HW3 Solutions

1. Problem 9.8 (see end of this document)
2. Consider the 4-bus system shown below. Both machines have subtransient reactances of 0.20 pu (you can combine the machine subtransient reactance with the transformer impedance to get a single reactance connecting the machine internal voltage with the network).

![Diagram of the 4-bus system with reactances](image)

a. Construct the Y-bus for this network (should be a 4×4 matrix).

b. Consider that there is a three-phase (symmetrical) fault at bus 2.
   ii. Use LU decomposition to obtain the 2 nd column of the Z-bus.
   iii. Compute the subtransient fault current.
   iv. Use eq. (12) to find the voltages during the fault.
   v. Use eq. (17) to find the subtransient currents in lines 3-2, 1-2, and 4-2.

Solution:
a). Compute the Y-Bus

\[
Y_{bus} = \begin{bmatrix}
\frac{1}{j.25} + \frac{1}{j.125} + \frac{1}{j.40} & -1 & -1 & -1 \\
-1 & \frac{1}{j.25} + \frac{1}{j.125} + \frac{1}{j.2} & -1 & -1 \\
-1 & -1 & \frac{1}{j.25} + \frac{1}{j.125} + \frac{1}{j.3} & 0 \\
-1 & -1 & \frac{1}{j.25} & 0
\end{bmatrix}
\]

\[
Y_{bus} = \begin{bmatrix}
-j14.5 & j8 & j4 & j2.5 \\
j8 & -j17 & j4 & j5 \\
j4 & j4 & -j11.3333 & 0 \\
j2.5 & j5 & 0 & -j10.83333
\end{bmatrix}
\]

b).

i). LU Decomposition to obtain \( Z_2 \)

\[
Y_{bus}Z_{bus} = I
\]

\[
Y_{bus}Z_2 = \begin{bmatrix}
0 \\
1 \\
0 \\
0
\end{bmatrix}
\]

Factorization of a matrix \( Y \) can be done efficiently and easily using the matlab command:

\([L, U] = \text{lu}(Y)\)

Then it is easy to find \( w \) by hand using forward substitution from:

\[
Lw = I_k
\]

\[
Lw = \begin{bmatrix}
0 \\
1 \\
0 \\
0
\end{bmatrix}
\]

And then it is easy to find \( Z_2 \) by hand using backwards substitution from:

\[
UZ_2 = w
\]

Alternatively, the manual steps of LU decomposition can be performed per the notes from “LU Decomposition.” I did it within Matlab, as follows. Here, \( y1 \) is the initial “augmented” matrix (see notes on LU decomposition for the meaning of this term).
>> y1=[-14.5i,8i,4i,2.5i,0;8i,-17i,4i,5i,1;4i,4i,-11.333333i,0,0;2.5i,5i,0,-10.833333i,0]

y1 =
0 -14.5000i 0 + 8.0000i 0 + 4.0000i 0 + 2.5000i 0
0 + 8.0000i 0 -17.0000i 0 + 4.0000i 0 + 5.0000i 1.0000
0 + 4.0000i 0 + 4.0000i 0 -11.3333i 0 0
0 + 2.5000i 0 + 5.0000i 0 0 -10.8333i 0

>> y2=[y1(1,:)/y1(1,1);y1(2,:);y1(3,:);y1(4,:)]

y2 =
1.0000 -0.5517 -0.2759 -0.1724 0
0 + 8.0000i 0 -17.0000i 0 + 4.0000i 0 + 5.0000i 1.0000
0 + 4.0000i 0 + 4.0000i 0 -11.3333i 0 0
0 + 2.5000i 0 + 5.0000i 0 0 -10.8333i 0

>> y3=[y2(1,:)*y1(2,2)+y2(2,:);y2(1,:)*y1(3,3)+y2(3,:);y2(1,:)*y1(4,4)+y2(4,:)]

y3 =
1.0000 -0.5517 -0.2759 -0.1724 0
0 +25.0000i 0 -26.3793i 0 - 0.6897i 0 + 2.0690i 1.0000
0 +15.3333i 0 - 2.2529i 0 -14.4598i 0 - 1.9540i 0
0 +13.3333i 0 - 0.9770i 0 - 2.9885i 0 -12.7011i 0

>> y3=[y2(1,:)*y1(2,1)+y2(2,:);y2(1,:)*y1(3,1)+y2(3,:);y2(1,:)*y1(4,1)+y2(4,:)]

y3 =
1.0000 -0.5517 -0.2759 -0.1724 0
0 0 -12.5862i 0 + 6.2069i 0 + 6.3793i 1.0000
0 0 + 6.2069i 0 -10.2299i 0 + 0.6897i 0
0 0 + 6.3793i 0 + 0.6897i 0 -10.4023i 0

>> y4=[y3(1,:)/y3(1,1);y3(2,:);y3(3,:);y3(4,:)]

y4 =
1.0000 -0.5517 -0.2759 -0.1724 0
0 1.0000 -0.4932 -0.5068 0 + 0.0795i
0 0 + 6.2069i 0 -10.2299i 0 + 0.6897i 0
0 0 + 6.3793i 0 + 0.6897i 0 -10.4023i 0
>> y5=[y4(1,:);y4(2,:);y4(2,:)*y3(3,2)+y4(3,:);y4(2,:)*y3(4,2)+y4(4,:)]
y5 =
1.0000  -0.5517  -0.2759  -0.1724  0
0  1.0000  -0.4932  -0.5068  0 + 0.0795i
0  0  0 - 7.1689i  0 + 3.8356i  0.4932
0  0  0 + 3.8356i  0 - 7.1689i  0.5068

>> y6=[y5(1,:);y5(2,:);y5(3,:)/y5(3,3);y5(4,:)]
y6 =
1.0000  -0.5517  -0.2759  -0.1724  0
0  1.0000  -0.4932  -0.5068  0 + 0.0795i
0  0  1.0000  -0.5350  0 + 0.0688i
0  0  0 + 3.8356i  0 - 7.1689i  0.5068

>> y7=[y6(1,:);y6(2,:);y6(3,:)-y5(4,3)+y6(4,:)]
y7 =
1.0000  -0.5517  -0.2759  -0.1724  0
0  1.0000  -0.4932  -0.5068  0 + 0.0795i
0  0  1.0000  -0.5350  0 + 0.0688i
0  0  0  0 - 5.1168i  0.7707

>> y8=[y7(1,:);y7(2,:);y7(3,:)/y7(4,4)]
y8 =
1.0000  -0.5517  -0.2759  -0.1724  0
0  1.0000  -0.4932  -0.5068  0 + 0.0795i
0  0  1.0000  -0.5350  0 + 0.0688i
0  0  0  1.0000  0 + 0.1506i

>> z4=y8(4,5)
z4 =
0 + 0.1506i

>> z3=y8(3,5)-y8(3,4)*z4
z3 =
\[ 0 + 0.1494i \]

\[
\gg \ z2=y8(2,5)-y8(2,4)*z4-y8(2,3)
\]

\[ z2 = 0.4932 + 0.1558i \]

\[
\gg \ z2=y8(2,5)-y8(2,4)*z4-y8(2,3)*z3
\]

\[ z2 = 0 + 0.2295i \]

\[
\gg \ z1=y8(1,5)-y8(1,4)*z4-y8(1,3)*z3-y8(1,2)*z2
\]

\[ z1 = 0 + 0.1938i \]

\[
\gg \ Z2=[z1;z2;z3;z4]
\]

\[ Z2 =
\begin{bmatrix}
0 + 0.1938i \\
0 + 0.2295i \\
0 + 0.1494i \\
0 + 0.1506i
\end{bmatrix}
\]

ii). Compute the subtransient fault current.

\[
I''_f = \frac{V_f}{Z_{22}} = \frac{1}{j.2295} = -j4.3573\ pu \ or \ 4.3573 \angle -90^\circ \ pu
\]

iii). Use eq. (12) to find the voltages during the fault.

\[
V_{if} = V_j - \frac{Z_{jk}}{Z_{kk}} V_f
\]

\[
V_{if} = V_1 - \frac{Z_{12}}{Z_{22}} V_f = 1 - \frac{j.1938}{j.2295} 1 = .1556 pu
\]

\[ V_{2f} = 0 \]
\[ V_{3f} = V_3 - \frac{Z_{32} V_f}{Z_{22}} = .34902 \text{pu} \]

\[ V_{4f} = V_4 - \frac{Z_{42} V_f}{Z_{22}} = .343791 \text{pu} \]

iv). Use eq. (17) to find the subtransient currents in lines 3-2, 1-2, and 4-2.

\[ I''_{ij} = -V_f \frac{Z_{ik} - Z_{jk}}{Z_b Z_{kk}} \]

\[ I''_{32} = -V_f \frac{Z_{32} - Z_{22}}{Z_b Z_{22}} = -j1.1494 - j.2295 \]

\[ I''_{12} = -V_f \frac{Z_{12} - Z_{22}}{Z_b Z_{22}} = -j1.244 \text{pu} \]

\[ I''_{42} = -V_f \frac{Z_{42} - Z_{22}}{Z_b Z_{22}} = -j1.71895 \text{pu} \]
3. A Y-connected load has balanced currents with a-c-b sequence given by

\[
I_{abc} = \begin{bmatrix}
I_a \\
I_b \\
I_c \\
\end{bmatrix} = \begin{bmatrix}
10 \angle 0^\circ \\
10 \angle +120^\circ \\
10 \angle -120^\circ \\
\end{bmatrix}
\]

Calculate the sequence currents. How does your answer differ from the answer obtained in Example 1 in these notes?

Solution:

\[
\begin{bmatrix}
I^0_a \\
I^+_a \\
I^-_a \\
\end{bmatrix} = A^{-1} \begin{bmatrix}
I_a \\
I_b \\
I_c \\
\end{bmatrix} = \frac{1}{3} \begin{bmatrix}
1 & 1 & 1 \\
1 & \alpha & \alpha^2 \\
1 & \alpha^2 & \alpha \\
\end{bmatrix} \begin{bmatrix}
I_a \\
I_b \\
I_c \\
\end{bmatrix}
\]

\[
= \frac{1}{3} \begin{bmatrix}
1 & 1 & 1 \\
1 & \alpha & \alpha^2 \\
1 & \alpha^2 & \alpha \\
\end{bmatrix} \begin{bmatrix}
10 \angle 0^\circ \\
10 \angle 120^\circ \\
10 \angle -120^\circ \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
0 \\
0 \\
10 \angle 0^\circ \\
\end{bmatrix}
\]

In this case the quantity that is non-zero is negative sequence component.
4. A feeder provides service to a delta-connected load having the following phase currents:

\[ I_{ab} = 208.3 \angle -18.19^\circ \]
\[ I_{bc} = 138.89 \angle -151.788^\circ \]
\[ I_{ca} = 131.94^\circ \angle 145.84 \]

a. For the phase currents:
   i. Are they balanced or unbalanced?
   ii. What is their sum?
   iii. Obtain their sequence quantities.
   iv. What is the 0-sequence quantity?

b. Obtain the line currents. For these currents:
   i. Are they balanced or unbalanced?
   ii. What is their sum?
   iii. Obtain their sequence quantities.
   iv. What is the 0-sequence quantity?

c. Use what you have learned in the parts (a) and (b) to answer the three questions (ii, iv) from part (b) for the following a-b-c quantities:
   i. Unbalanced currents into a grounded-Y.
   ii. Unbalanced currents into an ungrounded-Y.
   iii. Unbalanced line-to-line voltages.

**Solution:**

The situation is shown in the figure below.
Part a:

i. Phase currents are unbalanced.

ii. Their sum is:

\[ I_{ab} + I_{bc} + I_{ca} = \]
\[ 208.3 \angle -18.19^\circ + 138.89 \angle -151.788^\circ + 131.94^\circ \angle 145.84^\circ \]
\[ = 65.9252 \angle -121.01^\circ \]

iii. Their sequence quantities are:

\[
\begin{bmatrix}
I_{ab}^0 \\
I_{ab}^+ \\
I_{ab}^-
\end{bmatrix} = A^{-1}
\begin{bmatrix}
I_{ab} \\
I_{bc} \\
I_{ca}
\end{bmatrix} = \frac{1}{3}
\begin{bmatrix}
1 & 1 & 1 \\
1 & \alpha & \alpha^2 \\
1 & \alpha^2 & \alpha
\end{bmatrix}
\begin{bmatrix}
I_{ab} \\
I_{bc} \\
I_{ca}
\end{bmatrix}
\]

\[
= \frac{1}{3}
\begin{bmatrix}
1 & 1 & 1 \\
1 & \alpha & \alpha^2 \\
1 & \alpha^2 & \alpha
\end{bmatrix}
\begin{bmatrix}
208.3 \angle -18.19^\circ \\
138.89 \angle -151.788^\circ \\
131.94^\circ \angle 145.84^\circ 
\end{bmatrix}
\]

\[
= \begin{bmatrix}
21.9531 \angle -120.753^\circ \\
147.373 \angle -10.5147^\circ \\
67.0451 \angle -16.699^\circ 
\end{bmatrix}
\]

iv. The zero-sequence quantity is (21.9531 \angle -120.753^\circ) (it’s non-zero).

Part b: Consider the figure above and note that we may relate the line currents to the phase currents using KCL:

\[ I_a = I_{ab} - I_{ca} = (1)I_{ab} + (0)I_{bc} + (-1)I_{ca} \]
\[ I_b = -I_{ab} + I_{bc} = (-1)I_{ab} + (1)I_{bc} + (0)I_{ca} \]
\[ I_c = -I_{bc} + I_{ca} = (0)I_{ab} + (-1)I_{bc} + (1)I_{ca} \]
Writing in matrix form, we have:

\[
\begin{bmatrix}
I_a \\
I_b \\
I_c
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & -1 \\
-1 & 1 & 0 \\
0 & -1 & 1
\end{bmatrix}
\begin{bmatrix}
I_{ab} \\
I_{bc} \\
I_{ca}
\end{bmatrix}
\]

Denote the matrix as:

\[
K =
\begin{bmatrix}
1 & 0 & -1 \\
-1 & 1 & 0 \\
0 & -1 & 1
\end{bmatrix}
\]

Now let’s use it to obtain the line currents:

\[
\begin{bmatrix}
I_a \\
I_b \\
I_c
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & -1 \\
-1 & 1 & 0 \\
0 & -1 & 1
\end{bmatrix}
\begin{bmatrix}
208.3^\circ \angle -18.19^\circ \\
138.89^\circ \angle -151.788^\circ \\
131.94^\circ \angle 145.84^\circ
\end{bmatrix}
\]

\[
= \begin{bmatrix}
208.3^\circ \angle -18.19^\circ - 131.94^\circ \angle 145.84^\circ \\
-208.3^\circ \angle -18.19^\circ + 138.89^\circ \angle -151.788^\circ \\
-138.89^\circ \angle -151.788^\circ + 131.94^\circ \angle 145.84^\circ
\end{bmatrix}
\]

\[
= \begin{bmatrix}
337.108^\circ \angle -24.3718^\circ \\
320.282^\circ \angle -179.887^\circ \\
140.367^\circ \angle 84.5983^\circ
\end{bmatrix}
\]

i. They are unbalanced.
ii. Their sum is
\[337.108 \angle -24.3718^\circ + 320.282 \angle -179.887^\circ + 140.367 \angle 84.598^\circ = 0\]

iii. Obtain their sequence quantities.
\[
\begin{bmatrix}
I_a^0 \\
I_a^+ \\
I_a^-
\end{bmatrix} = A^{-1} \begin{bmatrix}
I_a \\
I_b \\
I_c
\end{bmatrix} = \frac{1}{3} \begin{bmatrix}
1 & 1 & 1 \\
1 & \alpha & \alpha^2 \\
1 & \alpha^2 & \alpha
\end{bmatrix} \begin{bmatrix}
I_a \\
I_b \\
I_c
\end{bmatrix}
\]
\[
= \frac{1}{3} \begin{bmatrix}
1 & 1 & 1 \\
1 & \alpha & \alpha^2 \\
1 & \alpha^2 & \alpha
\end{bmatrix} \begin{bmatrix}
337.1 \angle -24.37^\circ \\
320.3 \angle -168.81^\circ \\
140.4 \angle 84.56^\circ
\end{bmatrix}
\]
\[
= \begin{bmatrix}
0 \\
255.258 \angle -40.5147^\circ \\
116.125 \angle 13.301^\circ
\end{bmatrix}
\]

iv. The 0-sequence quantity is zero.

Part c:

i. Unbalanced currents into a grounded-Y will not sum to zero and therefore will have a non-zero zero-sequence component.

ii. Unbalanced currents into an ungrounded-Y will sum to zero and therefore will have a zero zero-sequence component.

iii. Unbalanced line-to-line voltages must sum to zero (do a KVL!) and therefore will never have a zero-sequence component.
Problem 9.8

The network considered in Example 9.5 together with the appropriate branch impedances is shown in the figure below.

We apply the $Z_{bus}$ building algorithm described in Section 9.5.

**Step 1**
Add node 1 to the reference node. The branch has a reactance of $j1.25$. This falls under the category Modification 1.

$$Z_{bus} = [j1.25]$$

**Step 2**
Add node 2 to node 1. The branch has a reactance of $j0.0533$. This falls under the category Modification 2.

$$Z_{bus} = \begin{bmatrix} j1.25 & j1.25 \\ j1.25 & j1.3033 \end{bmatrix}$$

**Step 3**
Add node 3 to node 2. The branch has a reactance of $j0.25$. This falls under the category Modification 2.

$$Z_{bus} = \begin{bmatrix} j1.25 & j1.25 \\ j1.25 & j1.3033 \\ j1.25 & j1.3033 \end{bmatrix}$$

**Step 4**
Add node 4 to node 3. The branch has a reactance of $j0.25$. This falls under category Modification 2.

$$Z_{bus} = \begin{bmatrix} j1.25 & j1.25 \\ j1.25 & j1.3033 \\ j1.25 & j1.3033 \\ j1.25 & j1.3033 \end{bmatrix}$$

**Step 5**
Add branch between node 2 and node 4. The branch has a reactance of $j0.15$. This falls under category Modification 2.

$$\begin{bmatrix} 0 \\ 0 \\ -j0.25 \\ -j0.50 \end{bmatrix}$$

$$Z_{bus}' = Z_{bus} - \text{diag}(b)^T Z_{bus} b$$

$$Z_{bus} = \begin{bmatrix} j1.25 & j1.25 \\ j1.25 & j1.3033 \\ j1.25 & j1.3033 \\ j1.25 & j1.3033 \\ j1.25 & j1.3033 \\ j1.25 & j1.4571 \\ j1.25 & j1.3610 \\ j1.25 & j1.3610 \end{bmatrix}$$

**Step 6**
Add node 5 to node 4. The branch has a reactance of $j0.08$. This falls under the category Modification 2.


**Step 7**
Add the branch between node 5 and the reference node. The branch has a reactance of $j1.25$. This falls under the category Modification 3.

We first augment the $Z_{bus}$ with an additional row and column as shown below.


Kron reduce the last row and column to obtain.

$$Z_{bus} = \begin{bmatrix} 0.6815 & 0.6573 & 0.6311 & 0.6048 & 0.5689 \\ 0.6573 & 0.6853 & 0.6580 & 0.6306 & 0.5927 \\ 0.6311 & 0.6853 & 0.7382 & 0.6558 & 0.6189 \\ 0.6048 & 0.6306 & 0.6585 & 0.6685 & 0.6452 \\ 0.5689 & 0.6306 & 0.6585 & 0.6685 & 0.6452 \\ 0.6815 & 0.6573 & 0.6311 & 0.6048 & 0.5689 \end{bmatrix}$$

**Step 8**
Add capacitive reactance between node 2 and the reference node. This branch has a reactance of $(j0.11+j0.055)^{-1} = -j0.061$.

As in step 7 we first augment the original $Z_{bus}$ with an additional row and column and Kron reduce. The augmented matrix and the Kron reduced matrix are given below.

$$Z_{bus} = \begin{bmatrix} 0.6815 & 0.6573 & 0.6311 & 0.6048 & 0.5689 \\ 0.6573 & 0.6853 & 0.6580 & 0.6306 & 0.5927 \\ 0.6311 & 0.6853 & 0.7382 & 0.6558 & 0.6189 \\ 0.6048 & 0.6306 & 0.6585 & 0.6685 & 0.6452 \\ 0.5689 & 0.6306 & 0.6585 & 0.6685 & 0.6452 \end{bmatrix}$$

**Step 9**
Add capacitive reactance between node 3 and the reference node. This branch has a reactance of $(j0.11+j0.11)^{-1} = -j0.355$.

Once again we first augment the original $Z_{bus}$ and Kron reduce.

$$Z_{bus} = \begin{bmatrix} 0.7669 & 0.7441 & 0.7115 & 0.6819 & 0.6409 \\ 0.7441 & 0.7722 & 0.7419 & 0.7111 & 0.6868 \\ 0.7115 & 0.7419 & 0.8637 & 0.7357 & 0.6915 \\ 0.6819 & 0.7111 & 0.7357 & 0.7004 & 0.7147 \\ 0.6409 & 0.6868 & 0.6915 & 0.6746 & 0.6915 \end{bmatrix}$$

After multiplying by $0.8944, 0.8845, 0.8796, 0.8785, 0.8785, 0.8845, 0.8768$, the matrix becomes.

$$Z_{bus} = \begin{bmatrix} 0.8944 & 0.8845 & 0.8768 & 0.821 & 0.7646 \\ 0.8845 & 0.9222 & 0.9159 & 0.8593 & 0.8076 \\ 0.8796 & 0.9222 & 0.9159 & 0.8593 & 0.8076 \\ 0.8785 & 0.9222 & 0.9159 & 0.8593 & 0.8076 \\ 0.8785 & 0.9222 & 0.9159 & 0.8593 & 0.8076 \\ 0.8785 & 0.9222 & 0.9159 & 0.8593 & 0.8076 \end{bmatrix}$$

**Step 10**
Add capacitive reactance between node 4 and the reference node. This branch has a reactance of $(j0.11+j0.055)^{-1} = -j0.061$.

We again augment the matrix and Kron reduce to obtain.

$$Z_{bus} = \begin{bmatrix} 0.8944 & 0.8845 & 0.8785 & 0.821 & 0.7746 \\ 0.8845 & 0.9222 & 0.9159 & 0.8593 & 0.8076 \\ 0.8785 & 0.9159 & 0.9065 & 0.8903 & 0.8377 \\ 0.8944 & 0.9222 & 0.9159 & 0.8593 & 0.8076 \\ 0.8796 & 0.9222 & 0.9159 & 0.8593 & 0.8076 \end{bmatrix}$$