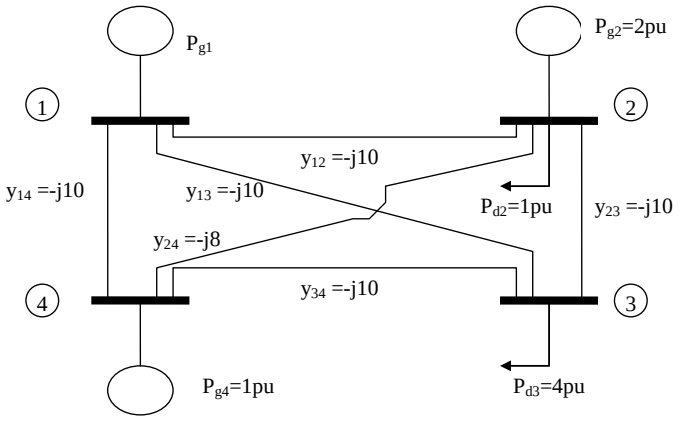
EE 458, HW 6 – Due Monday, October 24, 2019

1. For the system below:
    a. Obtain the Y-bus.
    b. Obtain the B' matrix.
    c. Set up and solve the DC power flow relation $\underline{P}=\underline{B}'\underline{\theta}$. Give all four angles.
    d. Obtain the $\underline{D}$ matrix and the node-arc incidence matrix $\underline{A}$.
    e. Compute all line flows. Compare the line flows to those for the solution given in the class notes, and comment on the effect of the added line in terms of loading in other lines.



Solution:

a. Relative to the Y-bus from the notes, we need to modify four elements of the matrix corresponding to entries for buses 2 and 4 between which the new line is connected. Specifically, we need to modify elements (2,2), (2, 4), (4,2), and (4,4). The resulting Y-bus is given by:

$$Y = j\begin{bmatrix} -30 & 10 & 10 & 10 \\ 10 & -28 & 8 & 10 \\ 10 & 10 & -30 & 10 \\ 10 & 8 & 10 & -28 \end{bmatrix}$$

b. The B' matrix is obtained by negating the Y-bus, removing the "j", and then removing column 1 and row 1.

$$\begin{bmatrix} 30 & -10 & -10 & -10 \\ -10 & 28 & -10 & -8 \\ -10 & -10 & 30 & -10 \\ -10 & -8 & -10 & 28 \end{bmatrix} \rightarrow B' = \begin{bmatrix} 28 & -10 & -8 \\ -10 & 30 & -10 \\ -8 & -10 & 28 \end{bmatrix}$$

c. The DC Power flow relations are:

$$\begin{bmatrix} P_2 \\ P_3 \\ P_4 \end{bmatrix} = \begin{bmatrix} 28 & -10 & -8 \\ -10 & 30 & -10 \\ -8 & -10 & 28 \end{bmatrix} \begin{bmatrix} \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix}$$

where

$$\begin{bmatrix} P_2 \\ P_3 \\ P_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix}$$

Solving the DC Power flow relations, we obtain:

$$\begin{bmatrix} \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} = \begin{bmatrix} 28 & -10 & -8 \\ -10 & 30 & -10 \\ -8 & -10 & 28 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.025 \\ -0.15 \\ -0.025 \end{bmatrix}$$

This is exactly the same solution we would obtain using the system without the line from bus 2 to bus 4. If you solve the system without the line between buses 2 and 4 (see class slides on LPOPF), you will find that the angles of buses 2 and 4 are the same, and so placing a line between them does not result in any flow and thus, it does not result in any change in the flows in the rest of the network.

d. The D-matrix is

$$\underline{D} = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 8 \end{bmatrix}$$

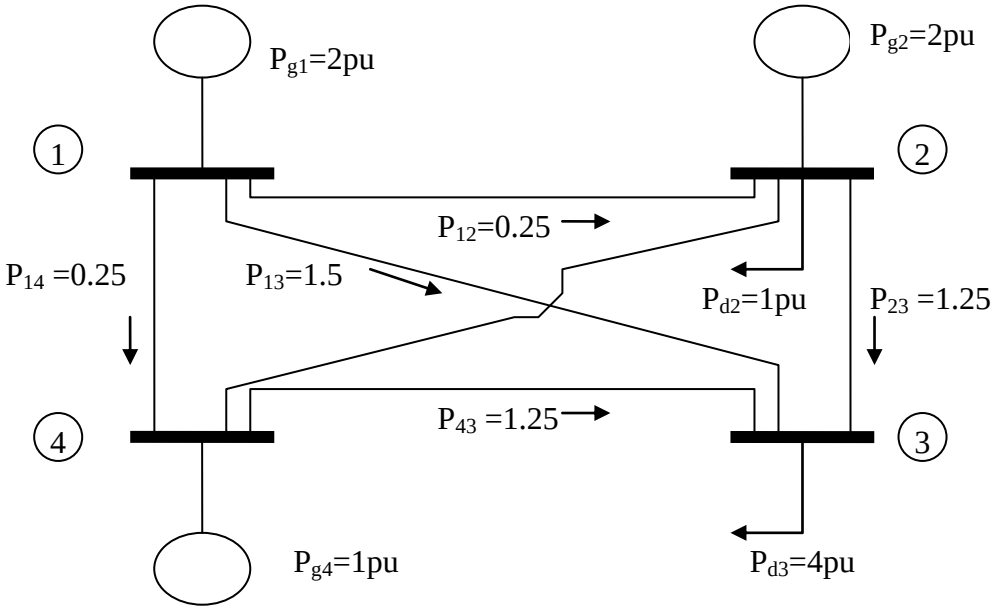
The node-arc incidence matrix is constructed as in the notes, except we need one more row corresponding to the extra branch between buses 2 and 4. We define directionality on this branch as positive from bus 2 to bus 4. The matrix is

$$\underline{A} = \begin{array}{c} \overbrace{\begin{array}{ccc} 2 & 3 & 4 \end{array}}^{\text{node number}} \\ \left.\begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}\right\} \text{branch number} \end{array}$$

e. Because the network angles did not change, and because the new line should carry no flow (because the angles at either terminal are the same), the line flows should be exactly the same, an expectation we can check via $\underline{P}_B = (\underline{D} \times \underline{A}) \times \underline{\theta}$.

$$\begin{bmatrix} P_{B1} \\ P_{B2} \\ P_{B3} \\ P_{B4} \\ P_{B5} \\ P_{B6} \end{bmatrix} = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} -0.025 \\ -0.15 \\ -0.025 \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0.25 \\ 1.25 \\ 1.25 \\ 1.5 \\ 0 \end{bmatrix}$$

Notice that the flow on the circuit we added is zero, which confirms that the solution is the same. The solution is illustrated in the figure below.



2. Using the same system you analyzed in problem 1, set up the optimal power flow as a linear program. Assume the objective function is exactly the same as used in the example in class, i.e., $Z = 1307P_{g1} + 1211P_{g2} + 1254P_{g4}$. Also, assume each unit has a lower limit of 100 MW and an upper limit of 300 MW, which will be (in per unit):

   $1 \le P_{g1} \le 3$

   $1 \le P_{g2} \le 3.$

   $1 \le P_{g3} \le 3$

   a. Write down the optimization problem you must solve. Assume infinite transmission capacity.

      Solution:

      First, let's obtain the power flow equations:

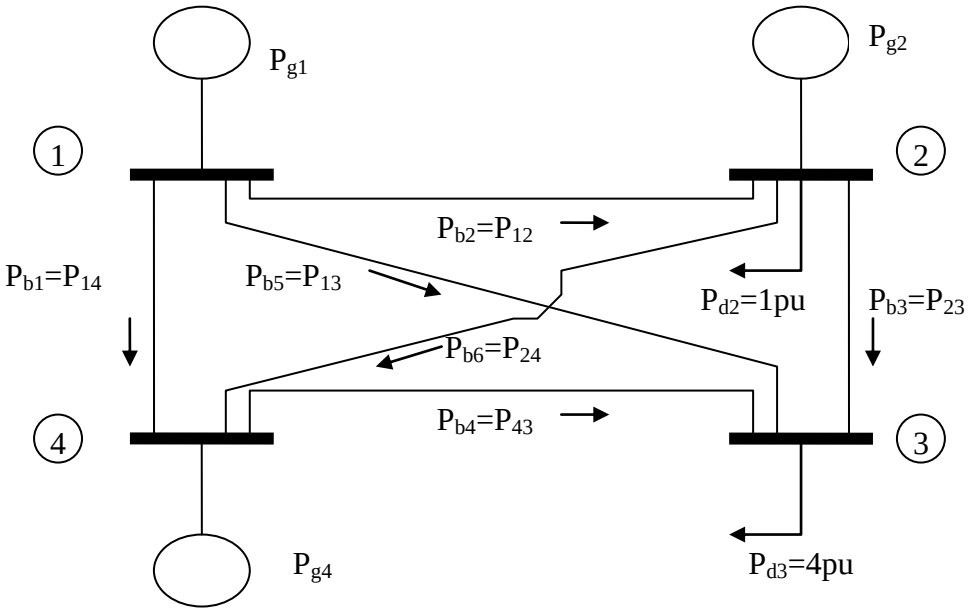
$$\begin{bmatrix} P_{g1} \\ P_{g2} - 1 \\ -4 \\ P_{g4} \end{bmatrix} = \begin{bmatrix} 30 & -10 & -10 & -10 \\ -10 & 28 & -10 & -8 \\ -10 & -10 & 30 & -10 \\ -10 & -8 & -10 & 28 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix}$$

Writing them out explicitly as they need to be used in CPLEX,

we obtain:

$$P_{g1} - 30\theta_1 + 10\theta_2 + 10\theta_3 + 10\theta_4 = 0$$
$$P_{g2} + 10\theta_1 - 28\theta_2 + 10\theta_3 + 8\theta_4 = 1$$
$$10\theta_1 + 10\theta_2 - 30\theta_3 + 10\theta_4 = 4$$
$$P_{g4} + 10\theta_1 + 8\theta_2 + 10\theta_3 - 28\theta_4 = 0$$

Now we need to obtain the line flow equations. Note that we define directionality according to the diagram below.



$$\underline{A} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \qquad \underline{D} = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 8 \end{bmatrix}$$

And so

$$(\underline{D}\times\underline{A})\times\begin{bmatrix}\theta_1\\ \theta_2\\ \theta_3\\ \theta_4\end{bmatrix}=\begin{bmatrix}10&0&0&0&0&0\\ 0&10&0&0&0&0\\ 0&0&10&0&0&0\\ 0&0&0&10&0&0\\ 0&0&0&0&10&0\\ 0&0&0&0&0&8\end{bmatrix}\begin{bmatrix}1&0&0&-1\\ 1&-1&0&0\\ 0&1&-1&0\\ 0&0&-1&1\\ 1&0&-1&0\\ 0&1&0&-1\end{bmatrix}\begin{bmatrix}\theta_1\\ \theta_2\\ \theta_3\\ \theta_4\end{bmatrix}=\begin{bmatrix}10&0&0&-10\\ 10&-10&0&0\\ 0&10&-10&0\\ 0&0&-10&0\\ 10&0&-10&0\\ 0&8&0&-8\end{bmatrix}\begin{bmatrix}\theta_1\\ \theta_2\\ \theta_3\\ \theta_4\end{bmatrix}=\begin{bmatrix}10\theta_1-10\theta_4\\ 10\theta_1-10\theta_2\\ 10\theta_2-10\theta_3\\ -10\theta_3+10\theta_4\\ 10\theta_1-10\theta_3\\ 8\theta_2-8\theta_4\end{bmatrix}$$

Therefore,

$$-\underline{P}_b+(\underline{D}\times\underline{A})\times\begin{bmatrix}\theta_1\\ \theta_2\\ \theta_3\\ \theta_4\end{bmatrix}=0\Rightarrow-\begin{bmatrix}P_{b1}\\ P_{b2}\\ P_{b3}\\ P_{b4}\\ P_{b5}\\ P_{b6}\end{bmatrix}+\begin{bmatrix}10\theta_1-10\theta_4\\ 10\theta_1-10\theta_2\\ 10\theta_2-10\theta_3\\ -10\theta_3+10\theta_4\\ 10\theta_1-10\theta_3\\ 8\theta_2-8\theta_4\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\\ 0\\ 0\\ 0\end{bmatrix}$$

$$-P_{b1}+10\theta_1-10\theta_4=0$$
$$-P_{b2}+10\theta_1-10\theta_2=0$$
$$-P_{b3}+10\theta_2-10\theta_3=0$$
$$-P_{b4}-10\theta_3+10\theta_4=0$$
$$-P_{b5}+10\theta_1-10\theta_3=0$$
$$-P_{b6}+8\theta_2-8\theta_4=0$$

And so here is the optimization problem we need to solve:

$\min\ Z=1307P_{g1}+1211P_{g2}+1254P_{g4}$

subject to

$$P_{g1}-30\theta_1+10\theta_2+10\theta_3+10\theta_4=0$$
$$P_{g2}+10\theta_1-28\theta_2+10\theta_3+8\theta_4=1$$
$$10\theta_1+10\theta_2-30\theta_3+10\theta_4=4$$
$$P_{g4}+10\theta_1+8\theta_2+10\theta_3-28\theta_4=0$$
$$-P_{b1}+10\theta_1-10\theta_4=0$$
$$-P_{b2}+10\theta_1-10\theta_2=0$$
$$-P_{b3}+10\theta_2-10\theta_3=0$$
$$-P_{b4}-10\theta_3+10\theta_4=0$$
$$-P_{b5}+10\theta_1-10\theta_3=0$$
$$-P_{b6}+8\theta_2-8\theta_4=0$$
$$1\le P_{g1}\le 3$$
$$1\le P_{g2}\le 3$$
$$1\le P_{g3}\le 3$$
$$-\infty\le P_{b1}\le\infty$$
$$-\infty\le P_{b2}\le\infty$$
$$-\infty\le P_{b3}\le\infty$$
$$-\infty\le P_{b4}\le\infty$$
$$-\infty\le P_{b5}\le\infty$$
$$-\infty\le P_{b6}\le\infty$$

b. Provide your CPLEX code used to solve the above optimization problem.

Solution:

```
minimize
  1307 pg1 + 1211 pg2 + 1254 pg4
subject to
  theta1=0
  -pb1 + 10 theta1 - 10 theta4 = 0
  -pb2 + 10 theta1 - 10 theta2 = 0
  -pb3 + 10 theta2 - 10 theta3 = 0
  -pb4 - 10 theta3 + 10 theta4 = 0
  -pb5 + 10 theta1 - 10 theta3 = 0
  -pb6 + 8 theta2  - 8 theta4  =0
   pg1 - 30 theta1 + 10 theta2 + 10 theta3 + 10 theta4 = 0
   pg2 + 10 theta1 - 28 theta2 + 10 theta3 +8 theta4 = 1.0
        10 theta1 + 10 theta2 - 30 theta3 + 10 theta4 = 4
   pg4 + 10 theta1 + 8 theta2 + 10 theta3 - 28 theta4 = 0
  -pg1 <= -1
   pg1 <= 3
  -pg2 <= -1
   pg2 <= 3
  -pg4<= -1
   pg4 <= 3
  -pb1 <= 500
   pb1 <= 500
  -pb2 <= 500
   pb2 <= 500
  -pb3 <= 500
   pb3 <= 500
  -pb4 <= 500
   pb4 <= 500
  -pb5 <= 500
   pb5 <= 500
  -pb6 <= 500
   pb6 <= 500
Bounds
 -500  <= pb1 <= 500
 -500  <= pb2 <= 500
 -500  <= pb3 <= 500
 -500  <= pb4 <= 500
 -500  <= pb5 <= 500
 -500  <= pb6 <= 500
 -3.14159  <= theta1 <= 3.14159
 -3.14159  <= theta2 <= 3.14159
 -3.14159  <= theta3 <= 3.14159
 -3.14159  <= theta4 <= 3.14159
end
```

c. Use CPLEX to solve the LP stated in part b. In answering the below questions, make sure you specify the units.

i. Provide the value of the objective function at the optimal solution.
Solution: Z*=$6194

ii. Provide the values of the decision variables at the optimal solution. Typing "display solution variables - " provides
Solution: Pg1=1.0 pu, Pg2=3.0 pu, Pg4=1.0 pu.

iii. Provide the values of the auxiliary variables (angles and line flows) at the optimal solution. Make sure you specify line flows as $P_{bk}=P_{ij}$ where the flow direction is defined positive from bus i to bus j.
Solution: Typing "display solution variables - " provides
$\theta_1=0$ rad, $\theta_2=0.026389$ rad,
$\theta_3=-0.125$ rad, $\theta_4=-0.001389$ rad.
$P_{b1}=P_{14}=0.013880$pu
$P_{b2}=P_{12}=-0.263889$pu
$P_{b3}=P_{23}=1.513889$pu
$P_{b4}=P_{43}=1.0$ pu
$P_{b5}=P_{13}=1.25$pu
$P_{b6}=P_{24}=0.222222$pu

iv. Identify the locational marginal prices (LMPs) at each bus.
Solution: Typing "display solution dual - " provides
LMP_1=$1211/pu-hr or $12.11/MWhr
and it is the same for buses 2, 3, and 4.

v. How much will the objective function increase if the load at bus 2 changes from 1.0 pu to 1.01 pu?
Solution:
Under most circumstances, this is given by the LMP at bus 2, which is $12.11. However, if you actually resolve the optimization under the condition that the bus 2 load is 1.01 pu, you will find the objective function increases by $12.54, indicating that the additional load was served by P_{g4}. This is because in the original solution (when the bus 2 load was 1.0 pu), P_{g2} was at 3 pu, which is its upper

limit. But P_{g2}=3pu, not because the P_{g2} constraint was activated and held P_{g2} to the limit but because the solution settles to exactly P_{g2}=3pu. In other words, even if the constraint was increased to 3.01pu, P_{g2} would remain at 3pu (this would not happen if the constraint was active at the P_{g2}=3pu solution). And so the constraint is not binding and the computed LMP is $12.11. Yet, a 1 MW increase in load at bus 2 will cost $12.54/MWhr. The issue here is that LMPs are discontinuous due to the activation of constraints. In this case, the "jump" of the LMP at bus 2 (and indeed at all buses) from $12.11 to $12.54 occurs just when the system load is increased from 5 pu (bus 2 load of 1pu and bus 3 load of 4pu) to anything larger than 5 pu. Such discontinuities cause volatility in the LMPs. So although the LMP is $12.11, the answer is $12.54.

vi. How much will the objective function increase if the load at bus 3 changes from 4.0 pu to 4.01 pu?
Solution:
The situation here is the same as in part v. Although the dual variable of the bus 3 power injection equation is $12.11, the objective increases by $12.54 when the bus 3 load changes from 4.0 to 4.01 pu. So although the LMP is $12.11, the answer is $12.54.

vii. How much will the objective function increase if the lower generation limit for P_{g1} is increased to 101 MW?
Solution: By typing "display solution dual -" we find that the dual variable for the lower limit on Pg1 is -96$/pu-hr or -.96$/MWhr. But this constraint is written in CPLEX as "-pg1 <= -1" and so if the RHS increases by 0.01 pu (1 MW), to -.99, the objective will increase by -$.96, i.e., the objective will actually decrease by $.96. This means that if the RHS changes to -1.01, the objective will increase by $.96.

d. Constrain the flow limit on P_{b3}=P_{23} to 1.4 pu and resolve using CPLEX.

i. Provide the value of the objective function at the optimal solution.
   Solution: Z*=$6211.63
ii. Provide the values of the decision variables at the optimal solution. Typing “display solution variables - ” provides
   Solution: Pg1=1.0 pu, Pg2=2.59 pu, Pg4=1.41 pu.
iii. Provide the values of the auxiliary variables (angles and line flows) at the optimal solution. Make sure you specify line flows as $P_{bk}=P_{ij}$ where the flow direction is defined positive from bus i to bus j.
   Solution: Typing “display solution variables - ” provides
   $\theta_1=0$ rad, $\theta_2=0.015$ rad,
   $\theta_3=-0.125$ rad, $\theta_4=0.01$ rad.
   $P_{b1}=P_{14}=-0.1$pu
   $P_{b2}=P_{12}=-0.15$pu
   $P_{b3}=P_{23}=1.4$pu
   $P_{b4}=P_{43}=1.35$ pu
   $P_{b5}=P_{13}=1.25$pu
   $P_{b6}=P_{24}=0.04$pu
iv. Identify the locational marginal prices (LMPs) at each bus.
   Solution: Typing “display solution dual - ” provides
   LMP_1=$1251.85/pu-hr or $12.5185/MWhr
   LMP_2=$1211/pu-hr or $12.11/MWhr
   LMP_3=$1290.55/pu-hr or $12.9055/MWhr
   LMP_4=$1254/pu-hr or $12.54/MWhr
v. How much will the objective function increase if the load at bus 2 changes from 1.0 pu to 1.01 pu?
   Solution: This is given by the LMP at bus 2, which is $12.11.
vi. How much will the objective function increase if the load at bus 3 changes from 4.0 pu to 4.01 pu?
   Solution: This is given by the LMP at bus 3, which is $12.9055.

vii. How much will the objective function increase if the transmission limit on $P_{b3}=P_{23}$ is increased by 1 MW from 1.4 to 1.41 pu?

Solution: By typing “display solution dual -” we find that the dual variable for the transmission limit on P_{b3} is -154.8$/pu-hr or -1.548$/MWhr.

If the RHS increases by 0.01 pu (1 MW), to 1.41, the objective will increase by -$1.548, i.e., the objective will actually decrease by $1.548.