# Notes on Mutual Inductance and Transformers
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## 1.0 Use of Transformers

Transformers are one of the most common electrical devices. Perhaps the most familiar application today is for small electronic devices such as laptop computers. The circuit diagram for such an application is given in Fig. 1.

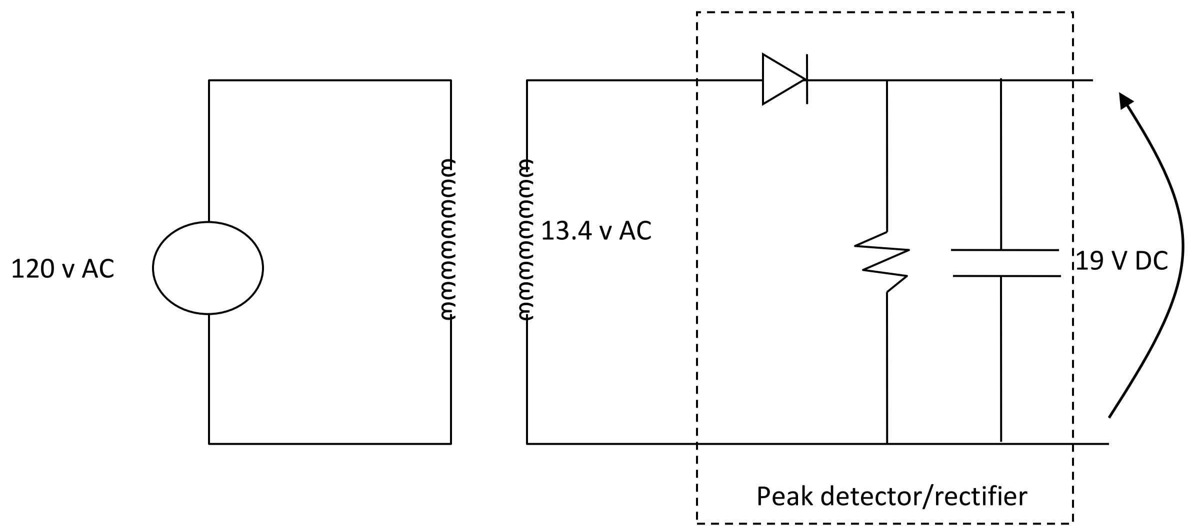


**Fig. 1**

Transformers are also used in devices with rechargeable batteries, e.g., drills, screwdrivers, cordless phones.  Another major transformer application is the ballast used in florescent lighting, as shown in Fig. 2.

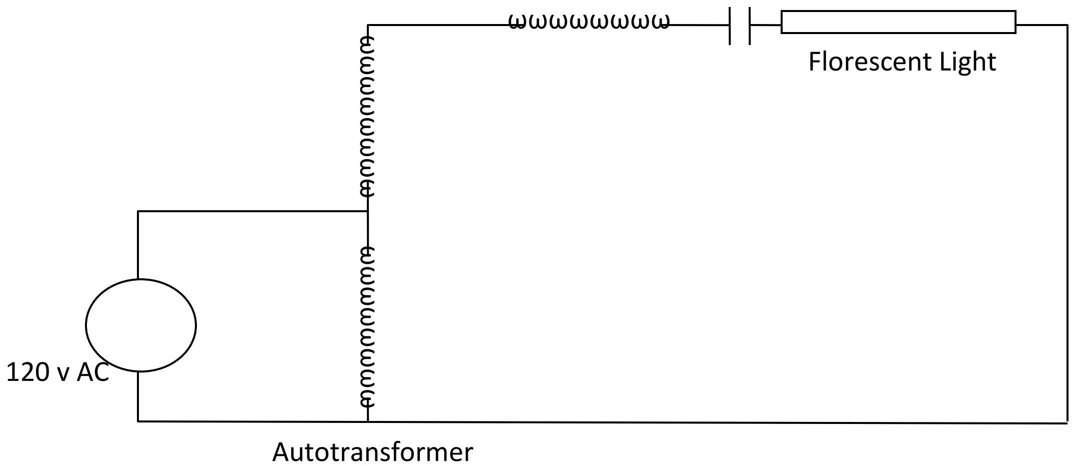


**Fig. 2**

In the electric power industry, several types of transformers are utilized, including the power transformer and the instrument transformer. Our interest is the power transformer. Power transformers are used in the following ways:

- Stepping up the voltage from a generator to high voltage transmission levels;
- Stepping down the voltage to distribution primary voltage levels;
- Stepping down the voltage to distribution secondary voltage levels;
- Interconnecting different system voltage levels in the HV and EHV systems.

## 2.0 Self inductance

Consider the arrangement of Fig. 3a.

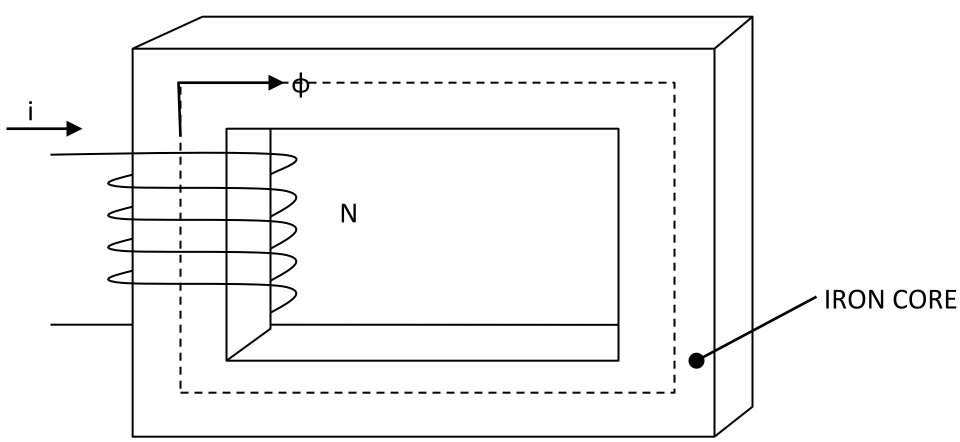


**Fig. 3a**

Ampere's law is

$$\oint \vec{H} \bullet d\vec{L} = I \tag{1}$$

Ampere's law says that the line integral of magnetic field intensity H about any closed path equals the current enclosed by that path. When (1) is applied to the arrangement of Fig. 3,

- The path is the dotted line;
- The magnetic field intensity is along the direction of ϕ, which is in the same direction as dL;
- The left hand side of (1) therefore becomes just Hl, where l is the mean length of the path around the core.
- The right-hand-side of (1) is the number of turns times the current, Ni.

Therefore, we obtain

$$Hl = Ni \tag{2}$$

We recall from basic electromagnetics that

$$B = \mu H \tag{3}$$

where B is the magnetic flux density (webers/$m^2$ or tesla), and μ is the permeability of the iron with units of Henry/m or Newtons/$ampere^2$ (μ for a given material is the amount of flux density B that will flow in that material for a unit value of magnetic field strength H. For most types of iron used in transformers, $\mu=5000\mu_0$ N/$A^2$, where $\mu_0=4\pi\times10^{-7}$N/$A^2$ is the permeability of free space).

We also know that flux ϕ (webers) is related to flux density by

A weber is a unit of flux with SI base units of kg-$m^2$/$sec^2$/A.

$$\phi = BA \tag{4}$$

where A is the cross-sectional area of the iron core. Solving for B in (4) and substituting into (3), solving for H, and substituting into (2) yields

$$\frac{\phi}{\mu A} l = Ni \tag{5}$$

Solving for ϕ results in

$$\phi = \frac{\mu A}{l} Ni \tag{6}$$

Now we define:

- Magnetomotive Force, $\mathscr{F} = Ni$ (units of ampere-turns)
- Reluctance: $\mathscr{R} = \dfrac{l}{\mu A}$ (units of amperes/weber)

Then (6) becomes

$$\varphi = \frac{\mathscr{F}}{\mathscr{R}} \tag{7}$$

Equation (7) should remind you of a familiar relation… Ohm's Law!

Ohm's Law is I=V/R and so the analogy is

- I → ϕ (flux "flows" like current)
- V → F (MMF provides the "push" like voltage)
- R → R (Reluctance "resists" like resistance)

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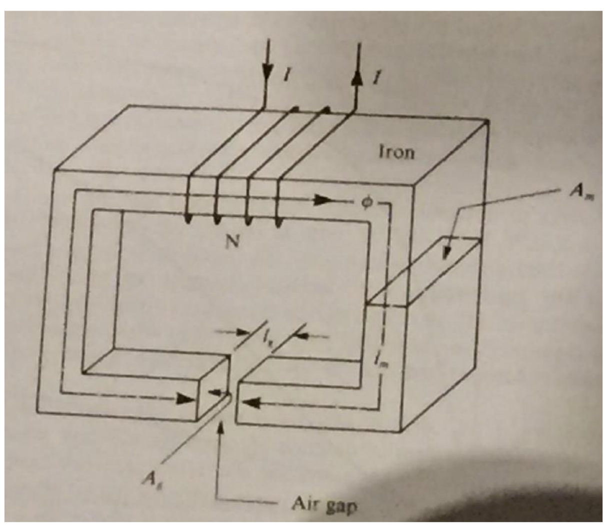
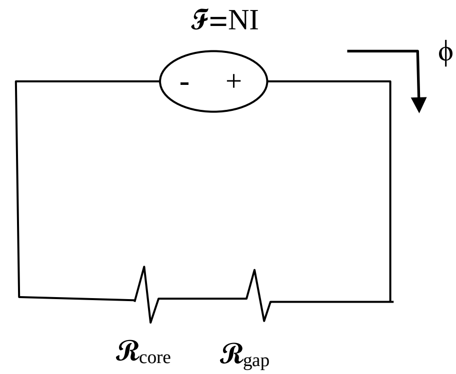
**Example 1** [1]: The magnetic circuit shown in the below figure has N=100 turns, a cross-section area of $A_m$=$A_g$=40cm$^2$, an air gap length of $l_g$=0.5mm, and a mean core length of $l_c$=1.2m. The relative permeability of the iron is $\mu_r$=2500. The current in the coil is $I_{DC}$=7.8 amperes. Determine the flux and flux density in the air gap.



**Solution**: We may think of this magnetic circuit in terms of its electric analogue, as shown below.




The electric analogue makes the solution immediately clear, where

$$\varphi = \frac{\mathcal{F}}{\mathcal{R}}$$

The MMF is computed as

$$\mathcal{F} = NI = 100 * 7.8 = 780 \text{ ampere-turns}$$

The reluctance of the air-gap is computed as

$$\mathcal{R}_{gap} = \frac{l_g}{\mu A_g} = \frac{0.5/1000}{(4\pi \times 10^{-7})(40/100^2)} = 99,472 \text{amperes/Weber}$$

The reluctance of the core is computed as

$$\mathcal{R}_{core} = \frac{l_c}{\mu A_c} = \frac{1.2}{2500(4\pi \times 10^{-7})(40/100^2)} = 95,492 \text{amperes/Weber}$$

The flux is then computed as

$$\varphi = \frac{\mathcal{F}}{\mathcal{R}} = \frac{780}{99,472 + 95,492} = 0.004 \text{Webers}$$

The flux density is given by

$$B = \frac{\varphi}{A} = \frac{0.004}{40/(100)^2} = 1 \text{ Weber/m}^2 = 1 \text{Tesla}$$

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Now let's return to (6) and multiply both sides by N to obtain

$$\phi = \frac{\mu A}{l} Ni \quad \Rightarrow \quad N\phi = \frac{\mu A}{l} N^2 i \tag{8}$$

Define:

- Flux linkage:

$$\lambda = N\phi \tag{9}$$

- Self inductance:

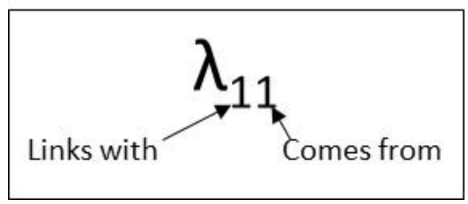
$$L = \frac{\mu A N^2}{l} = \frac{N^2}{\mathscr{R}} \tag{10}$$

Substituting (9) and (10) into (8), we obtain

$$\lambda = Li \tag{11}$$

We will introduce some additional notation that will help us later, as follows:

$$L_{11} = \frac{\lambda_{11}}{i_1} \tag{12}$$




And so we can see that the self-inductance $L_{11}$ is the ratio of

- the flux from coil 1 linking with coil 1, $\lambda_{11}$
- to the current in coil 1, $i_1$.

Here the first subscript of $\lambda_{11}$, 1 in this case, indicates "links with coil 1" and the second subscript, 1 in this case, indicates "flux from coil 1."

Observe from (12) that a large $L_{11}$ means that a little current $i_1$ generates a lot of flux linkages $\lambda_{11}$. What makes $L_{11}$ large? Recall:

$$L_{11} = \frac{\mu A N_1^2}{l}$$

And so we see that to make self inductance large, we need to

- make $N_1$, $\mu$, and A large

- make $\ell$ small

And so a large $L_{11}$ results from

- many turns ($N_1$)
- large μ (e.g., core made of iron)
- large cross section (A)
- compact construction (small $\ell$)

Figure 3b illustrates a configuration that would reflect a large cross section and a compact construction.

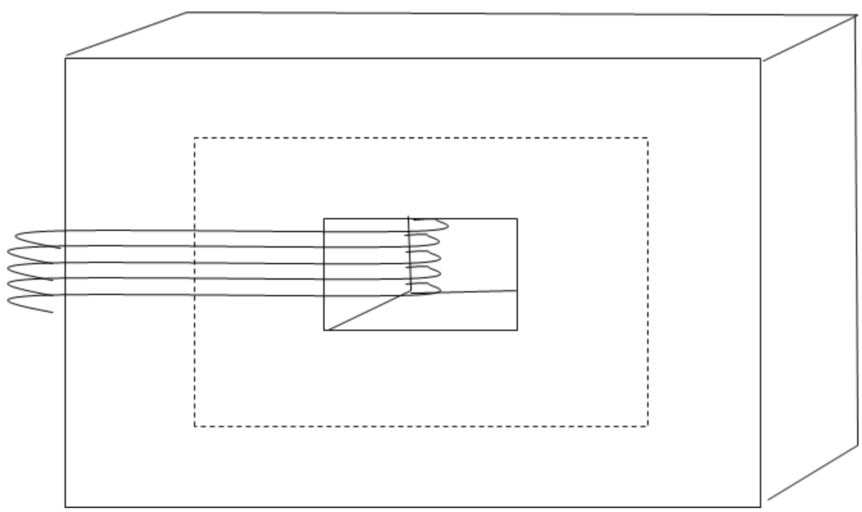


**Fig 3b**

Recalling that reluctance is given by $\mathscr{R} = \dfrac{l}{\mu A}$, we see that a magnetic circuit characterized by a large self-inductance will have a small magnetic path reluctance.

**Example 2** [1]: Compute the self-inductance of the magnetic circuit given in Example 1.

**Solution**: Here, we need to recognize that the magnetic field intensity, H, will be different in the iron core than in the air gap. We can see that this must be so because the air gap is in series with the core and so the flux ϕ in the air gap must be the same as the flux in the core. Since the cross-sectional area in the air gap and in the core are the same, the flux densities B must also be the same. But because B=μH, and the permeability of the air gap differs from the permeability of the core, the magnetic field intensities must differ as well. Thus, equation (2) (which is $Hl = Ni$) will be written as:

$$H_c l_c + H_g l_g = Ni$$

Using B=μH, we have

$$\frac{B}{\mu_c} l_c + \frac{B}{\mu_g} l_g = Ni \Rightarrow B\left[\frac{l_c}{\mu_c} + \frac{l_g}{\mu_g}\right] = Ni$$

And using B=ϕ/A, we have

$$\frac{\phi}{A}\left[\frac{l_c}{\mu_c} + \frac{l_g}{\mu_g}\right] = Ni$$

Solving for ϕ, we obtain

$$\phi = \frac{Ni}{\dfrac{l_c}{A\mu_c} + \dfrac{l_g}{A\mu_g}} = \frac{Ni}{\mathcal{R}_c + \mathcal{R}_g}$$

Using subscripted notation to identify the flux from coil 1 linking with coil 1 results in

$$\phi_{11} = \frac{N_1 i_1}{\dfrac{l_c}{A\mu_c} + \dfrac{l_g}{A\mu_g}} = \frac{N_1 i_1}{\mathcal{R}_c + \mathcal{R}_g}$$

Recalling that self-inductance is given by

$$L_{11} = \frac{\lambda_{11}}{i_1}$$

$L_{11}$? The ability of a current in coil 1, $i_1$, to create flux $\phi_{11}$ that links with coil 1.

and that $\lambda_{11}=N_1\phi_{11}$, we can write that

$$L_{11} = \frac{\lambda_{11}}{i_1} = \frac{N_1\phi_{11}}{i_1}$$

Substitution for $\phi_{11}$ results in

$$L_{11} = \frac{N_1^2}{\mathcal{R}_c + \mathcal{R}_g}$$

Recalling from Example 1 that $N_1=100$ and

$$\mathcal{R}_c = 95,492 \text{amperes/Weber} \,; \quad \mathcal{R}_g = 99,472 \text{amperes/Weber}$$

the self-inductance becomes:

$$L_{11} = \frac{100^2}{95492 + 99472} = 0.0513 \text{ henries}$$

## 3.0 Mutual inductance

Let's consider another arrangement as shown in Fig. 4.

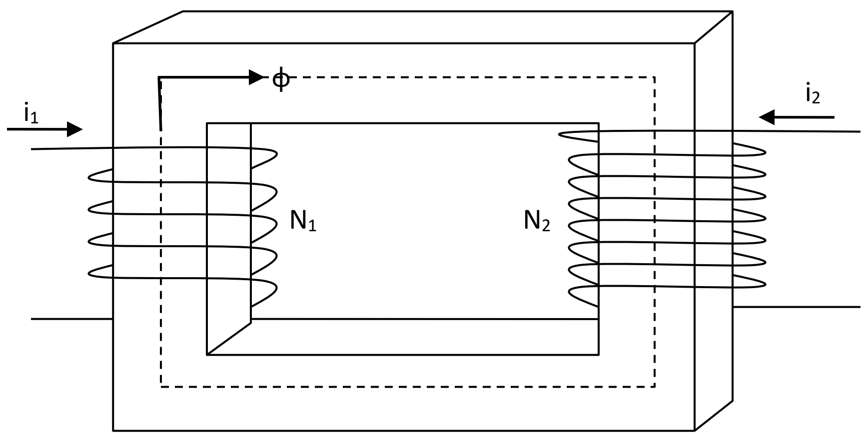



Fig. 4

We have for each coil:

$$L_{11} = \frac{\lambda_{11}}{i_1} \tag{13a}$$

$$L_{22} = \frac{\lambda_{22}}{i_2} \tag{13b}$$

We can also define $L_{12}$ and $L_{21}$.

$L_{12}$ is the ratio of

- the flux from coil 2 linking with coil 1, $\lambda_{12}$
- to the current in coil 2, $i_2$.

That is,

$$L_{12} = \frac{\lambda_{12}}{i_2} \tag{14a}$$

where the first subscript, 1 in this case, indicates "links with coil 1" and the second subscript, 2 in this case, indicates "flux from coil 2."

Here, we also have that

$$\lambda_{12} = N_1\phi_{12} \Rightarrow L_{12} = \frac{N_1\phi_{12}}{i_2} \tag{14b}$$

Likewise, we have that

$$L_{21} = \frac{\lambda_{21}}{i_1} \tag{15a}$$

$$\lambda_{21} = N_2\phi_{21} \Rightarrow L_{21} = \frac{N_2\phi_{21}}{i_1} \tag{15b}$$

Now let's assume that all flux produced by each coil links with the other coil. The implication of this is that there is no leakage flux, as illustrated in Fig. 5.

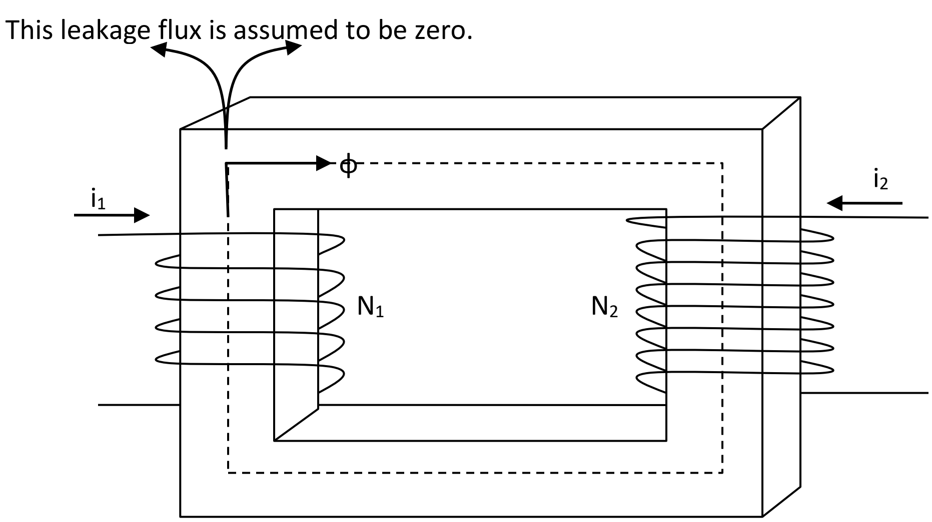


Fig. 5

Although in reality there is some leakage flux, it is quite small because the iron has much less reluctance than the air. With this assumption, then we can write:

- the flux from coil 2 linking with coil 1 is equal to the flux from coil 2 linking with coil 2, i.e.,

$$\phi_{12} = \phi_{22} = \frac{\mu A}{l} N_2 i_2 \tag{16a}$$

- the flux from coil 1 linking with coil 2 is equal to the flux from coil 1 linking with coil 1, i.e.,

$$\phi_{21} = \phi_{11} = \frac{\mu A}{l} N_1 i_1 \tag{16b}$$

Substitution of (16a) and (16b) into (14b) and (15b), respectively, results in:

$$L_{12} = \frac{N_1 \varphi_{12}}{i_2} = \frac{N_1 \frac{\mu A}{l} N_2 i_2}{i_2} = N_1 N_2 \frac{\mu A}{l} = \frac{N_1 N_2}{\mathcal{R}} \tag{17a}$$

$$L_{21} = \frac{N_2 \varphi_{21}}{i_1} = \frac{N_2 \frac{\mu A}{l} N_1 i_1}{i_1} = N_2 N_1 \frac{\mu A}{l} = \frac{N_2 N_1}{\mathcal{R}} \tag{17b}$$

Examination of (17a) and (17b) leads to

$$L_{21} = L_{12} = \frac{N_1 N_2}{\mathcal{R}} \tag{18}$$

Also recall

$$L = \frac{N^2}{\mathcal{R}} \tag{10}$$

or in subscripted notation

$$L_{11} = \frac{N_1^{\,2}}{\mathcal{R}} \tag{19a}$$

$$L_{22} = \frac{N_2^{\;2}}{\mathcal{R}} \tag{19b}$$

Solving for $N_1$ and $N_2$in (19a) and (19b) results in

$$N_1 = \sqrt{L_{11}\mathcal{R}} \tag{20a}$$

$$N_2 = \sqrt{L_{22}\mathcal{R}} \tag{20b}$$

Now substitute (20a) and (20b) into (18) to obtain

$$L_{21} = L_{12} = \frac{\sqrt{L_{11}\mathcal{R}}\sqrt{L_{22}\mathcal{R}}}{\mathcal{R}} = \sqrt{L_{11}L_{22}} \tag{21}$$

Definition: $L_{12}=L_{21}$ is the mutual inductance and is normally denoted M.

Mutual inductance gives the ratio of

- flux from coil k linking with coil j, $\lambda_{jk}$
- to the current in coil k, $i_k$,

That is,

$$M = \begin{cases} \dfrac{\lambda_{12}}{i_2} \\ \dfrac{\lambda_{21}}{i_1} \end{cases} \tag{22}$$

## 4.0 Polarity and dot convention for coupled circuits

Consider Fig. 6 illustrating two coupled circuits. Assume the voltage $v_1$ is DC, but you have a dial you can turn to increase $v_1$. Also assume that the secondary is open (i.e., the dashed line connecting the secondary terminals to a load is not really there). The coil 1 has very small resistance so that in the steady-state, the current is not infinite.

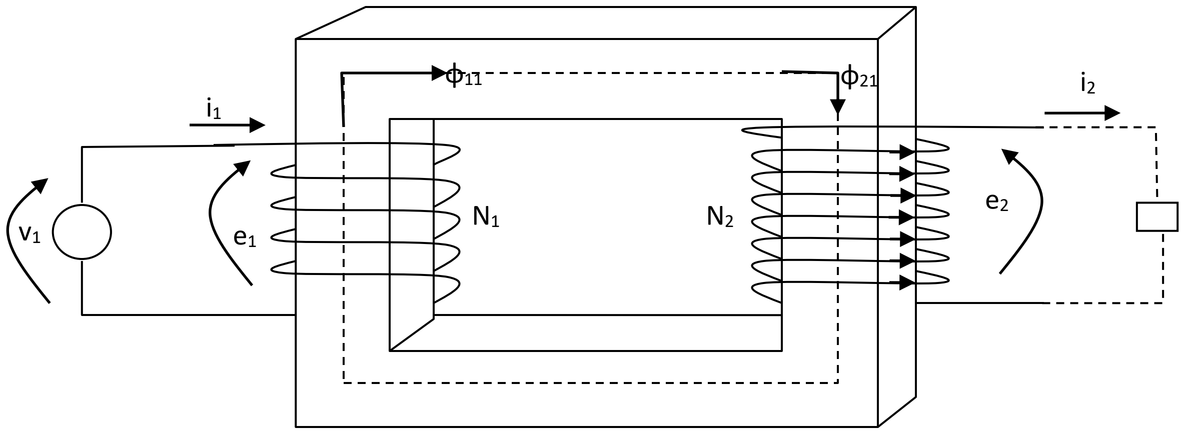



**Fig. 6**

Now assume that we increase the voltage $v_1$ to some higher value. This causes the current $i_1$ to increase with time which causes the flux from coil 1, $\phi_{11}$, to also increase with time (which means that the flux linkages $\lambda_{11}$ also increase with time). By Faraday's Law,

$$e_1 = N_1 \frac{d\phi_{11}}{dt} = \frac{d(N_1\phi_{11})}{dt} = \frac{d\lambda_{11}}{dt} = L_{11}\frac{di_1}{dt} \tag{23}$$

We know the sign of the right-hand-side of (23) is positive because the self-induced voltage across a coil is always positive at the terminal in which the current enters, and $e_1$ is defined positive at this terminal. If $e_1$ would have been defined negative at the terminal in which the current was entering, then the sign of the right-hand-side of (23) would have been negative.

Now let's consider coil 2. Coil 2 sees that same flux increase that coil 1 saw, which we will denote by $\phi_{21}$ (and correspondingly, the flux linkages are denoted as $\lambda_{21}$). Again, by Faraday's Law,

$$e_2 = N_2 \frac{d\phi_{21}}{dt} = \frac{d(N_2\phi_{21})}{dt} = \frac{d\lambda_{21}}{dt} = L_{21}\frac{di_1}{dt} = M\frac{di_1}{dt} \tag{24}$$

Question: How do we know the sign of the right-hand-side of (24)? That is, how do we know which of the below are correct?

$$\begin{aligned} e_2 &= +M\frac{di_1}{dt} \\ e_2 &= -M\frac{di_1}{dt} \end{aligned} \tag{25}$$

Here is another way to ask our question:

➔Does the assumed polarity of our $e_2$ match the actual polarity of the voltage that would be induced by the changing current $i_1$? If so, we should choose the equation in (25) with the positive sign. If not, we should choose the equation in (25) with the negative sign.

And so what is the answer? To obtain the answer, we need to recall Lenz's Law. This law states that the induced voltage $e_2$ must be in a direction so as to establish a current in a direction to produce a flux opposing the change in flux that produced $e_2$. (You can find a good explanation/illustration of Lenz's Law at https://www.khanacademy.org/science/physics/magnetic-forces-and-magnetic-fields/magnetic-flux-faradays-law/v/lenzs-law).

When $e_1$ increases, $i_1$ increases, and by the right-hand-rule (RHR), $\phi_{21}$ increases.

In Fig. 6, our assumed polarity of $e_2$ would cause current to flow into the load in the direction shown. How do we know if this polarity is correct or not? We know it is correct because the RHR says that a current in the direction of $i_2$ would cause flux in the direction opposite to the direction of *the* $\phi_{21}$ *increase* (we emphasize that this is "*the* $\phi_{21}$ *increase*," i.e., it is "the change in flux that produced $e_2$" and not necessarily the direction of $\phi_{21}$ itself, although in this case, "*the* $\phi_{21}$ *increase*" is the same as the direction of $\phi_{21}$ itself). Thus, for the given polarity of $e_2$, the sign of (25) should be positive, i.e.,

$$e_2 = +M\frac{di_1}{dt}$$

How might we obtain a different answer?

There are two ways.

**<u>First way</u>**: Switch the sign of $e_2$, as in Fig. 7. In this case, we also must switch the direction of current $i_2$ that would flow into the load (in using Lenz's Law, the $i_2$ direction must be consistent with the $e_2$ direction).

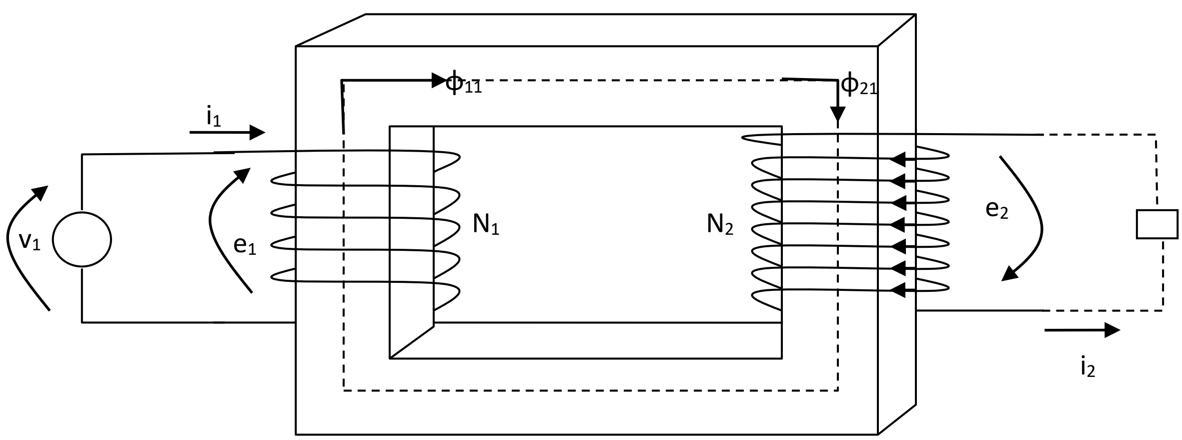



**Fig. 7**

In Fig. 7, the current $i_2$, by the RHR, would produce a flux in the same direction as the $\phi_{21}$ increase, which is in violation of Lenz's Law. Therefore, in this case, we should use a negative sign in (25) according to

$$e_2 = -M\frac{di_1}{dt}$$

**Second way**: Switch the sense of the coil 2 wrapping, as in Fig. 8 (while keeping the directions of $e_2$ and $i_2$ as they were in Fig. 6).

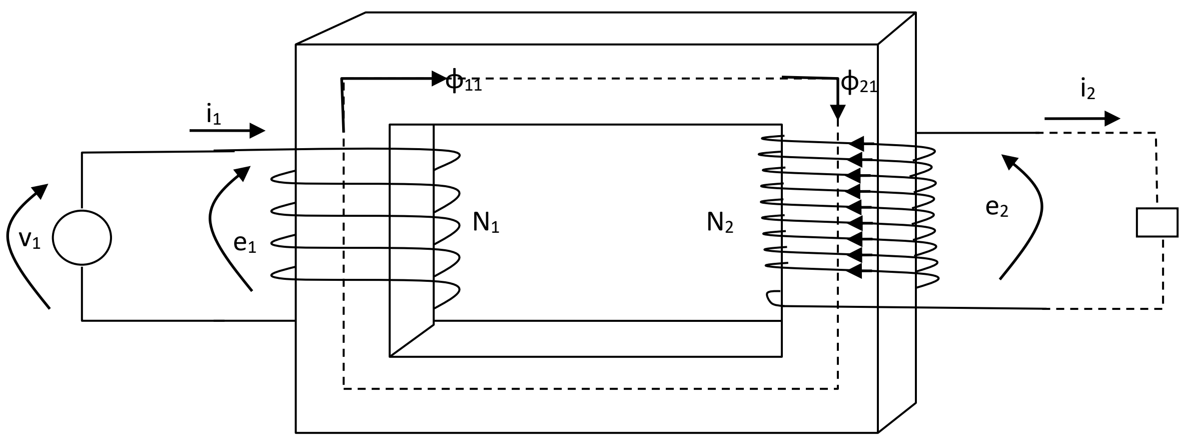



**Fig. 8**

In Fig. 8, the current $i_2$, by the RHR, would produce a flux in the same direction as the $\phi_{21}$ increase. In this case, we should again use the negative sign in (25). The main point here is that we want to be able to know which secondary terminal, when defined with positive voltage polarity, results in using the form of (25) with a positive sign.

On paper, there are two approaches for doing this. The first is to draw the physical winding and to go through the Lenz's Law analysis as we have done above.

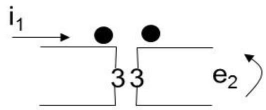
The second approach is easier, and it is to use the so-called "dot convention." A simplifying feature to the dot convention is that there is no need to be concerned with the $i_2$ direction, a feature that is consistent with the fact that (25) is independent of $i_2$.

In the dot convention, we mark one terminal on either side of the transformer so that

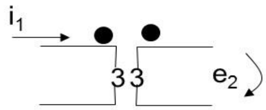
- when $e_2$ is defined positive at the dotted terminal of coil 2 and
- $i_1$ is into the dotted terminal of coil 1, then

$$e_2 = +M\frac{di_1}{dt}$$

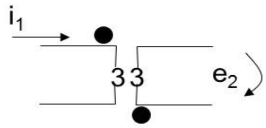
**Example 3**:



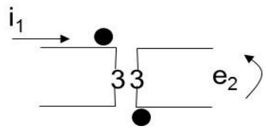

$$e_2 = +M\frac{di_1}{dt}$$




$$e_2 = -M\frac{di_1}{dt}$$




$$e_2 = +M\frac{di_1}{dt}$$




$$e_2 = -M\frac{di_1}{dt}$$

So far, we have focused on answering the following question: given the dotted terminals, how to determine the sign to use in (25)?

Here is another question: If you are given the physical layout, how do you obtain the dot-markings? There are two approaches, described below. In both approaches, we need to start with a defined voltage direction $e_2$; if this is not predefined for us, then we should define it ourselves at the beginning.

**First approach**: Use Lenz's Law and the right-hand-rule (RHR) to determine if a defined voltage direction at the secondary produces a current in the secondary that generates flux opposing the flux change that caused that voltage. (This is actually a conceptual summary of the second approach.)

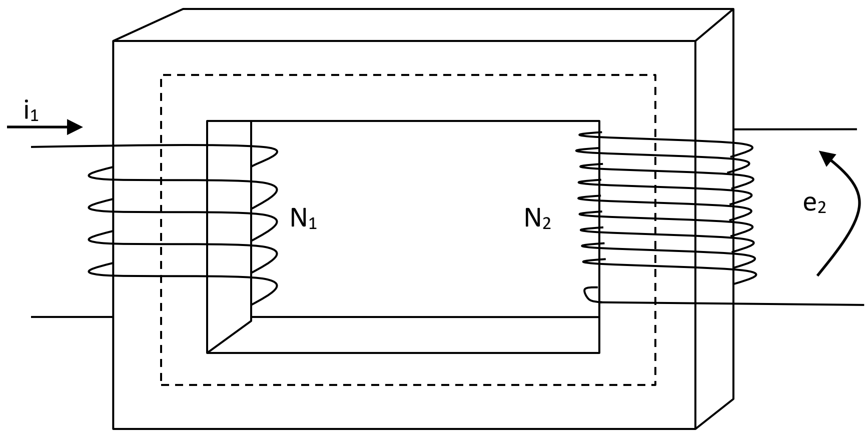
**Second approach**: Do it by steps. (This is actually a step-by-step articulation of the first approach.)

1. Arbitrarily pick a terminal on one side and dot it.
2. Assign a current into the dotted terminal.
3. Use RHR to determine flux direction for current assigned in step 2.
4. Arbitrarily pick a terminal on the other side and assign a current *out of* **(into)** it.
5. Use RHR to determine flux direction for current assigned in Step 4.
6. Compare the direction of the two fluxes (the one from Step 3 and the one from Step 5). If the two flux directions are *opposite* **(same)**, then the terminal chosen in Step 4 is correct. If the two flux directions are *same* **(opposite)**, then the terminal chosen in Step 4 is incorrect – dot the other terminal.

*This approach depends on the following principle (consistent with words in italics in above steps): Current entering one dotted terminal and leaving the other dotted terminal should produce fluxes inside the core that are in opposite directions.*

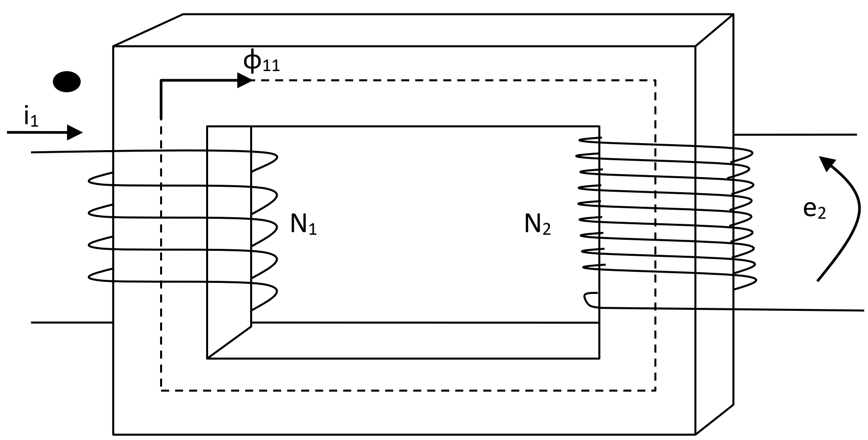
<u>**An alternative statement of this principle is as follows (consistent with words in underline bold in above steps): Currents entering the dotted terminals should produce fluxes inside the core that are in the same direction.**</u>

<u>**Example 4**</u>: Determine the dotted terminals for the configuration below, and then write the relation between $i_1$ and $e_2$.

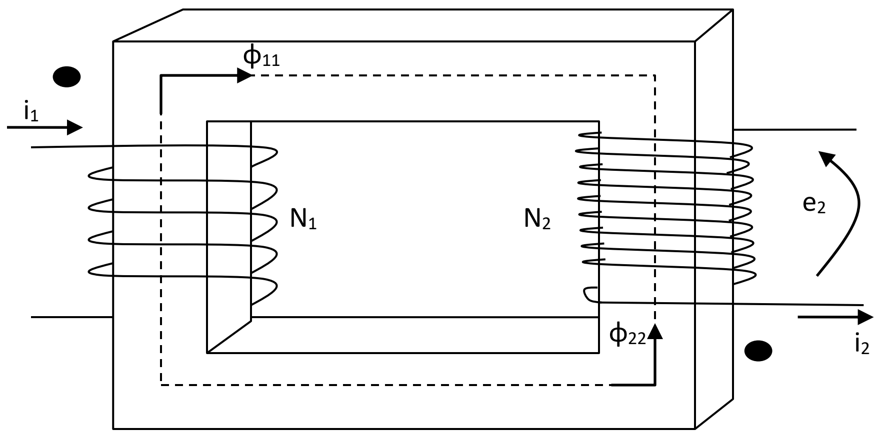



<u>**Solution**</u>:

Steps 1-3:




Steps 4-6:




Now we can write the equation for the above coupled circuits. Recall that in the dot convention, we will mark one terminal on either side of the transformer so that

- when $e_2$ is defined positive at the dotted terminal of coil 2 and
- $i_1$ is into the dotted terminal of coil 1, then

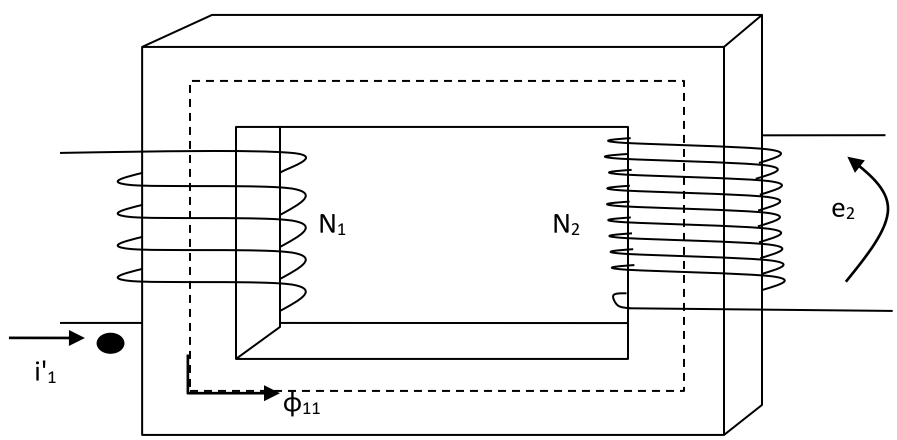
$$e_2 = +M\frac{di_1}{dt}$$

In the above, however, although $i_2$ is into the dotted terminal of coil 1, $e_2$ is defined negative at the dotted terminal of coil 2. Therefore

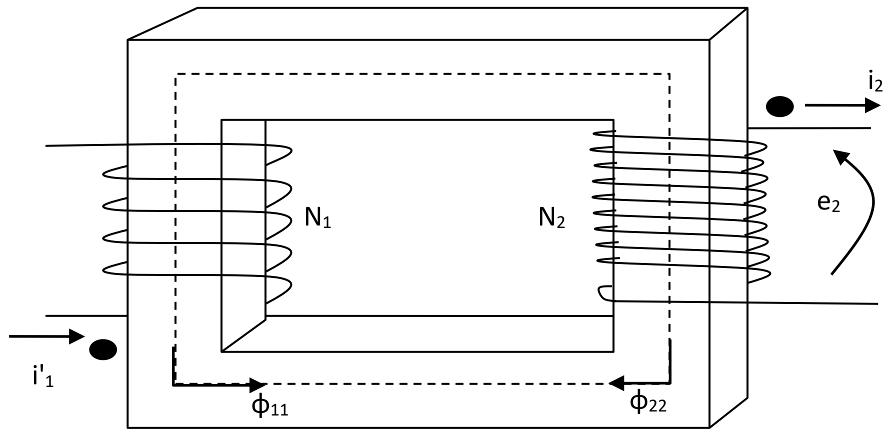
$$e_2 = -M\frac{di_1}{dt}$$

But note, there is another way we could have solved this problem, as follows:

Steps 1-3:



Steps 4-6:




Now we can write the equation for the above coupled circuits. Recall that in the dot convention, we will mark one terminal on either side of the transformer so that

- when $e_2$ is defined positive at the dotted terminal of coil 2 and
- $i_1$ is into the dotted terminal of coil 1, then

$$e_2 = +M\frac{di_1}{dt}$$

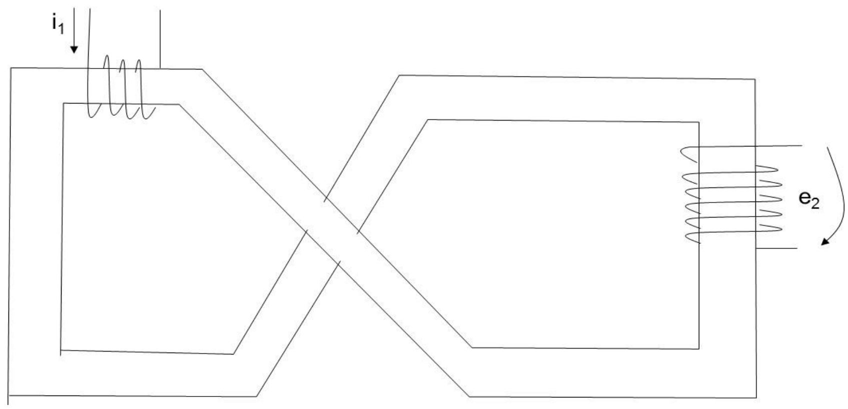
In the above, we have $i'_1$ flowing into the dotted terminal of coil 1, and $e_2$ is defined positive at the dotted terminal of coil 2. Therefore

$$e_2 = +M\frac{di_1'}{dt}$$

If, however, we wanted to express $e_2$ as a function of $i_1$ (observing that $i_1=-i'_1$) then we would have

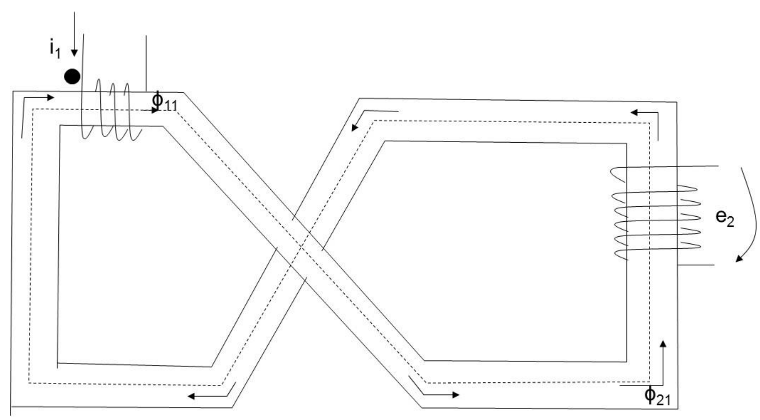
$$e_2 = -M\frac{di_1}{dt}$$

**<u>Example 5</u>**: For the configuration below, determine the dotted terminals and write the relation between $i_1$ and $e_2$.

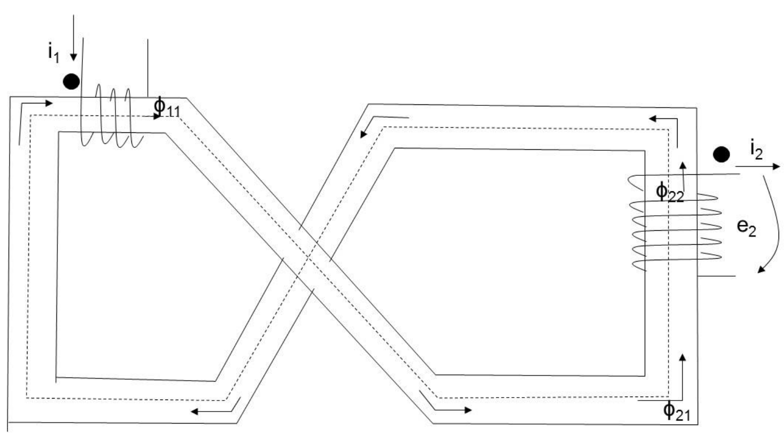



**Solution**:

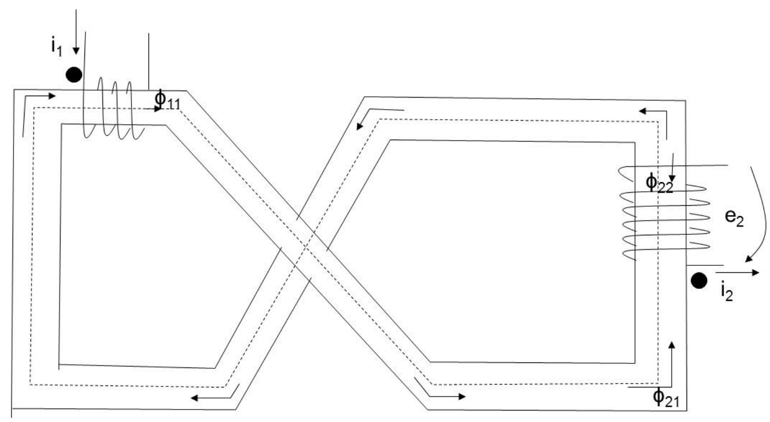
Steps 1-3:




Steps 4-6: Here we arbitrarily assign the dot to the upper terminal of coil 2 and then, with $i_2$ out of this dotted terminal, we use the RHR to determine that the flux $\phi_{22}$ is in the same direction as the flux from coil 1. This means our choice of the coil 2 terminal location dot was wrong.



Therefore we know the dot must be at the other terminal, and the below shows clearly this is the case, since the flux from coil 2, $\phi_{22}$, is opposite to the flux from coil 1, $\phi_{21}$.




Now we can write the equation for $e_2$. Recall that in the dot convention, we mark one terminal on either side so that

- when $e_2$ is defined positive at the dotted terminal of coil 2 and
- $i_1$ is into the dotted terminal of coil 1, then

$$e_2 = +M\frac{di_1}{dt}$$

## 5.0 Writing circuit equations for coupled coils

In our treatment so far, we have focused on transformers or similar circuits having magnetic coupling between coils. We may also encounter other kinds of circuits having elements that are magnetically coupled. We are well-positioned to handle such circuits by combining our (new) knowledge of the dot convention with our (old) knowledge of circuit analysis. The issue here, as in all circuits with mutual coupling, is that when we write a voltage equation, we must account for the self-induced voltage in an inductor from its own current as well as any mutually-induced voltage in the inductor from a current in a coupled coil.

Let's begin with the simplest of examples. Consider the circuit illustrated in Fig. 9.

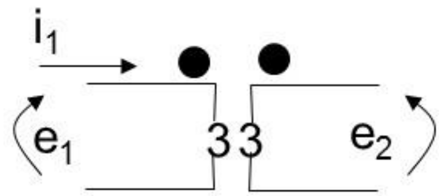



**Fig. 9**

We know that

$$e_1 = \frac{d\lambda_1}{dt} \tag{26}$$

where

$$\lambda_1 = \lambda_{11} + \lambda_{12} \tag{26}$$

Here, $\lambda_1$ is the total flux seen by coil 1, $\lambda_{11}$ is the flux from coil 1 linking coil 1 (i.e., the component of $\lambda_1$ that is produced by coil 1), and $\lambda_{12}$ is the flux from coil 2 linking with coil 1 (i.e., the component of $\lambda_1$ that is produced by coil 2). And so to compute the voltage induced across coil 1, we apply Faraday's Law to obtain

$$e_1 = \frac{d\lambda_1}{dt} = \frac{d(\lambda_{11} + \lambda_{12})}{dt} = \frac{d\lambda_{11}}{dt} + \frac{d\lambda_{12}}{dt} \tag{26}$$

Recalling that

$$\lambda_{11} = L_1 i_1 \qquad \lambda_{12} = M i_2 \tag{27}$$

we can substitute (27) into (26) to obtain

$$e_1 = \pm\frac{d(L_1 i_1)}{dt} \pm \frac{d(Mi_2)}{dt} = \pm L_1 \frac{di_1}{dt} \pm M\frac{di_2}{dt} \tag{28}$$

Observe in (28) the presence of the "±" signs preceding each term; these signs are made necessary by the fact we have equated the sum of the derivatives to a voltage $e_1$ with a certain assumed polarity. The question we need to answer at this point is "how do we determine the sign of each of these terms?" We begin with the first ("self") term because it is the easiest.

**Rule for determining the sign of the self term**: The polarity of the self term is determined entirely by the direction of the current $i_1$:

- when this current is into the positive terminal (as defined by the polarity of $e_1$), then the sign of the self term is positive;
- when this current is out of the positive terminal (as defined by the polarity of $e_1$), then the sign of the self term is negative.

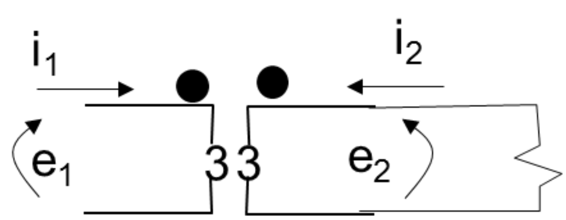
Now we need to determine how to know whether to add or subtract the mutual term from the self term. We should not be surprised to learn that we will make this determination using the dot convention.

**Rule for determining the sign of the mutual term**: We assume both coils have been appropriately dotted.

1. Choose reference current directions for each coil (if not chosen for you).
2. Apply the following to determine the reference polarity of the voltage induced by the mutual effects:
    a. If the reference current direction enters the dotted terminal of a coil, the reference polarity of the voltage that it induces in the other coil is positive at its dotted terminal.
    b. If the reference current direction leaves the dotted terminal of a coil, the reference polarity of the voltage that it induces in the other coil is negative at its dotted terminal.

**Example 6**:

Express the voltages $e_1$ and $e_2$ as a function of the currents $i_1$and $i_2$ in the following circuit.




First, let's express $e_1$. Here, we observe two things:

1. $i_1$ enters the positive terminal, and therefore the self term is positive.
2. $i_2$ enters the dotted terminal of coil 2, therefore the reference polarity of the voltage it induces in coil 1 is positive at its dotted terminal, and its dotted terminal is the positive terminal.

So $e_1$ is expressed as:

$$e_1 = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}$$

Now let's express $e_2$. Here, we observe two things:

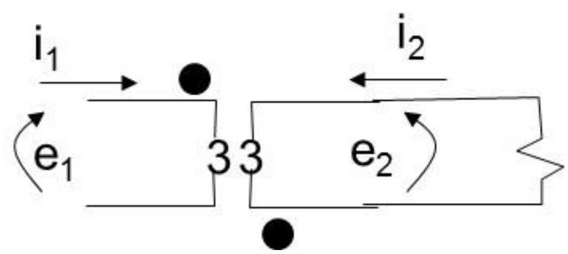
1. $i_2$ enters the positive terminal, and therefore the self term is positive.
2. $i_1$ enters the dotted terminal of coil 1, therefore the reference polarity of the voltage it induces in coil 2 is positive at its dotted terminal, and its dotted terminal is the positive terminal.

So $e_2$ is expressed as:

$$e_2 = L_2 \frac{di_2}{dt} + M \frac{di_1}{dt}$$

<u>**Example 7**</u>:

Express the voltages $e_1$ and $e_2$ as a function of the currents $i_1$and $i_2$ in the following circuit.




As in Example 6, we first express $e_1$. Here, we observe two things:

1. $i_1$ enters the positive terminal, and therefore the self term is positive.
2. $i_2$ leaves the dotted terminal of coil 2, therefore the reference polarity of the voltage it induces in coil 1 is negative at its dotted terminal, and its dotted terminal is the positive terminal.

So $e_1$ is expressed as:

$$e_1 = L_1 \frac{di_1}{dt} - M \frac{di_2}{dt}$$

Now let's express $e_2$. Here, we observe two things:

1. $i_2$ enters the positive terminal, and therefore the self term is positive.

2. $i_1$ enters the dotted terminal of coil 1, therefore the reference polarity of the voltage it induces in coil 2 is positive at its dotted terminal, but its dotted terminal is the negative terminal.

So $e_2$ is expressed as:

$$e_2 = L_2 \frac{di_2}{dt} - M \frac{di_1}{dt}$$

## 6.0 Derivation of turns ratio relations for ideal transformers

Consider the circuit of Fig. 10; what is in the dashed box represents what is referred to as the *ideal transformer*.

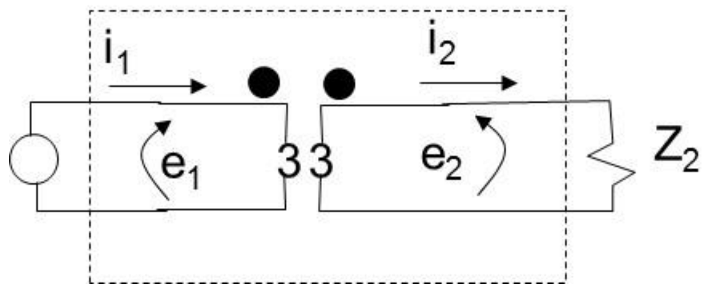



**Fig. 10**

Let's write the voltage equation for left-hand loop. Note that the current $i_2$ is leaving the dotted terminal, and so the voltage it induces in coil 1 must be negative at the coil 1 dotted terminal, which is the positive terminal. Therefore

$$e_1 = L_1 \frac{di_1}{dt} - M \frac{di_2}{dt} \tag{29}$$

Likewise, we obtain for the right-hand loop, given by:

$$e_2 = -L_2 \frac{di_2}{dt} + M \frac{di_1}{dt} = i_2 R_2 \tag{30}$$

Solving (30) for the mutual term results in

$$M \frac{di_1}{dt} = L_2 \frac{di_2}{dt} + i_2 Z_2 \tag{31}$$

We can write equations (29) and (31) in phasor representation by realizing that, for analysis of steady-state sinusoidal quantities, differentiation in the time-domain is equivalent to multiplication by jω in the phasor domain. This may be a familiar notion to those who have studied Fourier transforms. For those who have not studied Fourier transforms, it is easy to see from calculus, as follows. Let $i_1(t)=I_1\sin\omega t$, then we may express the phasor as $\mathbf{I_1}=I_1\angle 0°$. We observe then that the phasor transform of $i_1(t)$ can be expressed as

$$i_1(t) = I_1 \sin \omega t \leftrightarrow \boldsymbol{I}_1 = I_1 \angle 0° \tag{32}$$

We can differentiate the time-domain expression for $i_1(t)$ to obtain $di_1(t)/dt=I_1\omega \cos\omega t= I_1\omega \sin(\omega t+90°)$. Thus, we see that the phasor transform of $di_1(t)/dt$ may be expressed as

$$\frac{di_1(t)}{dt} = I_1\omega \sin(\omega t + 90°) \leftrightarrow I_1\omega\angle 90° = jI_1\omega\angle 0° = j\omega\boldsymbol{I}_1 \tag{33}$$

We use (32) and (33) to transform (29) and (31) into the phasor domain as follows:

$$\boldsymbol{E}_1 = j\omega L_1\boldsymbol{I}_1 - j\omega M\boldsymbol{I}_2 \tag{34}$$

$$j\omega M\boldsymbol{I}_1 = j\omega L_2\boldsymbol{I}_2 + Z_2\boldsymbol{I}_2 \tag{35}$$

We solve for $\mathbf{I}_2$ from (35) to obtain:

$$\boldsymbol{I}_2 = \frac{j\omega M}{j\omega L_2 + Z_2}\boldsymbol{I}_1 \tag{36}$$

Now recall from (10) that

$$L_2 = \frac{\mu A N_2^2}{l} = \frac{N_2^2}{\mathcal{R}} \tag{10}$$

$$\mathcal{R} = \frac{l}{\mu A} \tag{37}$$

Let’s assume that μ is very large (the material is highly permeable). This implies that $\mathcal{R}$ is very small, $L_2$ is very large, and so $|j\omega L_2|>>R_2$. Therefore (36) becomes

$$\boldsymbol{I}_2 = \frac{j\omega M}{j\omega L_2}\boldsymbol{I}_1 = \frac{M}{L_2}\boldsymbol{I}_1 \tag{38}$$

Also recall (18) that

$$M = \frac{N_1 N_2}{\mathscr{R}} \tag{18}$$

Substitution of (10) and (18) into (38) results in

$$\boldsymbol{I}_2 = \frac{M}{L_2}\boldsymbol{I}_1 = \frac{\frac{N_1 N_2}{\mathscr{R}}}{\frac{N_2^2}{\mathscr{R}}}\boldsymbol{I}_1 = \frac{N_1 N_2}{N_2^2}\boldsymbol{I}_1 = \frac{N_1}{N_2}\boldsymbol{I}_1 \tag{39}$$

That is,

$$\frac{\boldsymbol{I}_2}{\boldsymbol{I}_1} = \frac{N_1}{N_2} \tag{40}$$

Equation (40) is an important relation. It says the ratio of the currents in the coils on either side of an ideal transformer is in inverse proportion to the ratio of the coils' turns. We will use it heavily.

We may also derive from (34) and (35) the following relation

$$\boldsymbol{E}_1 = \left[\frac{\omega^2 M^2 - \omega^2 L_1 L_2 + j\omega L_1 Z_2}{Z_2 + j\omega L_2}\right]\boldsymbol{I}_1 \tag{41}$$

Using $M=\sqrt{(L_1 L_2)}$, we obtain

$$\boldsymbol{E}_1 = \left[\frac{j\omega L_1 Z_2}{Z_2 + j\omega L_2}\right]\boldsymbol{I}_1 \tag{42}$$

Again using $|j\omega L_2|>>|Z_2|$, we obtain

$$\boldsymbol{E}_1 = \frac{L_1 Z_2}{L_2}\boldsymbol{I}_1 \tag{43}$$

Substituting (10) and

$$L_1 = \frac{N_1^2}{\mathscr{R}} \tag{44}$$

into (43), we obtain

$$\boldsymbol{E}_1 = \frac{L_1 Z_2}{L_2}\boldsymbol{I}_1 = \frac{\frac{N_1^2}{\mathcal{R}} Z_2}{\frac{N_2^2}{\mathcal{R}}}\boldsymbol{I}_1 = \frac{N_1^2}{N_2^2} Z_2 \boldsymbol{I}_1 \tag{45}$$

But from (40), we can write that

$$\boldsymbol{I}_1 = \frac{N_2}{N_1}\boldsymbol{I}_2 \tag{46}$$

Substituting (46) into (45) results in

$$\boldsymbol{E}_1 = \frac{N_1^2}{N_2^2} R_2 \frac{N_2}{N_1}\boldsymbol{I}_2 = \frac{N_1}{N_2} R_2 \boldsymbol{I}_2 \tag{47}$$

But $R_2\mathbf{I}_2=\mathbf{E}_2$, and so (47) becomes

$$\boldsymbol{E}_1 = \frac{N_1}{N_2}\boldsymbol{E}_2 \tag{48}$$

That is,

$$\frac{\boldsymbol{E}_1}{\boldsymbol{E}_2} = \frac{N_1}{N_2} \tag{49}$$

Like (40), (49) is an important relation. It says the ratio of the voltage across the coils on either side of an ideal transformer is in proportion to the ratio of the coils' turns. We will also use this relation heavily.

## 7.0 Power for ideal transformers

Often, high voltage transformers are called "power transformers." Do they transform power? To answer this question, let's first realize that this discussion has nothing to do with losses that occur within a real transformer. Our device is "ideal," and thus suffers no losses. We return to the ideal transformer of Fig. 10 and the relations we derived for it in (40) and (49), repeated here for convenience:

$$\frac{\boldsymbol{I}_2}{\boldsymbol{I}_1} = \frac{N_1}{N_2} \tag{40}$$

$$\frac{\boldsymbol{E}_1}{\boldsymbol{E}_2} = \frac{N_1}{N_2} \tag{49}$$

We may express the power flowing into coil 1, and the power flowing out of coil 2 as

$$S_1 = \boldsymbol{E}_1\boldsymbol{I}_1^* \qquad\qquad S_2 = \boldsymbol{E}_2\boldsymbol{I}_2^*$$

From (40) and (49) we may express $\mathbf{E}_2$ and $\mathbf{I}_2$ as

$$\boldsymbol{E}_2 = \frac{N_2}{N_1}\boldsymbol{E}_1 \qquad\qquad \boldsymbol{I}_2 = \frac{N_1}{N_2}\boldsymbol{I}_1$$

Substitution of these expressions for $\mathbf{E}_2$ and $\mathbf{I}_2$ into the power expression for $S_2$ results in

$$S_2 = \boldsymbol{E}_2\boldsymbol{I}_2^* = \frac{N_2}{N_1}\boldsymbol{E}_1\frac{N_1}{N_2}\boldsymbol{I}_1^* = \boldsymbol{E}_1\boldsymbol{I}_1^*$$

We have just proved that, for an ideal transformer, $S_1=S_2$, enabling us to conclude that "power transformers" do not transform power. It is a good thing, because doing so would result in a violation of the conservation of energy (otherwise known as the first law of thermodynamics), since "power transformation" would imply that we could provide one side with a certain amount of power $P_1$ and get out a greater amount of power $P_2$ on the other side. If we allowed, then, such a device to operate for an amount of time T, the output energy $P_2T$ would be greater than the input energy $P_1T$, thus, the violation.

## 8.0 Referring quantities

We have heretofore referred to the two sides of an ideal transformer as "coil 1" and "coil 2"; from now on, we refer to these as the "primary" and "secondary" of the device.

It is of interest to determine what impedance is "seen" looking into the primary terminals of the ideal transformer, i.e., what is

$$Z_1 = \frac{\boldsymbol{E}_1}{\boldsymbol{I}_1} \tag{50}$$

But from (46) and (48), we have that

$$\boldsymbol{I}_1 = \frac{N_2}{N_1}\boldsymbol{I}_2 \qquad\qquad \boldsymbol{E}_1 = \frac{N_1}{N_2}\boldsymbol{E}_2$$

Substitution of these into (50) results in

$$Z_1 = \frac{\frac{N_1}{N_2}\boldsymbol{E}_2}{\frac{N_2}{N_1}\boldsymbol{I}_2} = \frac{N_1^2}{N_2^2}\frac{\boldsymbol{E}_2}{\boldsymbol{I}_2} = \frac{N_1^2}{N_2^2}Z_2 \tag{51}$$

More compactly, we have

$$Z_1 = \left(\frac{N_1}{N_2}\right)^2 Z_2 \tag{52}$$

Or,

$$\frac{Z_1}{Z_2} = \left(\frac{N_1}{N_2}\right)^2 \tag{53}$$

This equation, together with (40) and (49), are summarized below.

$$\frac{\boldsymbol{I}_2}{\boldsymbol{I}_1} = \frac{N_1}{N_2} \tag{40}$$

$$\frac{\boldsymbol{E}_1}{\boldsymbol{E}_2} = \frac{N_1}{N_2} \tag{49}$$

$$\frac{Z_1}{Z_2} = \left(\frac{N_1}{N_2}\right)^2 \tag{53}$$

Equations (40), (49), and (53) relate currents, voltages, and impedances that exist on one side of the transformer, i.e., the secondary, to corresponding currents, voltages, and impedances that exist on the other side of the transformer, i.e., the primary.

But from another perspective, they provide a way to *refer* quantities from one side of the transformer to the other. That is, these equations provide a way to represent on the primary side currents, voltages, and impedances that exist on one the secondary side. And vice-versa, i.e., these equations provide a way to represent on the secondary side currents, voltages, and impedances that exist on the primary side.

To clarify this, let's introduce some new notation.

- First, we observe that the subscripts "1" and "2" that we have been using (for turns, currents, voltages, and impedances) tell us whether the quantity actually exists on the primary side (having a subscript of "1") or the secondary side (having a subscript of "2").
- Second, we will use unprimed notation, i.e., $I_1$, $E_1$, $Z_1$, and $I_2$, $E_2$, $Z_2$, to denote the quantity represented on the side on which it actually exists. Thus, we say that
  - $I_1$, $E_1$, $Z_1$ are the current, voltage, and impedance of *primary side quantities referred to the primary side*, and
  - $I_2$, $E_2$, $Z_2$ are the current, voltage, and impedance of secondary side quantities referred to the secondary side.
- Third, we will use primed notation, i.e., $I''_1$, $E''_1$, $Z''_1$, and $I'_2$, $E'_2$, $Z'_2$, to denote the quantity represented on the opposite side from where it actually exists. Thus, we say that
  - $I''_1$, $E''_1$, $Z''_1$ are the current, voltage, and impedance of *primary side quantities referred to the secondary side*, and
  - $I'_2$, $E'_2$, $Z'_2$ are the current, voltage, and impedance of secondary side quantities referred to the primary side.

With this notation, we may adapt (40), (49), and (53) to facilitate the operation of referring quantities from the secondary side to the primary as follows:

$$\frac{\boldsymbol{I}_2}{\boldsymbol{I}'_2} = \frac{N_1}{N_2} \Rightarrow \boldsymbol{I}'_2 = \frac{N_2}{N_1}\boldsymbol{I}_2 \tag{54}$$

$$\frac{\boldsymbol{E}'_2}{\boldsymbol{E}_2} = \frac{N_1}{N_2} \Rightarrow \boldsymbol{E}'_2 = \frac{N_1}{N_2}\boldsymbol{E}_2 \tag{55}$$

$$\frac{Z'_2}{Z_2} = \left(\frac{N_1}{N_2}\right)^2 \Rightarrow Z'_2 = \left(\frac{N_1}{N_2}\right)^2 Z_2 \tag{56}$$

Relations for referring quantities from the primary side to the secondary side are as follows:

$$\frac{\boldsymbol{I}''_1}{\boldsymbol{I}_1} = \frac{N_1}{N_2} \Rightarrow \boldsymbol{I}''_1 = \frac{N_1}{N_2}\boldsymbol{I}_1 \tag{57}$$

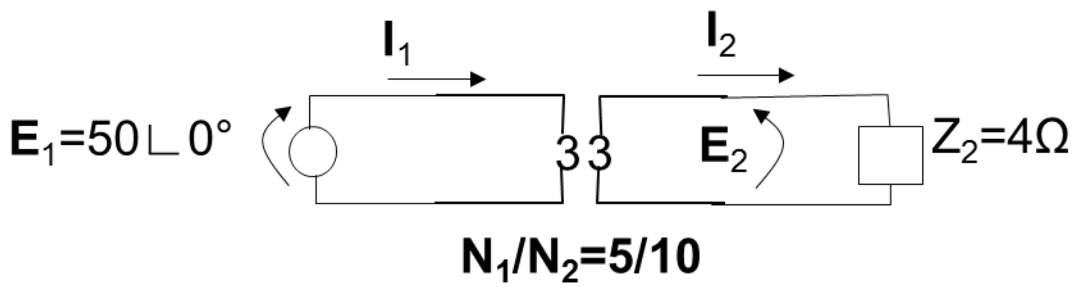
$$\frac{\boldsymbol{E}_1}{\boldsymbol{E}_1''} = \frac{N_1}{N_2} \Rightarrow \boldsymbol{E}_1'' = \frac{N_2}{N_1}\boldsymbol{E}_1 \tag{58}$$

$$\frac{Z_1}{Z_1''} = \left(\frac{N_1}{N_2}\right)^2 \Rightarrow Z_1'' = \left(\frac{N_2}{N_1}\right)^2 Z_1 \tag{59}$$

It is important to note that referring quantities affects only magnitudes; it does not affect phase angles.

**Example 8**:

Find the current in the primary side of the below circuit.

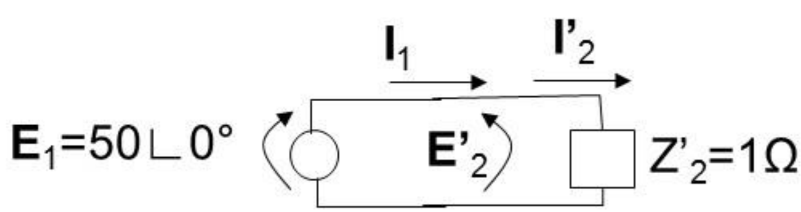



**Solution**: We can solve this problem in one of two ways. Either we refer all quantities to the secondary, solve for $\mathbf{I}_2$, and then refer this current to the primary, or else, we can refer all quantities to the primary and then solve for $\mathbf{I}_1$. Both ways will work, but the second way is a little easier.

Referring quantities to the primary requires the following calculation:

$$Z_2' = \left(\frac{N_1}{N_2}\right)^2 Z_2 = \left(\frac{5}{10}\right)^2 4 = \frac{25}{100}4 = 1$$

This results in the following circuit:




Use of Ohm's Law results in

$$I_1 = I_2' = \frac{50\angle 0°}{1} = 50\angle 0°$$

Now what if we wanted to obtain **I**$_2$ and **E**$_2$? Then we refer I'$_2$ and E'$_2$ (which are quantities we obtain on the primary side) back to the secondary side. We already know I'$_2$; we obtain E'$_2$ by inspection of the above circuit, observing that it is the same as E$_1$, i.e., E'$_2$=50∟0°. We use (54) and (55) to referring back to the secondary as follows:

$$\frac{I_2}{I_2'} = \frac{N_1}{N_2} \Rightarrow I_2 = \frac{N_1}{N_2} I_2' = \frac{5}{10} 50\angle 0° = 25\angle 0°$$

$$\frac{E_2'}{E_2} = \frac{N_1}{N_2} \Rightarrow E_2 = \frac{N_2}{N_1} E_2' = \frac{10}{5} 50\angle 0° = 100\angle 0°$$

Then it is satisfying to check that

$$Z_2 = \frac{E_2}{I_2} = \frac{100\angle 0°}{25\angle 0°} = 4\Omega$$

## 8.0 Exact and approximate transformer models

…(see in-class notes)

## 9.0 Three-phase transformers

A three-phase transformer will have six windings: three for the primary (phases A, B, and C on the primary) and three for the secondary (phases A, B, and C for the secondary).

There are two very different approaches to developing a three-phase transformer. One approach is to use just one “three phase bank,” where here the word “bank” refers to a single core, i.e., a single magnetic circuit. Figure 9 [1] illustrates, where each pair of primary and secondary windings (there are three pairs, one for each phase) are on the same leg.

[1] http://www.gamatronic.com/three-phase-transformers/

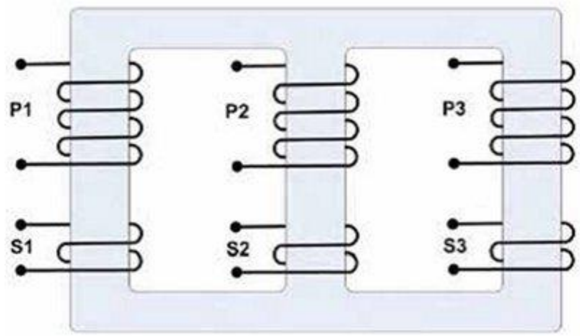



**Figure 9: A three-phase transformer bank**

The three-phase bank illustrated in Fig. 9 is utilizing a “core-type” of construction. In the core-type, primary and secondary windings are wound outside and surround their leg. In another type of construction, the so-called “shell-type,” windings pass inside the core, forming a shell around the windings. Figure 10 [2] illustrates the difference.

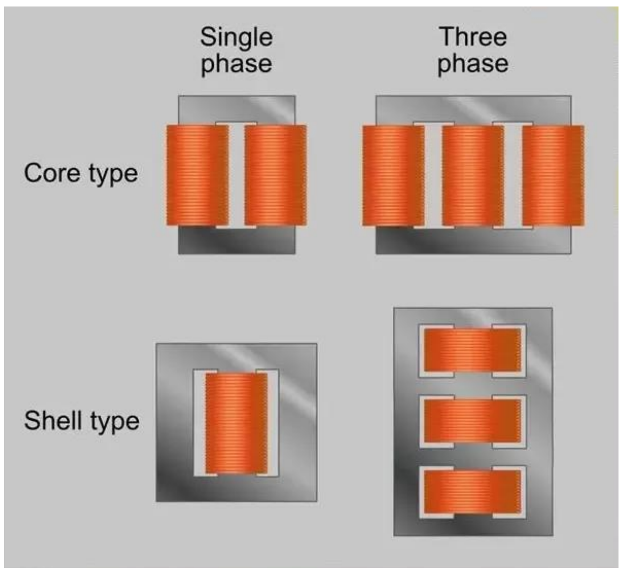



**Figure 10: Core type and shell type transformer construction**

The shell-type transformer requires more ferromagnetic material than does the core-type transformer and is therefore typically more costly on a per-MVA basis. However, its magnetic circuit provides multiple paths for flux to flow; this reduces flux density seen by each leg which is advantageous for short-circuit performance [3, 4], and so shell-type transformers are used more frequently when high capacity is required. Another approach to developing a three-phase transformer is to interconnect three single-phase transformers. This approach is illustrated in Fig. 11.

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[2] https://www.quora.com/What-is-the-difference-between-a-core-type-and-a-shell-type-transformer
[3] J. Harlow, editor, “Electric power transformer engineering,” CRC press, 2004, pp. 2-10.
[4] B. Kennedy, “Energy efficient transformers,” McGraw-Hill, 1998, p. 16.

Note that wires that cross are not connected unless they have a dot ⇒ —+—

There are actually coils in each phase of these pictures.

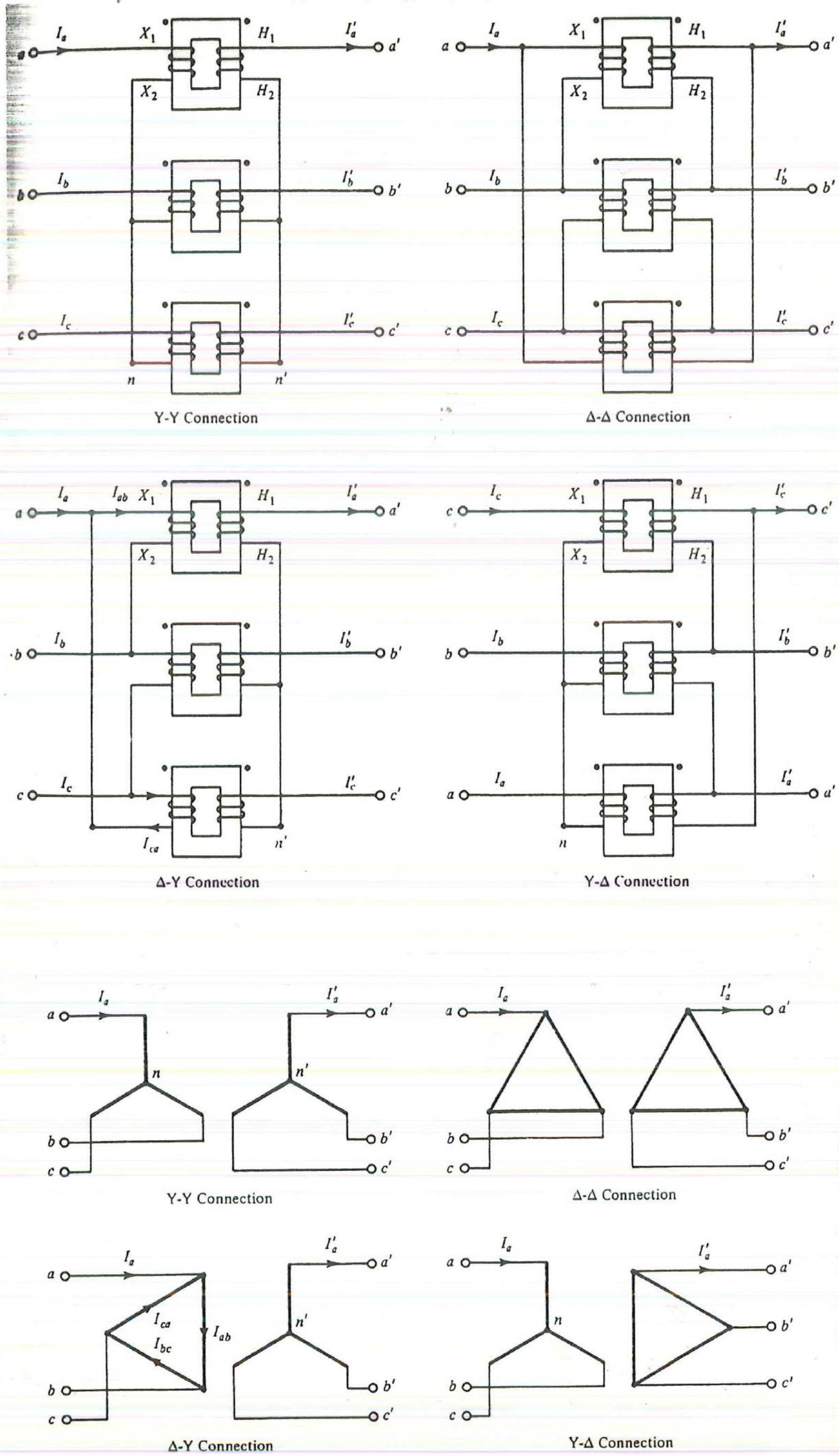


**Fig. 11: Single-phase transformer connections to form a three-phase transformer**

The voltage transformation ratio for three-phase transformers is always given as the ratio of the line-to-line voltage magnitude on either side. If a transformer is connected Y-Y or Δ-Δ, this ratio will be the same as the ratio of the winding voltages on either side (which is the same as the turns ratio $N_1/N_2$).

However, for Y-Δ or Δ-Y connected transformers, the ratio of the line-to-line voltages on either side is ***not*** the same as the ratios of the winding voltages. This means that you can take three single-phase transformers, each with the same turns ratio, and connect them for a three-phase configuration such that the three-phase configuration will have a different line-to-line ratio than the phase-to-phase ratio!

…see in-class examples.

It is the line-to-line ratio that you should use in performing per-phase analysis.