

**EE 458, Homework 2: Due Tuesday, September 17, 2019**

1. Assume that the fuel inputs  $R_1$  and  $R_2$ , in MBTU/hour for units 1 and 2, respectively, are a function of unit MW output powers  $P_1$  and  $P_2$ , respectively, and are given as

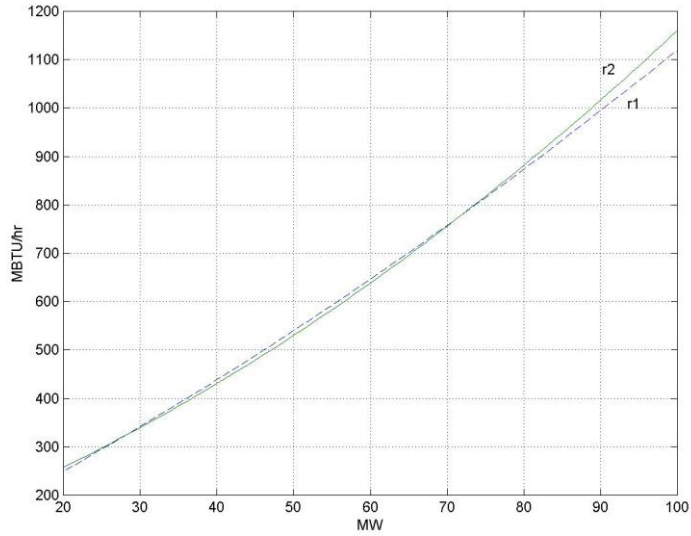
$$R_1 = 8P_1 + 0.024P_1^2 + 80$$

$$R_2 = 6P_2 + 0.044P_2^2 + 120$$

The minimum and maximum loadings for both units are 20 MW and 100 MW, respectively. All of the following plots should be given as a function of unit output power.

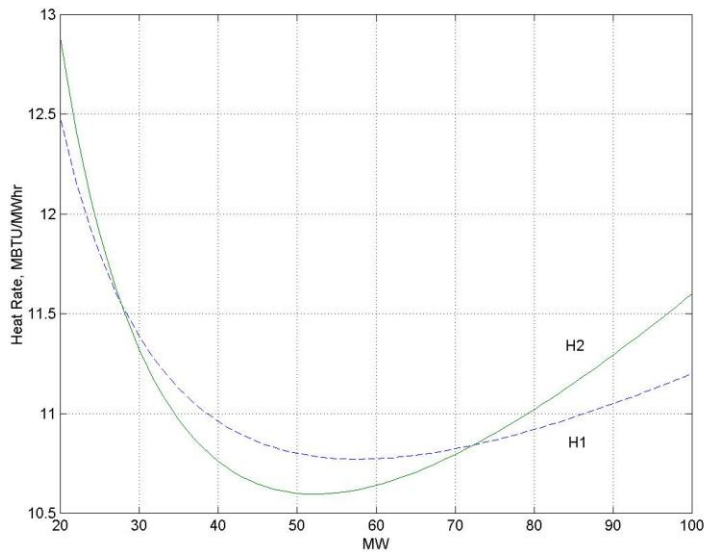
- a. Plot the input-output curve for each unit.

**Solution:** Here, we simply need to graph  $R_1$  and  $R_2$ , as below.



- b. Plot the heat rate curves for each unit.

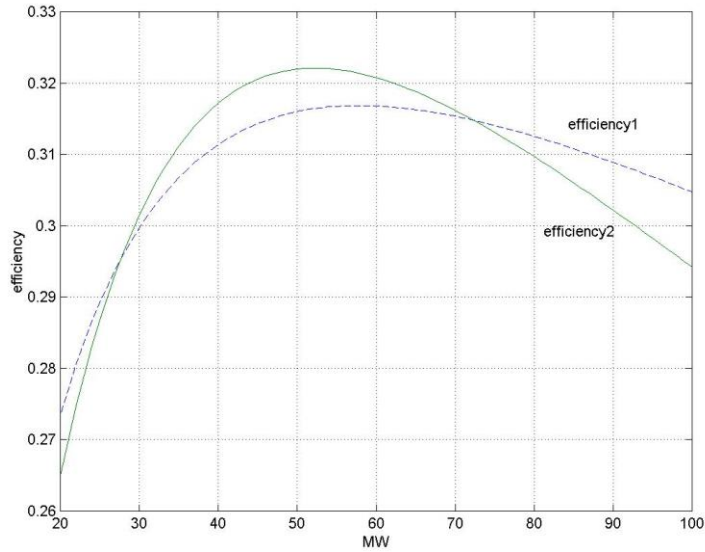
**Solution:** Here, we need to divide each value of  $R$  by corresponding value of  $P_g$ .



c. Plot efficiency for each unit.

**Solution:**

$$\eta = \frac{\text{MWhr} \times \left( 10^6 \frac{\text{watts}}{\text{MW}} \times 3600 \frac{\text{sec}}{\text{hr}} \right)}{H \text{ MBTU} \times \frac{10^6 \text{ BTU}}{\text{MBTU}} 1054.85 \frac{\text{joules}}{\text{BTU}}} = \frac{3.4128}{H}$$



d. Plot the incremental heat rate curves for each unit.

**Solution:** The fuel rates are:

$$R_1 = 8P_1 + 0.024P_1^2 + 80$$

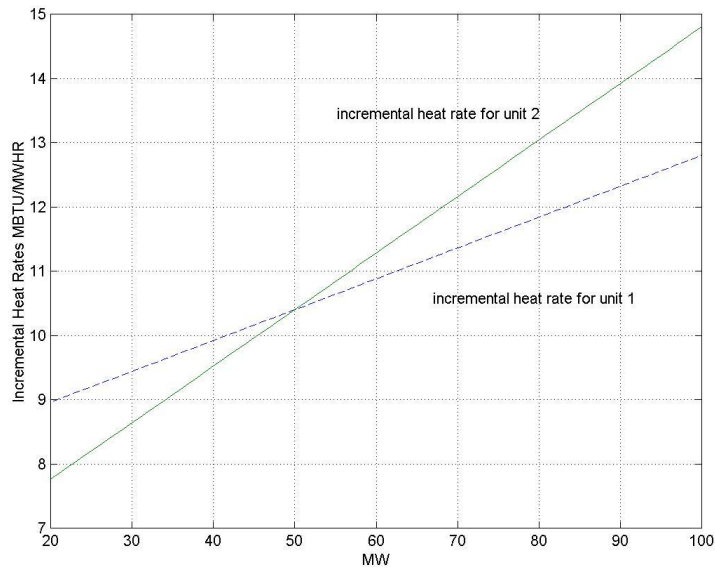
$$R_2 = 6P_2 + 0.044P_2^2 + 120$$

The incremental heat rates are just the derivatives:

$$IHR_1 = \frac{dR_1}{dP_1} = 8 + 0.048P_1$$

$$IHR_2 = \frac{dR_2}{dP_2} = 6 + 0.088P_2$$

And the plots are below.

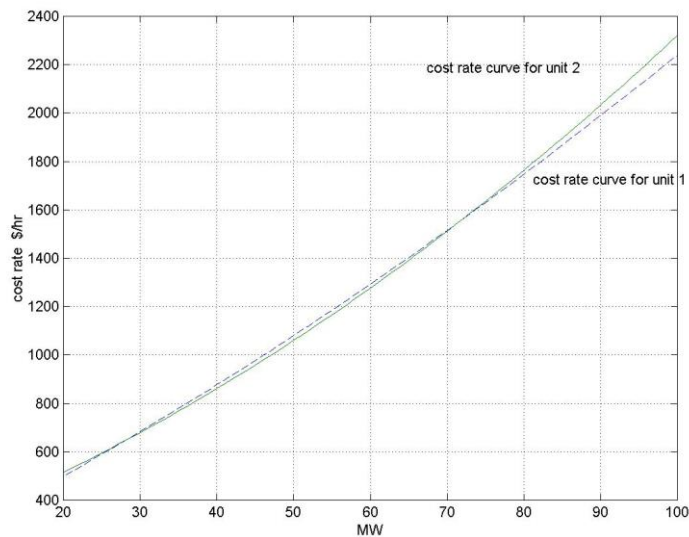


e. Assume the cost of fuel is \$2/MBTU. Plot the cost-rate curves for each unit.  
**Solution:** To get the cost rate curves, we simply multiply the fuel inputs by the fuel cost:

$$C_1 = KR_1 = 16P_1 + 0.048P_1^2 + 160$$

$$C_2 = KR_2 = 12P_2 + 0.088P_2^2 + 240$$

And the plots are as below.

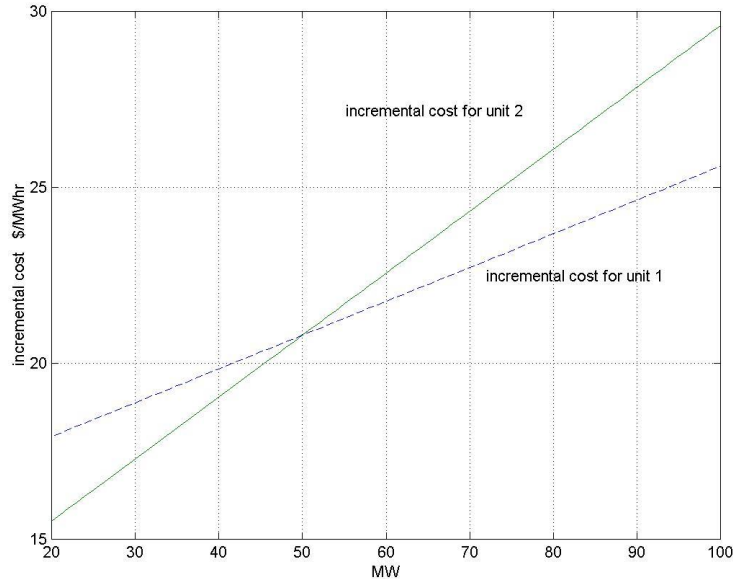


f. Plot the incremental cost-rate curves for each unit.  
**Solution:** The incremental cost-rate curves are derivatives of the cost-rate curves.

$$\frac{dC_1}{dP_1} = 16 + 0.096P_1$$

$$\frac{dC_2}{dP_2} = 12 + 0.176P_2$$

And the plots are below.



2. In class, we computed the emissions per unit electric energy produced by a power plant in lbs/MWhr as:

$$lbs / MWhr = EC \times \frac{3.41}{\eta}$$

where EC is the CO<sub>2</sub> emissions content in pounds per short ton of the coal and  $\eta$  is the average full load efficiency. For example, for subbituminous coal, we computed:

$$212.7 \frac{lbs}{MBTU_{IN}} \times \frac{1MBTU_{IN}}{.39MBTU_{OUT}} \times \frac{3.41MBTU}{MWhr} = 1859.8 lbs / MWhr$$

- a. Evaluate the average full load heat rate of the power plant used in the above calculation; make sure to give units.

**Solution:**

$$HR = \frac{3.41}{\eta} = \frac{3.41}{0.39} = 8.74 \text{ MMBTU/MWhr}$$

- b. Repeat the calculation for lbs/MWhr if the average full load heat rate is 10.0 MMBTU/MWhr.

**Solution:**

$$lbs / MWhr = EC \times HR = 212.7 \times 10 = 2127 lbs / MWhr$$

- c. Repeat the calculation for lbs/MWhr using the original heat rate, i.e., the one computed in part (a), but assume that bituminous coal is used.

**Solution:**

Reference to Table 5 in the notes shows that  $EC=227.4 \text{ lbs/MMBTU}$ . Therefore

$$lbs/MWhr = EC \times HR = 205.3 \times 8.74 = 1794 lbs/MWhr$$

- d. The Regional Greenhouse Gas Initiative ([www.rggi.org](http://www.rggi.org)) conducts an auction to sell GHG allowances. The clearing price at the June 2011 auction was \$1.89/allowance, where 1 allowance is a short ton of CO<sub>2</sub> emitted. Assume
  - the price of subbituminous (Powder River Basin) coal is \$15/ton and its energy content is 17.45MMBTU/ton;
  - the price of bituminous coal is \$60/ton and its energy content is 24MMBTU/ton.

For each type, compute the percentage increase in cost due to the emissions cost rate of producing a MWhr. Use the original heat rate, i.e., the one computed in part (a).

**Solution:**

When using subbituminous coal, the energy cost rate will be:

$$C_{energy} = \frac{\$15}{COALton} \times \frac{1COALton}{17.45MMBTU} \times \frac{8.74MMBTU}{MWhr} = \$7.51/MWhr$$

and the emissions cost rate will be

$$C_{emissions} = \frac{\$1.89}{CO2ton} \times \frac{1859.8lbs}{MWhr} \times \frac{CO2ton}{2000lbs} = \$1.76/MWhr$$

And so the percentage increase in cost of producing a MWhr will be

$$\frac{1.76}{7.51} = 23.3\%$$

When using bituminous coal, the energy cost rate will be:

$$C_{energy} = \frac{\$60}{COALton} \times \frac{1COALton}{24MMBTU} \times \frac{8.74MMBTU}{MWhr} = \$21.85/MWhr$$

and the emissions cost rate will be

$$C_{emissions} = \frac{\$1.89}{CO2ton} \times \frac{2217lbs}{MWhr} \times \frac{CO2ton}{2000lbs} = \$2.10/MWhr$$

And so the percentage increase in cost rate of producing a MWhr will be

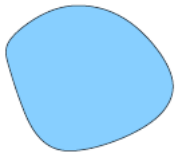
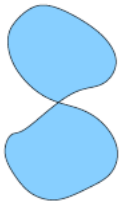



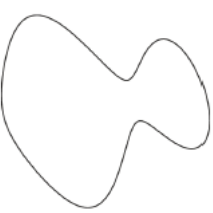
$$\frac{2.10}{21.85} = 9.61\%$$

- e. Go to the RGGI website and identify the clearing price for the 45<sup>th</sup> RGGI auction on September 4, 2019/allowance.

**Solution:**

From [www.rggi.org](http://www.rggi.org), we learn that the clearing price for the 45<sup>th</sup> RGGI auction was \$5.20/allowance. Seems the CO<sub>2</sub> price has gone up since 2011!

- 3. Determine which of the below are convex sets.

- a.  Convex
- b.  Nonconvex
- c.  Nonconvex
- d.  Convex
- e.  Convex
- f.  Nonconvex

4. Apply KKT conditions to the following optimization problems. Identify the solution(s) and the value of the objective function at the solution(s). Indicate whether your answer(s) is (are) a global optimum or not and how you know.

a.  $\text{Min } f(x_1, x_2) = x_1^2 + x_2^2$  subject to  $h(x_1, x_2) = x_1 + 2x_2 = 5$

**Solution:**

We form the Lagrangian and then apply the KKT conditions.

$$L(x_1, x_2, \lambda) = x_1^2 + x_2^2 - \lambda(x_1 + 2x_2 - 5)$$

$$\frac{\partial L}{\partial x_1} = 2x_1 - \lambda = 0$$

$$\frac{\partial L}{\partial x_2} = 2x_2 - 2\lambda = 0$$

$$\frac{\partial L}{\partial \lambda} = x_1 + 2x_2 - 5 = 0$$

$$\begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & -2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ \lambda \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

This is a global optimum because  $f$  is a convex function and the one equality constraint defines that the feasible region is a convex set, and so this is a convex programming problem.

- b. Min  $f(x_1, x_2) = x_1^2 + x_2^2$  subject to  $h(x_1, x_2) = 3x_1x_2 = 5$

**Solution:**

We form the Lagrangian and then apply the KKT conditions.

$$L(x_1, x_2, \lambda) = x_1^2 + x_2^2 - \lambda(3x_1x_2 - 5)$$

$$\frac{\partial L}{\partial x_1} = 2x_1 - 3\lambda x_2 = 0$$

$$\frac{\partial L}{\partial x_2} = 2x_2 - 6\lambda x_1 = 0$$

$$\frac{\partial L}{\partial \lambda} = 3x_1x_2 - 5 = 0$$

These are nonlinear equations. We can solve them by substitution:

$$2x_1 - 3\lambda x_2 = 0 \Rightarrow \lambda = \frac{2x_1}{3x_2}$$

$$2x_2 - 6\lambda x_1 = 0 \Rightarrow 2x_2 - 6 \left[ \frac{2x_1}{3x_2} \right] x_1 = 0 \Rightarrow 2x_2 - \left[ \frac{4x_1}{x_2} \right] x_1 = 0 \Rightarrow x_2^2 = 2x_1^2$$

$$\frac{\partial L}{\partial \lambda} = 3x_1x_2 - 5 = 0 \Rightarrow 3x_1(2x_1^2) - 5 = 0 \Rightarrow 6x_1^3 = 5 \Rightarrow x_1 = 0.9410$$

$$\Rightarrow x_2^2 = 2x_1^2 = 2(0.9410)^2 \Rightarrow x_2 = \pm 1.3308.$$

So there are two solutions:  $(x_1, x_2)^* = (0.9410, 1.3308), (0.9410, -1.3308)$ .

In this case, the objective is convex but the feasible region is not a convex set, and so, although we have two local solutions, we cannot guarantee either one is a global solution. But substitution of each of them into the objective function yields the same value (2.6566). Exploration of the

feasible region shows there are no better solutions than these two, but because we have two solutions, there is no single global best solution.

5. A three-unit system is given by the following data. The total system demand is 1100MW. Generator constraints are  $0 < P_{g1} < 550$ ,  $0 < P_{g2} < 300$ ,  $0 < P_{g3} < 300$

$$\begin{aligned} C_1(P_{g1}) &= 0.010 \cdot (P_{g1})^2 + 0.3 \cdot (P_{g1}) + 1 \\ C_2(P_{g2}) &= 0.030 \cdot (P_{g2})^2 + 0.2 \cdot (P_{g2}) + 3 \\ C_3(P_{g3}) &= 0.020 \cdot (P_{g3})^2 + 0.9 \cdot (P_{g3}) + 5 \end{aligned}$$

- Identify the objective function for this optimization problem.
- Identify the Lagrangian function assuming no constraints are binding.
- Identify the KKT conditions assuming no constraints are binding.
- Find the solution to the problem assuming no constraints are binding.
- Find the solution to the problem accounting for any binding constraints.
- Find the total cost of supplying the 1100MW using the solution found in part (e)
- Approximate the total cost of supplying the 1100MW change if the upper limit on generator 1 was increased from 550MW to 560MW.

**Solution:**

(a)  $f(P_g) = C_1(P_{g1}) + C_2(P_{g2}) + C_3(P_{g3})$

(b)  $L[\lambda, P_g] = f(P_g) - \lambda [P_{g1} + P_{g2} + P_{g3} - 1100]$

(c)

$$\frac{\partial L}{\partial P_{g1}} = 0.02P_1 + 0.3 - \lambda = 0$$

$$\frac{\partial L}{\partial P_{g2}} = 0.06P_2 + 0.2 - \lambda = 0$$

$$\frac{\partial L}{\partial P_{g3}} = 0.04P_3 + 0.9 - \lambda = 0$$

$$\frac{\partial L}{\partial \lambda} = P_1 + P_2 + P_3 - 1100 = 0$$

(d)

$$\begin{bmatrix} 0.02 & 0 & 0 & -1 \\ 0 & 0.06 & 0 & -1 \\ 0 & 0 & 0.04 & -1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} P_{g1} \\ P_{g2} \\ P_{g3} \\ \lambda \end{bmatrix} = \begin{bmatrix} -0.3 \\ -0.2 \\ -0.9 \\ 1100 \end{bmatrix}$$

$$\Rightarrow P_{g1} = 607.3MW$$

$$\Rightarrow P_{g2} = 204.1MW$$



$$\Rightarrow \underline{P_{g3} = 288.6MW}$$

$$\Rightarrow \underline{\lambda = 12.45\$ / MW - hr}$$

(a) We note that  $P_{g1}$  violates its upper limit. Therefore we add in the equation  $P_{g1}=550$  and also argument the KKT condition for the first equation to be  $0.02P_{g1} + 0.3 - \lambda - \mu_1 = 0$ . So our equation become

$$\begin{bmatrix} 0.02 & 0 & 0 & -1 & -1 \\ 0 & 0.06 & 0 & -1 & 0 \\ 0 & 0 & 0.04 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} P_{g1} \\ P_{g2} \\ P_{g3} \\ \lambda \\ \mu_1 \end{bmatrix} = \begin{bmatrix} -0.3 \\ -0.2 \\ -0.9 \\ 550 \\ 1100 \end{bmatrix}$$

$$\Rightarrow P_{g1} = 550MW$$

$$\Rightarrow P_{g2} = 227MW$$

$$\Rightarrow P_{g3} = 323MW$$

$$\Rightarrow \lambda = 13.82\$ / MW - hr$$

$$\Rightarrow \mu_1 = 2.52\$ / MW - hr$$

But  $P_{g3}$  violates its upper limit so we must reformulate again:

$$\begin{bmatrix} 0.02 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0.06 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0.04 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} P_{g1} \\ P_{g2} \\ P_{g3} \\ \lambda \\ \mu_1 \\ \mu_3 \end{bmatrix} = \begin{bmatrix} -0.3 \\ -0.2 \\ -0.9 \\ 550 \\ 300 \\ 1100 \end{bmatrix}$$

$$\Rightarrow \underline{P_{g1} = 550MW}$$

$$\Rightarrow \underline{P_{g2} = 250MW}$$

$$\Rightarrow \underline{P_{g3} = 300MW}$$

$$\Rightarrow \underline{\lambda = 15.2\$ / MW - hr}$$

$$\Rightarrow \underline{\mu_1 = -3.9\$ / MW - hr}$$

$$\Rightarrow \underline{\mu_3 = -2.3\$ / MW - hr}$$

(f)  $C_1(550) + C_2(250) + C_3(300) = 3191 + 1928 + 2075 = \underline{7194}$

(g)  $7194 - (3.9) \cdot (10) = \underline{7155}$