Module E3-a
Economic Dispatch
HW: Probs 1-5, 11
Due Tuesday 4/23

Min f(x)
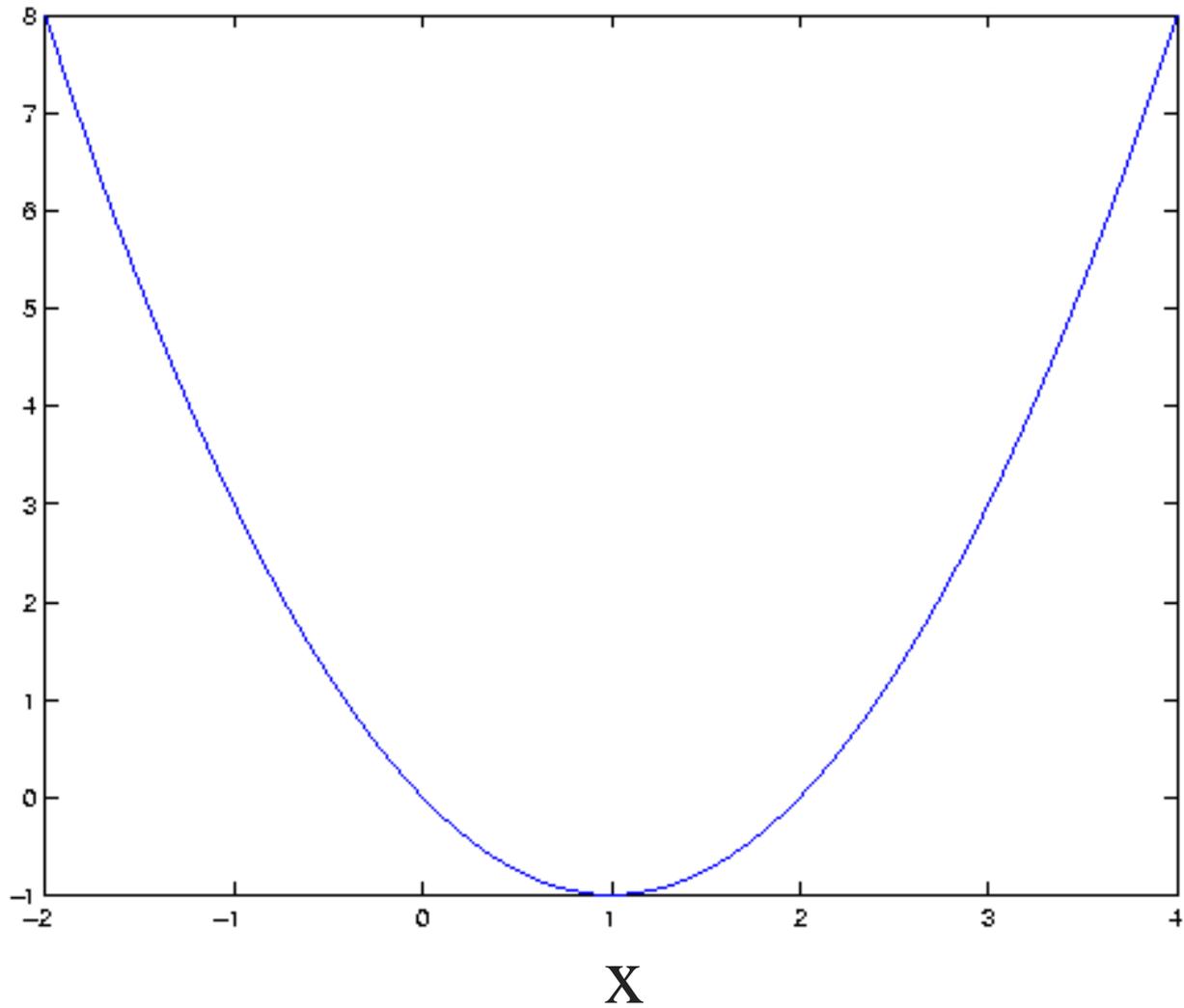
$$f(x) = x^2 - 2x$$

$$\frac{df(x)}{dx} = 2x - 2 = 0$$

$x = 1$ minimizes this function

(Also, f(1) = -1)

Min f(x) subject to h(x)=x=0.5

$$F(x) = f(x) - \lambda(h(x) - 0.5) = x^2 - 2x - \lambda(x - 0.5)$$

$$\frac{\partial F(x)}{\partial x} = 2x - 2 - \lambda = 0$$

$$\frac{\partial F(x)}{\partial \lambda} = -x + 0.5 = 0$$

$x = 0.5$ minimizes this function

(also, $\lambda = -1$, f(0.5) = −0.75)

Min f(x) subject to g(x)=x≤0.5

$$F(x) = f(x) - \mu(g(x) - 0.5) = x^2 - 2x - \mu(x - 0.5)$$

$$\frac{\partial F(x)}{\partial x} = 2x - 2 - \mu = 0$$

$$\frac{\partial F(x)}{\partial \mu} = -x + 0.5 = 0$$

$x = 0.5$ minimizes this function

(Also, $\mu = $ -1, f(0.5) = -0.75)

Min f(x) subject to g(x)=x$\leq$1.5

$$F(x) = f(x) - \mu(g(x) - 1.5) = x^2 - 2x - \mu(x - 1.5)$$

$$\frac{\partial F(x)}{\partial x} = 2x - 2 - \mu = 0$$

$$\frac{\partial F(x)}{\partial \mu} = -x + 1.5 = 0$$

$x = 1.5$ DOES NOT minimize this function!

Conclusion:

Our approach seems to work fine for

- **equality constraints**

- **and for inequality constraints when we know they are binding**

but not for inequality constraints when they are not binding!

General solution approach when we have inequality constraints:

Solve problem <u>without</u> them and then check to see if inequality constraints are satisfied. If not, add the violated constraint(s) as equality constraints.

The previous problems illustrated simple applications of the general approach to solving optimization problems.

Optimization problems can also be multivariable.

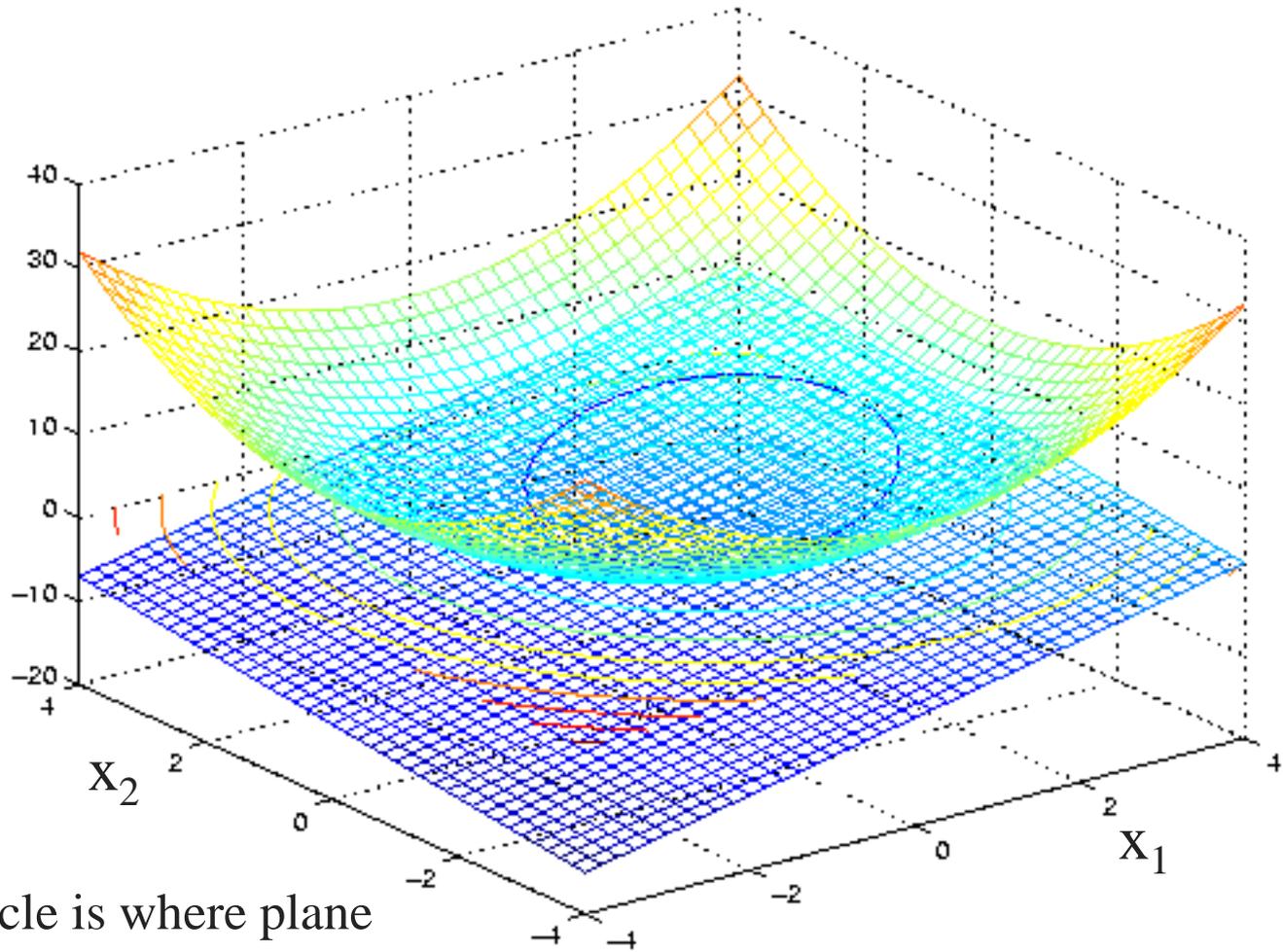They can also have both equality and inequality constraints.

Another example.

$$\min \ f(x_1, x_2) = x_1{}^2 + x_2{}^2$$

subject to

$$h(x_1, x_2) = 2x_1 + x_2 = 3$$

$$g(x_1, x_2) = x_2 \leq 0.5$$

$f(x_1,x_2)$

x_2

x_1

Circle is where plane
intersects the bowl and represents the set of feasible solutions
of the equality-constrained problem.

Solve, ignoring inequality constraint.

$$F(x_1, x_2) = x_1^{\,2} + x_2^{\,2} - \lambda(2x_1 + x_2 - 3)$$

$$\frac{\partial F}{\partial x_1} = 2x_1 - 2\lambda = 0$$

$$\frac{\partial F}{\partial x_2} = 2x_2 - \lambda = 0$$

$$\frac{\partial F}{\partial \lambda} = 2x_1 + x_2 - 3 = 0$$

$$\begin{pmatrix} 2 & 0 & -2 \\ 0 & 2 & -1 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ \lambda \end{pmatrix} = \begin{pmatrix} 1.2 \\ 0.6 \\ 1.2 \end{pmatrix} \qquad f(\underline{x}) = 1.8$$

But $x_2 > 0.5$, so this solution is infeasible.

So we must bring in inequality constraint.

$$F(x_1, x_2) = x_1^2 + x_2^2 - \lambda(2x_1 + x_2 - 3) - \mu(x_2 - 0.5)$$

$$\frac{\partial F}{\partial x_1} = 2x_1 - 2\lambda = 0$$

$$\frac{\partial F}{\partial x_2} = 2x_2 - \lambda - \mu = 0$$

$$\frac{\partial F}{\partial \lambda} = 2x_1 + x_2 - 3 = 0$$

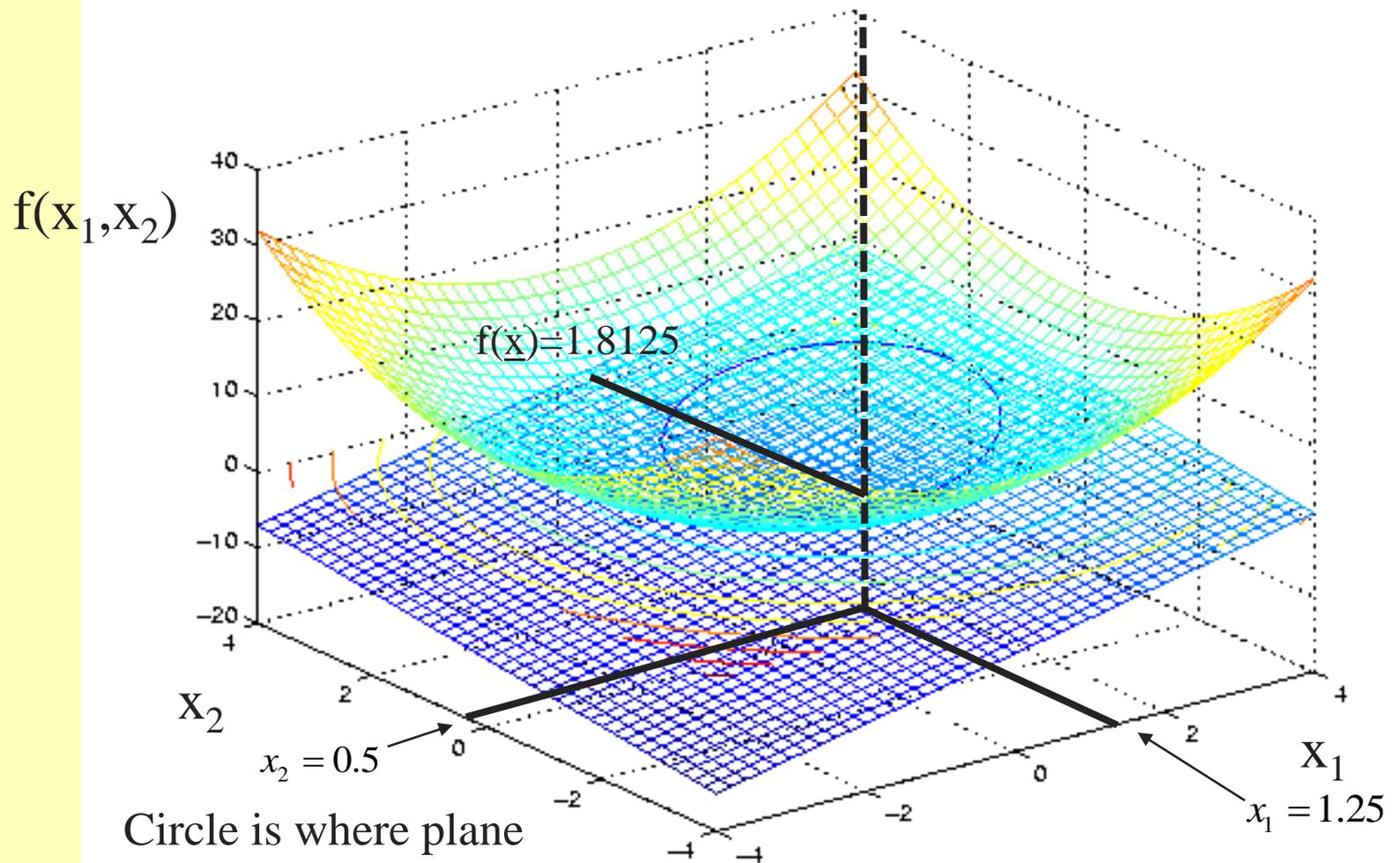$$\frac{\partial F}{\partial \mu} = x_2 - 0.5 = 0$$

$$\begin{pmatrix} 2 & 0 & -2 & 0 \\ 0 & 2 & -1 & -1 \\ 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x1 \\ x2 \\ \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3 \\ 0.5 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} 1.25 \\ 0.5 \\ 1.25 \\ -0.25 \end{pmatrix}$$

$f(\underline{x})=1.8125$

Note $f(\underline{x})$ is larger than without the inequality constraint (solution got worse), but that the inequality constraint $x_2 \leq 0.5$ is satisfied. This is therefore the optimal feasible solution.

$f(x_1, x_2)$

$f(\underline{x}) = 1.8125$

x_2

$x_2 = 0.5$

x_1

$x_1 = 1.25$

Circle is where plane
intersects the bowl and represents the set of feasible solutions
of the equality-constrained problem.

Some definitions:

- $\underline{x}$ are decision variables
- $f(\underline{x})$ is the objective function
- $\underline{h}(\underline{x}) = \underline{c}$ are the equality constraints
- $\underline{g}(\underline{x}) \leq \underline{b}$ are the inequality constraints.

General optimization problem.

Min f $(\underline{x})$

subject to:

$$\underline{g}\,(\underline{x}) \leq \underline{b}$$
$$\underline{h}\,(\underline{x}) = \underline{c}$$

and: $\underline{x} \geq \underline{0}$

Aside:
- The non-negativity constraint on decision-variables is imposed to ensure the feasible region is bounded from below (reflecting engineering constraints imposing non-negativity).
- Doing so does not cause loss of generality because you can always make a non-positive decision variable x to be non-negative by defining y=-x.

General solution strategy
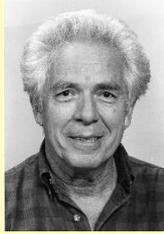
Form the LaGrangian Function:

$$F(\underline{x}, \lambda) \equiv f(\underline{x}) - \underline{\lambda}^T \left[\underline{h}(\underline{x}) - \underline{c} \right] - \underline{\mu}^T \left[\underline{g}(\underline{x}) - \underline{b} \right]$$

The underlines indicate vectors.

Theorem: The solution to the previous problem is found when the Karush-Kuhn-Tucker (KKT) conditions are satisfied.

Albert Tucker, 1905-1995
Harold Kuhn, 1925-2014
Published conditions in 1951.

William Karush, 1917-1997
Developed conditions in 1939 in his (unpublished) MS thesis.

$$\frac{\partial F}{\partial x_i} = 0 \qquad \forall\, i = 1, n$$

$$\frac{\partial F}{\partial \lambda_j} = 0 \qquad \forall\, j = 1, m$$

$$\mu_k \left[\, g_k(\underline{x}) - b_k \,\right] = 0 \qquad \forall\, k = 1, r$$

$$x_i \geq 0 \qquad \forall\, i = 1, n$$

$$\mu_k \geq 0 \qquad \forall\, k = 1, r$$

Aside: We assert (here) that the μ_k are the decrease in the objective function for an increase in b_k; then we see that non-negativity constraints are imposed on the μ_k because a negative μ_k would indicate increasing b_k (which enlarges the feasible region) makes the objective function larger (gets worse).

The complementarity condition (also known as the complimentary slackness condition):

$$\mu_k\left[g_k(x)-b\right]=0 \qquad \forall\, k=1,r$$

Complementarity: the state of working usefully together.

Complementary slackness: There cannot be slack in both a constraint and the associated dual variable (this one becomes clearer by studying duality, which we will not do).

If $\mu_k \neq 0$, then $g_k(\underline{x})-b_k=0$ (binding)

If $g_k(\underline{x})-b_k \neq 0$, then $\mu_k=0$ (not binding)

This is a math way of saying:

ignore constraint if not binding
and use it as equality constraint if binding.

Economic dispatch problem

Unit cost function:

$$\text{COST}_i = C_i(P_i) = a_i(P_i)^2 + b_iP_i + c_i \qquad (E3.40)$$

where COST_i = production cost

C_i = energy to cost conversion curve

P_i = production power

Unit capacity limits

$$P_i \geq \underline{\underline{P}}_i \qquad P_i \leq \overline{\overline{P}}_i$$

where :

$\underline{\underline{P_i}}$ = P_{min} = min *generator level*

$\overline{\overline{P_i}}$ = P_{max} = max *generator level*

Notation: double underline means lower bound.
Double overline means upper bound.

Transmission model

$$\sum_{i=1}^{n} P_i = P_D + P_{LOSS} + P_{tie} \qquad (E3.29)$$

General EDC problem statement.

$$\min f(P_i) = \sum_{i=1}^{n} C_i(P_i)$$

Subject to

$$\sum_{i=1}^{n} P_i \; = \; P_D + P_{LOSS} + P_{tie} \quad = \quad P_T$$

$$P_i \geq \underline{\underline{P_i}} \quad \Rightarrow \quad -P_i \leq -\underline{\underline{P_i}}$$

$$P_i \leq \overline{\overline{P_i}}$$

$$P_i \geq 0$$

General case, LaGrangian Function

$$F(\underline{P}, \lambda, \underline{\mu}) \;=\; \sum_{i=1}^{n} C_i(P_i) - \lambda \left[\sum_{i=1}^{n} P_i - P_T \right]$$

$$- \underline{\underline{\mu}}_1 [- P_1 + \underline{\underline{P}}_1] - \overline{\overline{\mu}}_1 [P_1 - \overline{\overline{P}}_1]$$

$$- \underline{\underline{\mu}}_2 [- P_2 + \underline{\underline{P}}_2] - \overline{\overline{\mu}}_2 [P_2 - \overline{\overline{P}}_2]$$

$$- \bullet \bullet \bullet \bullet$$

$$- \underline{\underline{\mu}}_n [- P_n + \underline{\underline{P}}_n] - \overline{\overline{\mu}}_n [P_n - \overline{\overline{P}}_n] \qquad (E3.31)$$

Notation: double underline for lower bound multiplier
double overline for upper bound multiplier.

General case, KKT conditions:

$$\frac{\partial F}{\partial P_i} = 0 \quad \Rightarrow \quad \frac{\partial C_i(P_i)}{\partial P_i} - \lambda + \underline{\mu_i} - \overline{\mu_i} = 0 \quad \forall i = 1, n \qquad (E3.32)$$

$$\frac{\partial F}{\partial \lambda} = 0 \quad \Rightarrow \quad \sum_{i=1}^{n} P_i - P_D - P_{LOSS} - P_{tie} = 0 \qquad (E3.33)$$

$$\underline{\mu}^{\mathrm{T}}\left[\underline{g}(\underline{x}) - \underline{c}\right] = \underline{0} \quad \Rightarrow \quad \underline{\mu_1}\left[-P_1 + \underline{P_1}\right] = 0, \ \overline{\mu_1}\left[P_1 - \overline{P_1}\right] = 0$$

$$\underline{\mu_2}\left[-P_2 + \underline{P_2}\right] = 0, \ \overline{\mu_2}\left[P_2 - \overline{P_2}\right] = 0$$

$$....$$

$$\underline{\mu_n}\left[-P_n + \underline{P_n}\right] = 0, \ \overline{\mu_n}\left[P_n - \overline{P_n}\right] = 0 \qquad (E3.34)$$

$$\mu_i \geq 0 \quad \forall \, i = 1, n \quad \Rightarrow \quad \underline{\mu_i} \geq 0, \overline{\mu_i} \geq 0 \quad \forall \, i = 1, n$$

Two unit system, LaGrangian Function:

$$F\left(P_1, P_2, \lambda, \underline{\underline{\mu_1}}, \overline{\overline{\mu_1}}, \underline{\underline{\mu_2}}, \overline{\overline{\mu_2}}\right) = C_1(P_1) + C_2(P_2)$$

$$- \lambda\left[P_1 + P_2 - P_T\right]$$

$$- \underline{\underline{\mu_1}}\left[-P_1 + \underline{\underline{P_1}}\right] - \overline{\overline{\mu_1}}\left[P_1 - \overline{\overline{P_1}}\right]$$

$$- \underline{\underline{\mu_2}}\left[-P_2 + \underline{\underline{P_2}}\right] - \overline{\overline{\mu_2}}\left[P_2 - \overline{\overline{P_2}}\right] \qquad (E3.35)$$

Two unit system, KKT conditions:

$$\frac{\partial F}{\partial P_1} = 0 \quad \Rightarrow \quad \frac{\partial C_1(P_1)}{\partial P_1} - \lambda + \underline{\underline{\mu_1}} - \overline{\overline{\mu_1}} = 0 \qquad (E3.36)$$

$$\frac{\partial F}{\partial P_2} = 0 \quad \Rightarrow \quad \frac{\partial C_2(P_2)}{\partial P_2} - \lambda + \underline{\underline{\mu_2}} - \overline{\overline{\mu_2}} = 0 \qquad (E3.37)$$

$$\frac{\partial F}{\partial \lambda} = 0 \quad \Rightarrow \quad P_1 + P_2 - P_T = 0 \qquad (E3.38)$$

$$\underline{\mu}^{\mathrm{T}}[\underline{h}(\underline{x}) - \underline{c}] = \underline{0} \quad \Rightarrow \quad \underline{\underline{\mu_1}}[-P_1 + \underline{\underline{P_1}}] = 0, \ \overline{\overline{\mu_1}}[P_1 - \overline{\overline{P_1}}] = 0$$

$$\underline{\underline{\mu_2}}[-P_2 + \underline{\underline{P_2}}] = 0, \ \overline{\overline{\mu_2}}[P_2 - \overline{\overline{P_2}}] = 0$$

Now let's add some numbers.

Table E3.4 Dispatch Data for Example E3.2

	Unit 1	Unit 2
Generation Specifications:		
Minimum Generation	200 MW	100 MW
Maximum Generation	380 MW	200 MW
Cost Curve Coefficients:		
Quadratic Term	0.016	0.019
Linear Term	2.187	2.407
Constant Term	120.312	74.074

Minimum load limitations are caused by fuel combustion stability and inherent steam generator design constraints. For example, most supercritical units cannot operate below 30% of design capability. A minimum flow of 30% is required to cool the tubes in the furnace of the steam generator adequately. – A. Wood and B. Wollenberg, "Power generation, operation, and control," 2nd edition, Wiley, 1996.

$$C_1(P_1) = 0.016(P_1)^2 + 2.187(P_1) + 120.312$$

$$C_2(P_2) = 0.019(P_2)^2 + 2.407(P_2) + 74.074$$

$$h(P_1, P_2) = P_1 + P_2 = 400$$

$$200 \leq P_1 \leq 380 \Rightarrow P_1 \leq 380, \ -P_1 \leq -200$$

$$100 \leq P_2 \leq 200 \Rightarrow P_2 \leq 200, \ -P_2 \leq -100$$

LaGrangian function

$$F\left(P_1, P_2, \lambda, \underline{\underline{\mu_1}}, \overline{\overline{\mu_1}}, \underline{\underline{\mu_2}}, \overline{\overline{\mu_2}}\right) = 0.016\ (P_1)^2 + 2.187\ (P_1) + 120.312$$

$$+ 0.019\ (P_2)^2 + 2.407\ (P_2) + 74.074$$

$$- \lambda[P_1 + P_2 - 400]$$

$$- \underline{\underline{\mu_1}}[-P_1 + 200] - \overline{\overline{\mu_1}}[P_1 - 380]$$

$$- \underline{\underline{\mu_2}}[-P_2 + 100] - \overline{\overline{\mu_2}}[P_2 - 200]$$

KKT conditions:

$$\frac{\partial F}{\partial P_1} = 0 \quad \Rightarrow \quad 0.032(P_1) + 2.187 - \lambda + \underline{\mu_1} - \overline{\mu_1} = 0$$

$$\frac{\partial F}{\partial P_2} = 0 \quad \Rightarrow \quad 0.038(P_2) + 2.407 - \lambda + \underline{\mu_2} - \overline{\mu_2} = 0$$

$$\frac{\partial F}{\partial \lambda} = 0 \quad \Rightarrow \quad P_1 + P_2 - 400 = 0$$

$$\underline{\mu}^{\mathrm{T}}\left[\underline{g}(\underline{x}) - \underline{c}\right] = \underline{0} \quad \Rightarrow \quad \underline{\mu_1}\left[-P_1 + 200\right] = 0, \ \overline{\mu_1}\left[P_1 - 380\right] = 0$$

$$\underline{\mu_2}\left[-P_2 + 100\right] = 0, \ \overline{\mu_2}\left[P_2 - 200\right] = 0$$

Assume all inequality constraints are non-binding.
This means that

$$\mu_i = 0 \qquad and \qquad \overline{\overline{\mu}}_i = 0 \qquad \forall\, i = 1, n$$

And KKT conditions become

$$0.032(P_1) + 2.187 - \lambda = 0$$

$$0.038(P_2) + 2.407 - \lambda = 0$$

$$P_1 + P_2 - 400 = 0$$

Rewrite them as:

$$0.032(P_1) + 0(P_2) - \lambda = -2.187$$

$$(P_1) + 0.038(P_2) - \lambda = -2.407$$

$$P_1 + P_2 = 400$$

And it is easy to see how to put them into matrix form for solution in matlab.

$$\begin{bmatrix} 0.032 & 0 & -1 \\ 0 & 0.038 & -1 \\ 1 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} P_1 \\ P_2 \\ \lambda \end{bmatrix} = \begin{bmatrix} -2.187 \\ -2.407 \\ 400 \end{bmatrix}$$

Solution yields: $\begin{bmatrix} P_1 \\ P_2 \\ \lambda \end{bmatrix} = \begin{bmatrix} 220.29 \\ 179.71 \\ 9.24 \end{bmatrix}$

What is λ ?

= $9.24/MW-hr from previous problem

It is the system "incremental cost."

It is the cost if the system provides an additional MW over the next hour.

It is the cost of "increasing" the RHS of the equality constraint by 1 MW for an hour.

We can verify this.

Verification for meaning of lambda.

- Compute total costs/hr for Pd=400 MW
- Compute total costs/hr for Pd=401 MW
- Find the difference in total costs/hr for the two demands.

If our interpretation of lambda is correct, this difference should be $9.24.

Get cost/hr for each unit.

$$C_1(P_1) = 0.016(220.29)^2 + 2.187(220.29) + 120.312$$

$$C_1(P_1) = 1378.53 \ \$/hr$$

$$C_2(P_2) = 0.019(179.71)^2 + 2.407(179.71) + 74.074$$

$$C_2(P_2) = 1120.25 \ \$/hr$$

Total cost/hr are C1+C2

$$C_T = C_1(P_1) + C_2(P_2) = 1378.53 + 1120.25 = 2498.78$$

Now solve EDC for Pd=401 MW to get P1,P2

$$
\begin{bmatrix} 0.032 & 0 & -1 \\ 0 & 0.038 & -1 \\ 1 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} P_1 \\ P_2 \\ \lambda \end{bmatrix} = \begin{bmatrix} -2.187 \\ -2.407 \\ 401 \end{bmatrix}
$$

$$
\begin{bmatrix} P_1 \\ P_2 \\ \lambda \end{bmatrix} = \begin{bmatrix} 220.83 \\ 180.17 \\ 9.25 \end{bmatrix}
$$

Get cost/hr for each unit.

$$C_1(P_1) = 0.016(220.83)^2 + 2.187(220.83) + 120.312$$
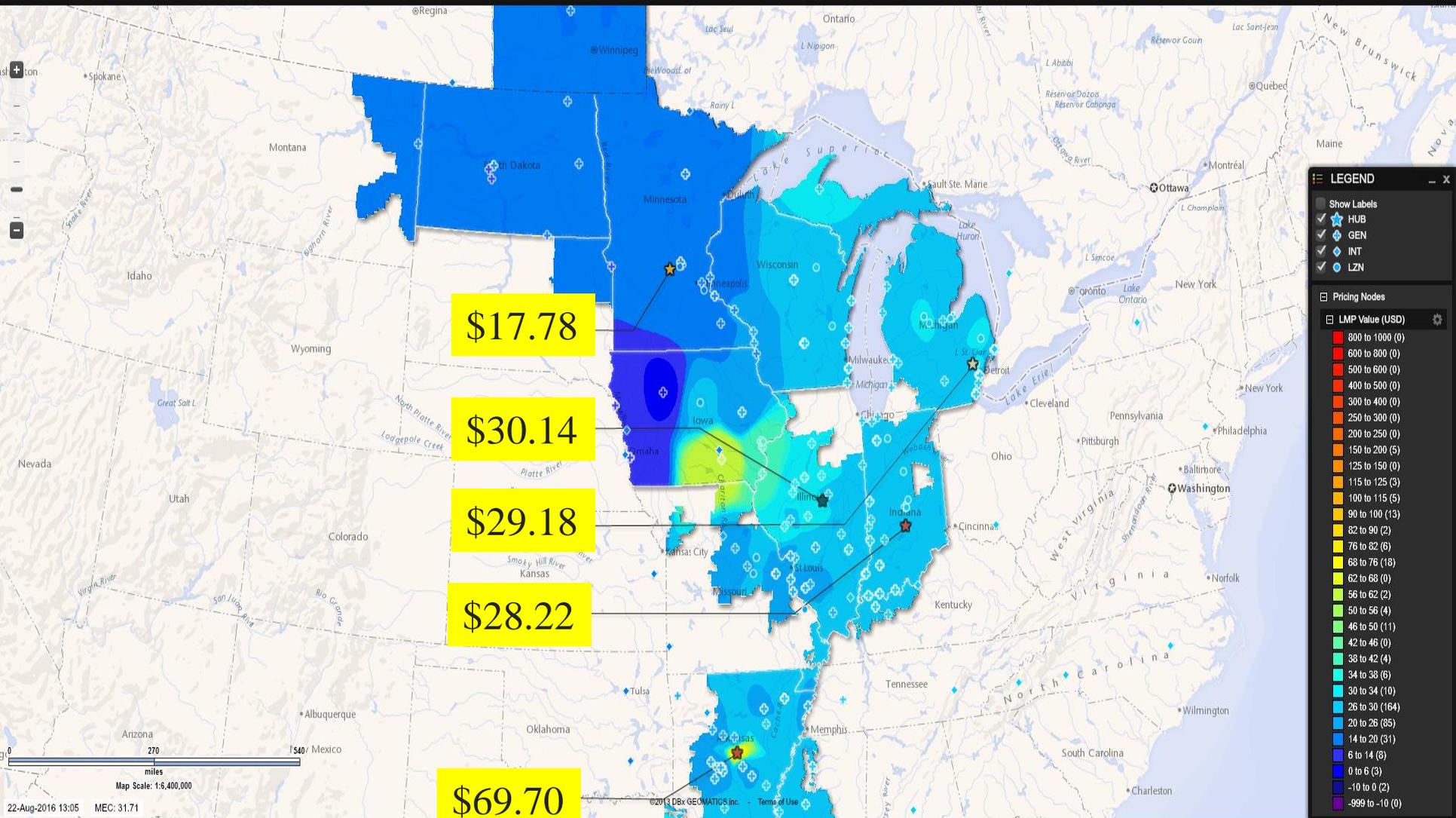
$$C_1(P_1) = 1383.52 \; \$ / hr$$

$$C_2(P_2) = 0.019(180.17)^2 + 2.407(180.17) + 74.074$$

$$C_2(P_2) = 1124.51 \; \$ / hr$$

Total cost/hr are C1+C2

$$C_T = C_1(P_1) + C_2(P_2) = 1383.52 + 1124.51 = 2508.03$$

Total cost/hr changed by 2508.03-2498.78 = 9.25 $/hr, which is in agreement with our interpretation of lambda.

https://www.misoenergy.org/markets-and-operations/real-time-
-market-data/real-time-displays/