour, which is exactly what this question is asking, therefore the answer is \$8.375.
  - a. Because the unconstrained solution is feasible, there are no inequality constraints that are binding. Therefore, by the complementary condition, it must be the case that all Lagrange multipliers on the inequality constraints must be zero, i.e.,  $\mu_i = 0 \forall i$ .
2. Starting from the end of the notes “LPSimplex2,” (and the slides LPSimple1-2.ppt), take the next iterative step following the exact same approach as described in those notes (and slides). Perform a second iteration just as we did in these notes (and slides) for the first iteration, and then test your solution for optimality. You do not need to perform additional iterations.

### Solution:

The place to begin is the system of equations on page 25 of LPSimplex2 notes, as given below:

$$F \quad -3x_1 \qquad \qquad \qquad + 2.5x_4 \qquad = \quad 30 \quad (12)$$

$$x_1 \qquad \qquad \qquad + x_3 \qquad = \quad 4 \quad (13)$$

$$\qquad \qquad \qquad x_2 \qquad \qquad \qquad + 0.5x_4 \qquad = \quad 6 \quad (14)$$

$$3x_1 \qquad \qquad \qquad - x_4 \qquad + x_5 \qquad = \quad 6 \quad (15)$$

Recall from the LPSimplex2 notes that in the above,  $(x_2, x_3, x_5)$  are basic variables (with non-zero values), and  $(x_1, x_4)$  are non-basic variables (with zero values).

We see that  $x_1$  is the only variable with a negative coefficient in the objective function expression, and so that is our only alternative for an entering variable.

So let's identify the leaving variable. Equation (14) does not contain  $x_1$  and so is not of interest here.

Equation (13):  $x_1=4-x_3$ , and if we make  $x_3=0$ , then  $x_1=4$ .

Equation (15):  $3x_1-x_4+x_5=6$ . With  $x_4$  already zero (it is a nonbasic), if we make  $x_5$  zero, then  $x_1=2$ .

And so eq. (15) is more restrictive on how much  $x_1$  can increase than eq. (13), and therefore eq. (15)'s

basic variable,  $x_5$ , should become nonbasic. Therefore  $x_5$  is the leaving variable.

So eq. (15) will be the pivot row and the element corresponding to the variable  $x_1$  will be the pivot element. We rewrite the system of equations (12)-(15) as a matrix below, and indicate that the last row is the pivot row and the “3” in it is the pivot element.

$$\begin{bmatrix} 1 & -3 & 0 & 0 & 2.5 & 0 & 30 \\ 0 & 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 0.5 & 0 & 6 \\ 0 & \textcircled{3} & 0 & 0 & -1 & 1 & 6 \end{bmatrix} \leftarrow$$

Performing Gaussian elimination, we first divide the last row by 3 to get:

$$\begin{bmatrix} 1 & -3 & 0 & 0 & 2.5 & 0 & 30 \\ 0 & 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 0.5 & 0 & 6 \\ 0 & 1 & 0 & 0 & -0.333 & 0.333 & 2 \end{bmatrix} \leftarrow$$

Now multiple row 4 by 3 and add it to the first row, as illustrated below:

$$\begin{bmatrix} 1 & -3 & 0 & 0 & 2.5 & 0 & 30 \\ 0 & 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 0.5 & 0 & 6 \\ 0 & 1 & 0 & 0 & -0.333 & 0.333 & 2 \end{bmatrix} \leftarrow \begin{matrix} \leftarrow \\ \\ \\ \leftarrow 3 \times \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1.5 & 1 & 36 \\ 0 & 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 0.5 & 0 & 6 \\ 0 & 1 & 0 & 0 & -0.333 & 0.333 & 2 \end{bmatrix}$$

Now multiple row 4 by -1 and add it to the second row, as illustrated below:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1.5 & 1 & 36 \\ 0 & 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 0.5 & 0 & 6 \\ 0 & 1 & 0 & 0 & -0.333 & 0.333 & 2 \end{bmatrix} \begin{matrix} \\ \leftarrow \\ \\ \leftarrow -1 \times \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1.5 & 1 & 36 \\ 0 & 0 & 0 & 1 & +0.333 & -0.333 & 2 \\ 0 & 0 & 1 & 0 & 0.5 & 0 & 6 \\ 0 & 1 & 0 & 0 & -0.333 & 0.333 & 2 \end{bmatrix}$$

We observe that all coefficients in the objective function equation (row 1) are positive, indicating there is no variable that can enter which will improve the objective. And so this solution is optimal.

Given that  $(x_1, x_2, x_3)$  are basic, and  $(x_4, x_5)$  are non-basic, we may immediately read off the solution from the above matrix as

$$(x_1, x_2, x_3) = (2, 6, 2)$$

$$F = 36$$

3. Starting from the end of the notes “LPSimplex3” (and the end of slides LPSimplex3.ppt), take the next iterative step in the problem described in those notes (and slides). Continue to perform iterations just as we did in these notes (and slides), and test for optimality until you find the optimal solution.

**Solution:**

Recall our LP example problem we have been working on, in equality form, is given below.

$$\max F = 3x_1 + 5x_2$$

s.t.

$$x_1 + x_3 = 4$$

$$2x_2 + x_4 = 12$$

$$3x_1 + 2x_2 + x_5 = 18$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0$$

which, when written in a slightly different form, is

$$F - 3x_1 - 5x_2 = 0$$

$$x_1 + x_3 = 4$$

$$2x_2 + x_4 = 12$$

$$3x_1 + 2x_2 + x_5 = 18$$

Recall that we performed the first iteration of the simplex method for it, which resulted in the following Tableau:

Tableau 1

Basic variable	Eq. #	Coefficients of						Right side	
		$F$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$		
$F$	0	1	-3	0	0	2.5	0	30	← Add $5 \times$ pivot row
$x_3$	1	0	1	0	1	0	0	4	
$x_2$	2	0	0	1	0	0.5	0	6	← Add $-2 \times$ pivot row
$x_5$	3	0	3	0	0	-1	1	6	

Then when we tested for optimality, we discovered that we must do another iteration because the coefficient of  $x_1$  is negative in the above tableau. I asked you to do this for homework. I will do it here, to provide you with the solution to the homework.

So our only choice for the entering variable is  $x_1$ . To determine the leaving variable, we identify the constraints that most limit the increase in  $x_1$ , as shown in Tableau 2 below.

Tableau 2

Basic variable	Eq. #	Coefficients of						Right side	
		$F$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$		
$F$	0	1	-3	0	0	2.5	0	30	← $\frac{4}{1} = 4$
$x_3$	1	0	1	0	1	0	0	4	
$x_2$	2	0	0	1	0	0.5	0	6	← $\frac{6}{3} = 2$
$x_5$	3	0	3	0	0	-1	1	6	

And so the constraint identifying the leaving variable is the last one, since its ratio is smallest ( $2 < 4$ ).

The leaving variable is therefore  $x_5$ , since it is the variable in the last constraint which gets pushed to 0 as  $x_1$  increases. And so the tableau is shown below with the pivot row, pivot column, and pivot element identified. We have also modified the last basic variable (column 1) to be  $x_1$ .

Tableau 3

Basic variable	Eq. #	Coefficients of						Right side
		$F$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	
$F$	0	1	-3	0	0	2.5	0	30
$x_3$	1	0	1	0	1	0	0	4
$x_2$	2	0	0	1	0	0.5	0	6
$x_1$	3	0	3	0	0	-1	1	6

Dividing the last equation by 3, we obtain:

Tableau 4

Basic variable	Eq. #	Coefficients of						Right side
		$F$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	
$F$	0	1	-3	0	0	2.5	0	30
$x_3$	1	0	1	0	1	0	0	4
$x_2$	2	0	0	1	0	0.5	0	6
$x_1$	3	0	1	0	0	-0.333	0.333	2

Using the last equation to eliminate the -3 in the top equation, we get:

Tableau 5

Basic variable	Eq. #	Coefficients of						Right side
		$F$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	
$F$	0	1	0	0	0	1.5	1	36
$x_3$	1	0	1	0	1	0	0	4
$x_2$	2	0	0	1	0	0.5	0	6
$x_1$	3	0	1	0	0	-0.333	0.333	2

Using the last equation to eliminate the 1 from the second equation, we get:

Tableau 6

Basic variable	Eq. #	Coefficients of						Right side
		$F$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	
$F$	0	1	0	0	0	1.5	1	36
$x_3$	1	0	0	0	1	0.333	-0.333	2
$x_2$	2	0	0	1	0	0.5	0	6
$x_1$	3	0	1	0	0	-0.333	0.333	2

The solution above is optimal because all coefficients in the objective function expression are positive.

- Use CPLEX to solve the optimization problem you addressed in problem #2 and #3 of this assignment.