

# Symmetrical Components 1

## 1.0 Introduction

These notes should be read together with Section 12.1 of your text.

When performing steady-state analysis of high voltage transmission systems, we make use of the per-phase equivalent circuit.

Also, when performing *symmetrical* fault (three-phase fault) analysis of high-voltage transmission systems, we make use of the per-phase equivalent circuit.

But for unsymmetrical faults (single line to ground, two line to ground, and line to line) analysis, the three phases no longer see the same impedance, which violates the basic requirement of per-phase analysis (phases must be balanced).

There is a very elegant approach available for analyzing unsymmetric three-phase circuits. The approach was developed by a man named Charles Fortescue and reported in a famous paper in 1918. It is now called the method of symmetrical components. We will spend a little time studying this method in order to understand how to use it in unsymmetrical fault analysis.

Charles Le Geyt FORTESCUE Professor was born about 1878 in Keewantin, Northwest Territories, Canada. Hudson Bay Factory where the Hayes River enters Hudson Bay He immigrated in 1901 to USA. He appeared in the census on 11 April 1930 in Pittsburgh, USA. Charles died on 4 December 1936 in Pittsburgh, USA. The Charles LeGeyt Fortescue Scholarship was established in 1939 at MIT as a memorial to Charles LeGeyt in recognition of his valuable contributions to the field of electrical engineering

Charles LeGeyt Fortescue, born at York Factory, Manitoba, 1876, son of chief factor of Hudson Bay Company-was the first electrical engineering graduate of Queen's University.

After graduation Fortescue joined Westinghouse Electric and Manufacturing Company at East Pittsburgh and attained universal fame for his contributions to the engineering principles and analysis of power transmission and distribution systems. He is especially noted for development of polyphase systems analysis by the symmetrical components method. He made his way, eventually, to MIT where he became a very well known and respected professor.

Its fascinating that Cecil Lewis Fortescue born 1881 also became a Professor of Electrical Engineering in London University, in the same period. One wonders if they heard about each other?

Charles Le Geyt FORTESCUE Professor and Louise Cameron WALTER were married about 1905. Louise Cameron WALTER<sup>113</sup> was born about 1885 in Pennsylvania, USA. She was a Sculptor

Charles Le Geyt FORTESCUE Professor and Louise Cameron WALTER had the following children:  
Jane Faithful FORTESCUE.



## **METHOD OF SYMMETRICAL CO-ORDINATES APPLIED TO THE SOLUTION OF POLYPHASE NETWORKS**

BY C. L. FORTESCUE

### **ABSTRACT OF PAPER**

In the introduction a general discussion of unsymmetrical systems of co-planar vectors leads to the conclusion that they may be represented by symmetrical systems of the same number of vectors, the number of symmetrical systems required to define the given system being equal to its degrees of freedom. A few trigonometrical theorems which are to be used in the paper are called to mind. The paper is subdivided into three parts, an abstract of which follows. It is recommended that only that part of Part I up to formula (33) and the portion dealing with star-delta transformations be read before proceeding with Part II.

*Part I* deals with the resolution of unsymmetrical groups of numbers into symmetrical groups. These numbers may represent rotating vectors of systems of operators. A new operator termed the sequence operator is introduced which simplifies the manipulation. Formulas are derived for three-phase circuits. Star-delta transformations for symmetrical co-ordinates are given and expressions for power deduced. A short discussion of harmonics in three-phase systems is given.

*Part II* deals with the practical application of this method to unsymmetrical rotating machines operating on unsymmetrical circuits. General formulas are derived and such special cases, as the single-phase induction motor, synchronous motor-generator, phase converters of various types, are discussed.

## **2.0 Symmetrical Components: Motivation**

Def: A symmetrical set of phasors have equal magnitude & are  $120^\circ$  out of phase.

Goal: Decompose a set of three unsymmetrical phasors into

- One unsymmetric but equal set of 3
- Two symmetrical sets of 3

Then we can analyze each set individually and use superposition to obtain the composite result.

In what follows, we demonstrate that:

- **Step 1:** An unsymmetrical set, not summing to 0, can be decomposed into two unsymmetrical sets:
  - an “equal” set and an
  - unsymmetrical set that does sum to 0;
- **Step 2:** An unsymmetrical set that sums to 0 can be decomposed into two symmetrical sets

**Step 1:** Consider a set of phasors that do not add to zero (because of different magnitudes or because of angular separation different than  $120^\circ$  or because of both). Assume that they have phase sequence a-b-c.

Add them up, as in Fig. 1, i.e.,

$$V_R^0 = V_a + V_b + V_c \quad (1)$$

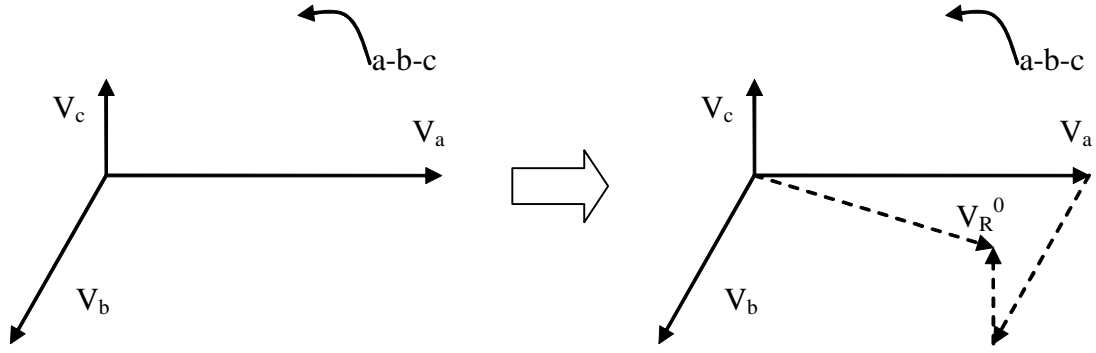


Fig. 1: Addition of Unsymmetrical Phasors

So we see from (1) that

$$V_a + V_b + V_c - V_R^0 = 0 \quad (2)$$

Define:

$$V_0 = \frac{1}{3} V_R^0 \quad (3)$$

Then:

$$V_a + V_b + V_c - 3V_0 = 0 \quad (4)$$

$$(V_a - V_0) + (V_b - V_0) + (V_c - V_0) = 0 \quad (5)$$

Define:

$$\begin{aligned} V_A &= (V_a - V_0) \\ V_B &= (V_b - V_0) \\ V_C &= (V_c - V_0) \end{aligned} \quad (6)$$

Then:

$$V_A + V_B + V_C = 0 \quad (7)$$

Conclusion: We obtain an unsymmetrical set of voltages that sum to 0 by subtracting  $V_0$  from each original phasor, where  $V_0$  is  $1/3$  of the resultant phasor, illustrated in Fig. 2.

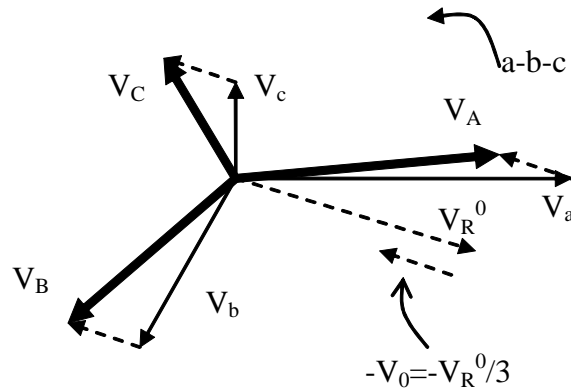


Fig. 2: Subtracting  $V_0$  from unsymmetrical phasors

**Step 2:** How to decompose  $V_A$ ,  $V_B$ , and  $V_C$  into two symmetrical sets?

Can we decompose  $V_A$ ,  $V_B$ ,  $V_C$  into 2 a-b-c symmetrical sets?

As a test, try to add any 2 a-b-c symmetrical sets and see what you get. See Fig. 3.

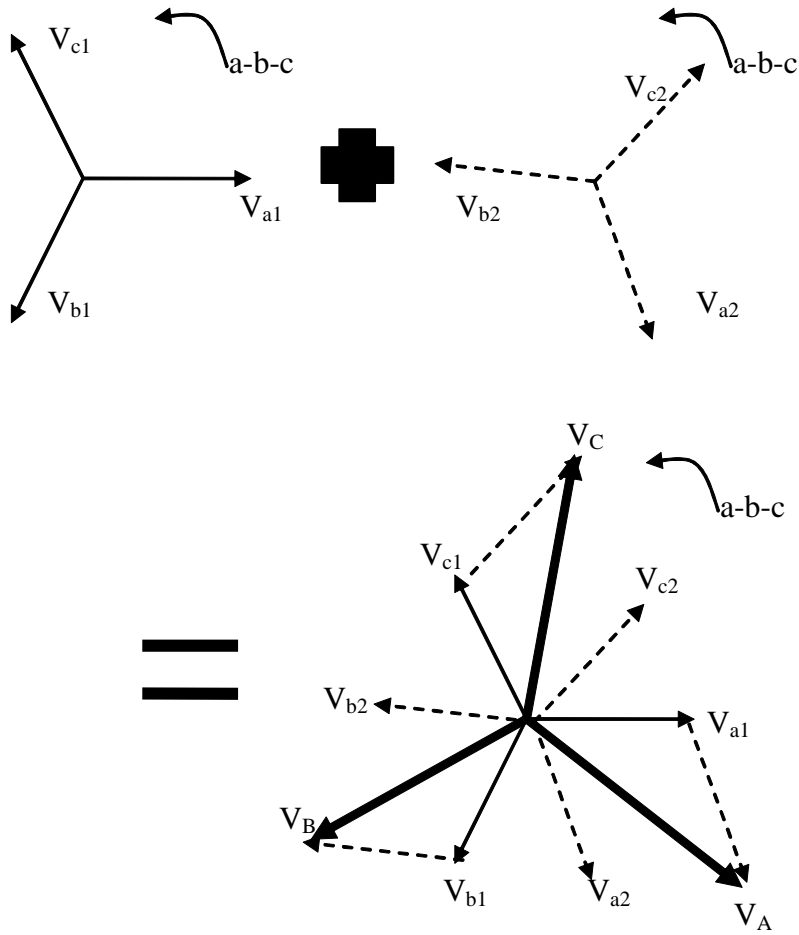


Fig. 3: Adding 2 symmetrical a-b-c sets

Note that in adding the 2 phasor sets, we add the two a-phase phasors, the two b-phase phasors, and the two c-phase phasors.

One can observe from Fig. 3 that the resultant phasor set, denoted by the solid lines, are in fact symmetrical!

It is possible to prove mathematically that the sum of any 2 a-b-c symmetrical sets is always another symmetrical set.

Let's try a different thing. Let's try to add two symmetrical sets, but let's have one be a-b-c (called *positive sequence*) and another be a-c-b (called *negative sequence*).

As before, in adding the 2 phasor sets, we add the two a-phase phasors, the two b-phase phasors, and the two c-phase phasors. The result of our efforts is shown in Fig. 4.



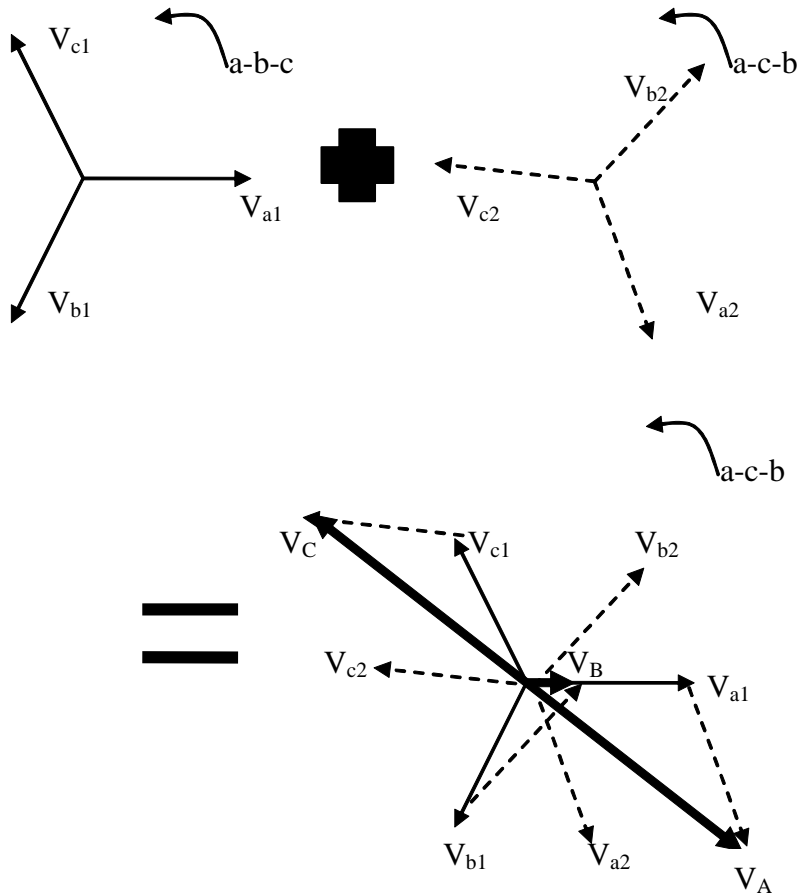


Fig. 4: Adding a symmetrical a-b-c set to a symmetrical a-c-b set

The resultant phasor set is unsymmetrical!

We can guarantee that the three phasors in this unsymmetrical phasor set sums to zero, since we obtained it by adding two phasor sets that sum to zero, i.e.,

$$\begin{array}{r}
V_{a1} + V_{b1} + V_{c1} = 0 \\
V_{a2} + V_{b2} + V_{c2} = 0 \\
\hline
V_A + V_B + V_C = 0
\end{array}
\tag{8}$$

Now consider Fig. 4 again. Assume that someone hands you the unsymmetrical set of phasors  $V_A$ ,  $V_B$ , and  $V_C$ .

Can you decompose them into the two symmetrical sets?

Can you be assured that two such symmetrical sets exist?

The answer is yes, you can be assured that two such symmetrical sets exist. Fortescue's paper contains the proof.

I simply argue that the three phasors given in Fig. 4,  $V_A$ ,  $V_B$ , and  $V_C$ , are quite general (there is nothing special about them), with the single exception that they sum to zero.

Claim: We can represent ANY unsymmetrical set of 3 phasors that sum to 0 as the sum of 2 constituent symmetrical sets:

- A positive (a-b-c) sequence set and
- A negative (a-c-b) sequence set.

Given this claim, then the following theorem holds.

Theorem: We can represent ANY unsymmetrical set of 3 phasors as the sum of 3 constituent sets, each having 3 phasors:

- A positive (a-b-c) sequence set and
- A negative (a-c-b) sequence set and
- An equal set

These three sets we will call, respectively,

- Positive  $(V_a^+, V_b^+, V_c^+)$
- Negative  $(V_a^-, V_b^-, V_c^-)$
- zero  $(V_a^0, V_b^0, V_c^0)$

sequence components.

The implication of this theorem is that any unsymmetrical set of 3 phasors  $V_a$ ,  $V_b$ ,  $V_c$  can be written in terms of the above sequence components in the following way:

$$\begin{aligned}V_a &= V_a^0 + V_a^+ + V_a^- \\V_b &= V_b^0 + V_b^+ + V_b^- \\V_c &= V_c^0 + V_c^+ + V_c^-\end{aligned}\tag{9}$$

We can write the equations of (9) in a more compact fashion, but first, we must describe a mathematical operator that is essential.

### 3.0 The $\alpha$ -operator

To begin on familiar ground, we are all conversant with the operator “j” which is used in complex numbers.

Remember that “j” is actually a vector with a magnitude and an angle:

$$j = 1\angle 90^\circ\tag{10}$$

In the same way, we are going to define the “ $\alpha$ ” operator as:

$$\alpha = 1\angle 120^\circ = -0.5000 + j0.8660 \quad (11)$$

It is easy to show the following relations:

$$\alpha^2 = 1\angle -120^\circ \quad (12)$$

$$\alpha^3 = 1\angle 0^\circ \quad (13)$$

$$\alpha^4 = 1\angle 120^\circ = \alpha \quad (14)$$

We also have that:

$$1 + \alpha = -\alpha^2 = 1\angle 60^\circ \quad (15)$$

as illustrated in Fig. 5.

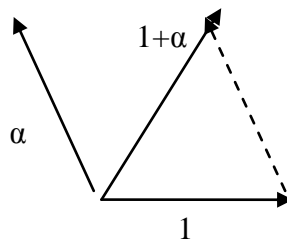


Fig. 5: Illustration of  $1+\alpha$

Note that

$$-\alpha^2 = -1\angle 240^\circ = 1\angle 60^\circ \quad (16)$$

Similarly, we may show that:

$$1 + \alpha^2 = -\alpha = 1 \angle -60^\circ \quad (17)$$

$$1 - \alpha = \sqrt{3} \angle -30^\circ \quad (18)$$

$$1 - \alpha^2 = \sqrt{3} \angle 30^\circ \quad (19)$$

$$\alpha - 1 = \sqrt{3} \angle 150^\circ \quad (20)$$

$$\alpha^2 - 1 = \sqrt{3} \angle -150^\circ \quad (21)$$

And there are many more relations like this that are sometimes helpful when dealing with symmetrical components. (See the text called “Analysis of faulted power systems” by Paul Anderson, pg. 17.)

#### **4.0 Symmetrical components: the math**

We repeat equations (9) below for convenience:

$$\begin{aligned} V_a &= V_a^0 + V_a^+ + V_a^- \\ V_b &= V_b^0 + V_b^+ + V_b^- \\ V_c &= V_c^0 + V_c^+ + V_c^- \end{aligned} \quad (9)$$

We can relate the three different quantities having the same superscript.

- Zero sequence quantities: These quantities are all equal, i.e.,

$$V_a^0 = V_b^0 = V_c^0 \quad (22)$$

- Positive sequence quantities: The relation between these quantities can be observed immediately from the phasor diagram and can be expressed using the  $\alpha$ -operator.

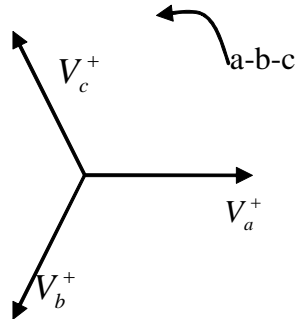


Fig. 6: Positive sequence components

$$V_b^+ = \alpha^2 V_a^+$$

$$V_c^+ = \alpha V_a^+ \quad (23)$$

- Negative sequence quantities: The relation between these quantities can be observed immediately from the phasor diagram and can be expressed using the  $\alpha$ -operator.

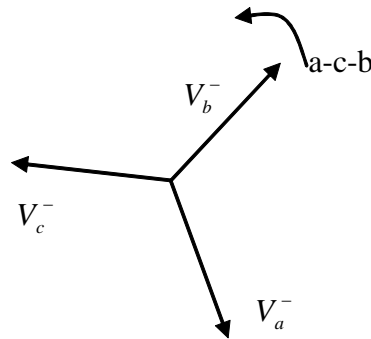


Fig. 8: Negative sequence components

$$V_b^- = \alpha V_a^-$$

$$V_c^- = \alpha^2 V_a^- \quad (24)$$

Now let's use equations (22), (23), and (24) to express the original phasors  $V_a$ ,  $V_b$ ,  $V_c$  in terms of only the a-phase components

$$V_a^0, V_a^+, V_a^-,$$

i.e., we will eliminate the b-phase components

$$V_b^0, V_b^+, V_b^-$$



and the c-phase components

$$V_c^0, V_c^+, V_c^-$$

This results in

$$\begin{aligned} V_a &= V_a^0 + V_a^+ + V_a^- \\ V_b &= V_a^0 + \alpha^2 V_a^+ + \alpha V_a^- \\ V_c &= V_a^0 + \alpha V_a^+ + \alpha^2 V_a^- \end{aligned} \quad (9)$$

So we have written the abc quantities (phase quantities) in terms of the 0+- quantities (sequence quantities) of the a-phase. We can write this in matrix form as:

$$\begin{bmatrix} V_a \\ V_b \\ V_c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha^2 & \alpha \\ 1 & \alpha & \alpha^2 \end{bmatrix} \begin{bmatrix} V_a^0 \\ V_a^+ \\ V_a^- \end{bmatrix} \quad (25)$$

Defining

$$\underline{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha^2 & \alpha \\ 1 & \alpha & \alpha^2 \end{bmatrix} \quad (26)$$

we see that eq. (25) can be written as:

$$\begin{bmatrix} V_a \\ V_b \\ V_c \end{bmatrix} = \underline{A} \begin{bmatrix} V_a^0 \\ V_a^+ \\ V_a^- \end{bmatrix} \quad (27)$$

We may also obtain the 0+- (sequence) quantities from the abc (phase) quantities:

$$\begin{bmatrix} V_a^0 \\ V_a^+ \\ V_a^- \end{bmatrix} = \underline{A}^{-1} \begin{bmatrix} V_a \\ V_b \\ V_c \end{bmatrix} \quad (28)$$

where

$$\underline{A}^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha \end{bmatrix} \quad (29)$$

Equations 22-29 hold for

- Line-to-line voltages
- Line-to-neutral voltages
- Line currents
- Phase currents