Synchronous Machine Modeling – Notes2
1.0 Inductances

Recall the relation between flux linkages and currents developed in the previous notes.

$$\begin{bmatrix} \lambda_a \\ \lambda_b \\ \lambda_c \\ \hline \lambda_F \\ \lambda_D \\ \lambda_Q \end{bmatrix} = \begin{bmatrix} L_a & L_{ab} & L_{ac} & L_{aF} & L_{aD} & L_{aQ} \\ L_{ab} & L_b & L_{bc} & L_{bF} & L_{bD} & L_{bQ} \\ L_{ac} & L_{bc} & L_c & L_{cF} & L_{cD} & L_{cQ} \\ \hline L_{aF} & L_{bF} & L_{cF} & L_F & L_{FD} & L_{FQ} \\ L_{aD} & L_{bD} & L_{cD} & L_{FD} & L_D & L_{DQ} \\ L_{aQ} & L_{bQ} & L_{cQ} & L_{FQ} & L_{DQ} & L_Q \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_c \\ \hline i_F \\ i_D \\ i_Q \end{bmatrix} \quad (1)$$

We saw there are four blocks:
- $\underline{L}_{11}$ (stator-stator inductances)
- $\underline{L}_{12}$ (stator-rotor inductances)
- $\underline{L}_{21}$ (rotor-stator inductances)
- $\underline{L}_{22}$ (rotor-rotor inductances)

We want to investigate these inductances.

Some important relations to remember are:

Self: $$L_{ii} = \frac{\lambda_i}{i_i} = \frac{n_i^2}{\mathcal{R}_i} \qquad (2)$$

Mutual: $$L_{ij} = \frac{\lambda_i}{i_j} = \frac{\lambda_j}{i_i} = \frac{n_i n_j}{\mathcal{R}_{ij}} \qquad (3)$$

where, for self inductance, $\mathcal{R}_i$ is the reluctance of the path seen by λ_i in linking with circuit i, and n_i is the number of turns of circuit i.

For mutual inductance, $\mathcal{R}_{ij}$ is the reluctance of the path seen by λ_i in linking with coil i or the path seen by λ_j in linking with coil j, and n_i is the number of turns of circuit i.

The above use i and j (numbers 1-6) to denote the coils, but we will actually use letters a, b, c, F, D, and Q.

The question we must ask is, for each inductance: "Is it constant, or does it depend on the angle of the rotor, θ?"

We answer this question by considering what happens to the reluctance of the path seen by the associated flux.

The reluctance of a path in a magnetic circuit is given by

$$\mathcal{R}_i = \frac{l}{\mu A} \qquad (4)$$

where l is the length of the path, μ is the permeability of the material, and A is the cross-sectional area.

If a path consists of different materials (e.g., air and iron), then the reluctance for the part of the path corresponding to each material must be computed, e.g.,

$$\mathcal{R}_i = \frac{l_1}{\mu_1 A_1} + \frac{l_2}{\mu_2 A_2} \qquad (5)$$

We are not going to derive expressions for reluctances. Rather, for each coil and coil pair in the synchronous machine, we are going to see if any of the terms (on which reluctance depends) change as the rotor turns.

We will do this by determining if the proportion of the path in air, relative to the proportion of the path in iron, changes as the rotor turns.

2.0 Rotor-rotor inductances

Here, we are considering the inductances in the lower right-hand sub-matrix $\underline{L}_{22}$, given by

$$\underline{L}_{22} = \begin{bmatrix} L_F & L_{FD} & L_{FQ} \\ L_{FD} & L_D & L_{DQ} \\ L_{FQ} & L_{DQ} & L_Q \end{bmatrix} \qquad (6)$$

Each of these terms is inversely proportional to the reluctance of the path seen by the associated flux linkage.

Let's use Fig. 1 to consider the reluctance seen by the flux from the main field winding F, linking with itself, to determine whether L_F depends on θ.
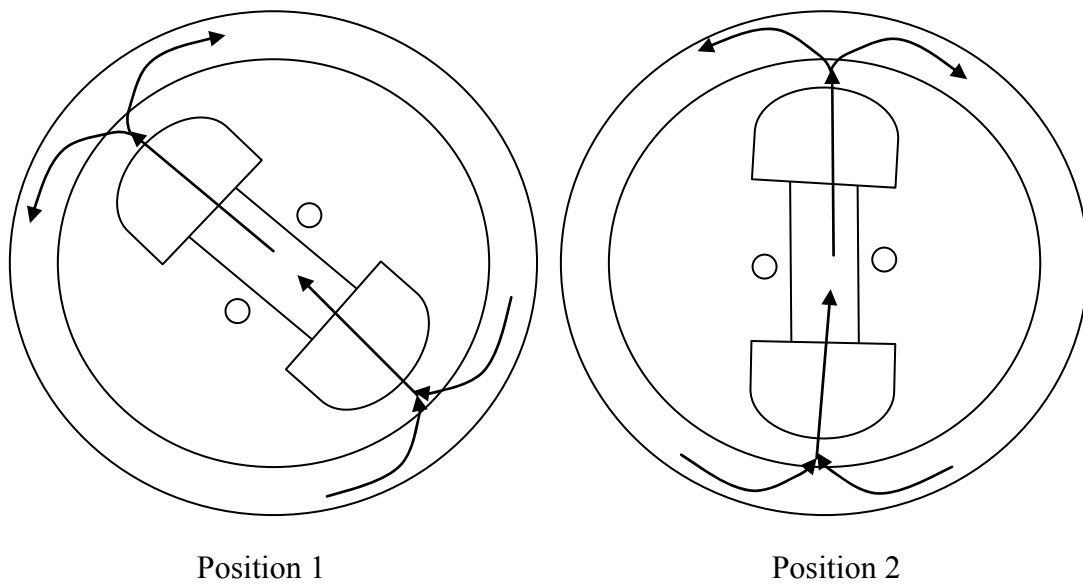

Position 1   Position 2

Fig. 1

Compare position 1 with position 2: does the percentage of flux path in air change relative to the percentage of flux path in iron?

Answer is NO. Therefore L_F is constant.

By the same reasoning, L_D is constant, since it has location identical to the main field winding F (see Fig. 2).
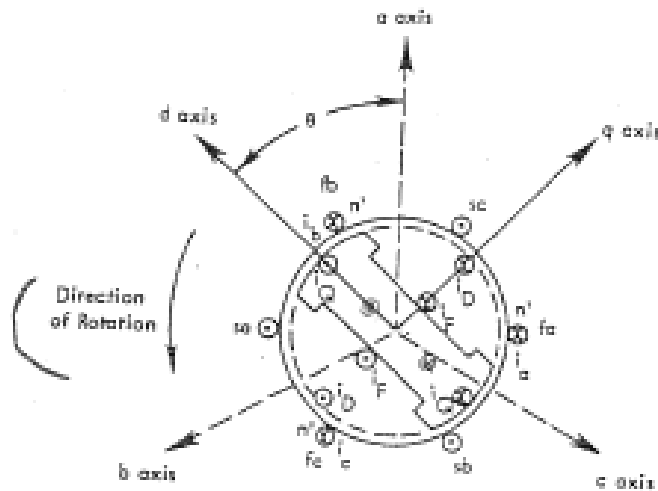

Fig. 2

What about L_Q? Let's use Fig. 3 to consider the reluctance seen by the flux from the damper winding Q, linking with itself, to determine whether L_Q depends on θ.

Compare position 1 with position 2: does the percentage of flux path in air change relative to the percentage of flux path in iron?
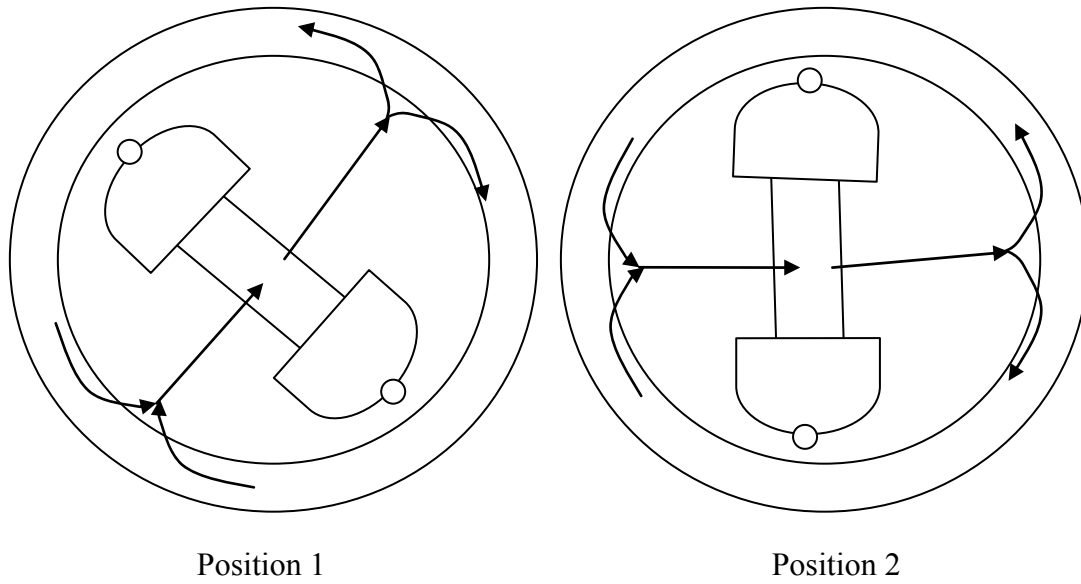
Position 1   Position 2

Fig. 3

Answer is once again NO. So L_Q is constant.

Now consider the mutuals. There are three of them to consider: L_{FD}, L_{FQ}, and L_{DQ}.

For L_{FD}, consider that the two windings are have collinear axes, i.e., they are aligned, and the flux of one completely links with the other, similar to the situation of winding two different coils around the same nail, as illustrated in Fig. 4.
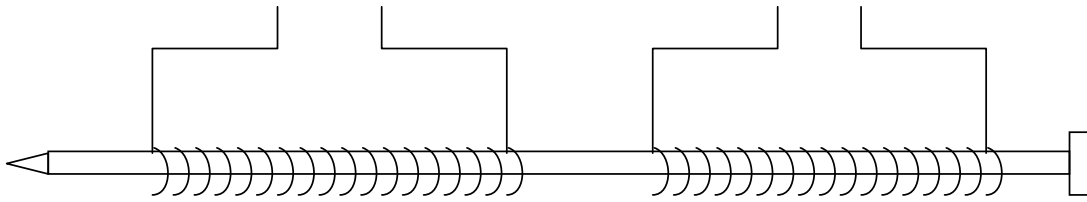
Fig. 4

And this situation will not change with θ. Therefore, L_{FD} is constant. It is conventional in the literature to denote this mutual inductance as M_R.

For L_{FQ} and L_{DQ}, consider that the two pairs of windings each have orthogonal (perpendicular) axes, i.e., they are wound so that the flux of one has zero linkage with the flux of the other, similar to the situation of winding two different coils around nails that are positioned perpendicular to each other, as illustrated in Fig. 5. Sometimes we say that orthogonal windings are *in quadrature*.
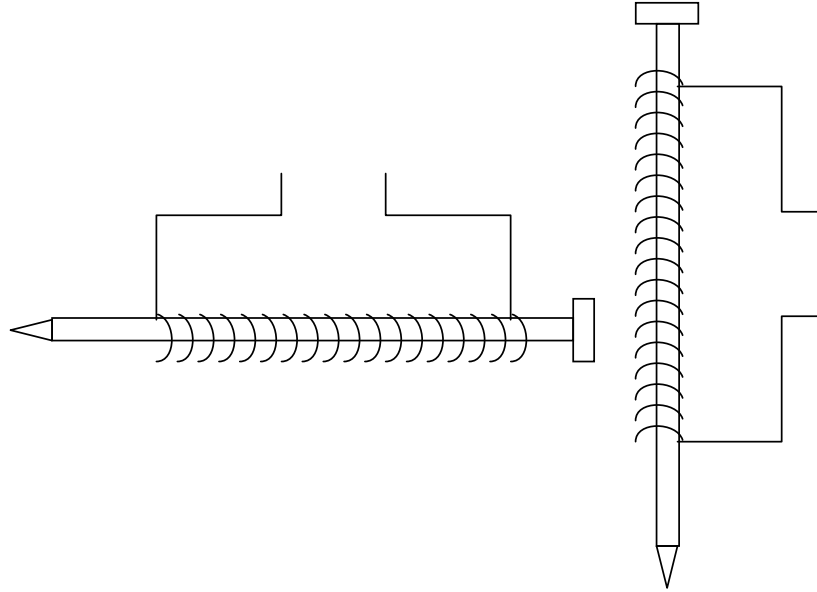
Fig. 5

When the flux from a current does not link with a circuit, there can be no inductance. And this situation does not change with θ. Therefore both L_{FQ} and L_{DQ} are zero.

From the above considerations, the $\underline{L}_{22}$ submatrix becomes:

$$\underline{L}_{22} = \begin{bmatrix} L_F & M_R & 0 \\ M_R & L_D & 0 \\ 0 & 0 & L_Q \end{bmatrix} \tag{7}$$

3.0 Stator-stator inductances

Here, we are considering the inductances in the upper left-hand sub-matrix $\underline{L}_{11}$, given by

$$\underline{L}_{11} = \begin{bmatrix} L_a & L_{ab} & L_{ac} \\ L_{ab} & L_b & L_{bc} \\ L_{ac} & L_{bc} & L_c \end{bmatrix}$$

3.1 Stator self-inductances
We consider L_a, L_b, and L_c. We focus on L_a first. There are 3 important concepts to grasp.

a. *Effect of rotor position*: Inspection of Fig. 6 shows the flux produced by the a-phase stator winding. Clearly, the reluctance of the path seen by this flux in position 1 is quite different than the reluctance of the path seen by this flux in position 2 (position 1 is a path mainly of iron, whereas position 2 is a path mainly of air).
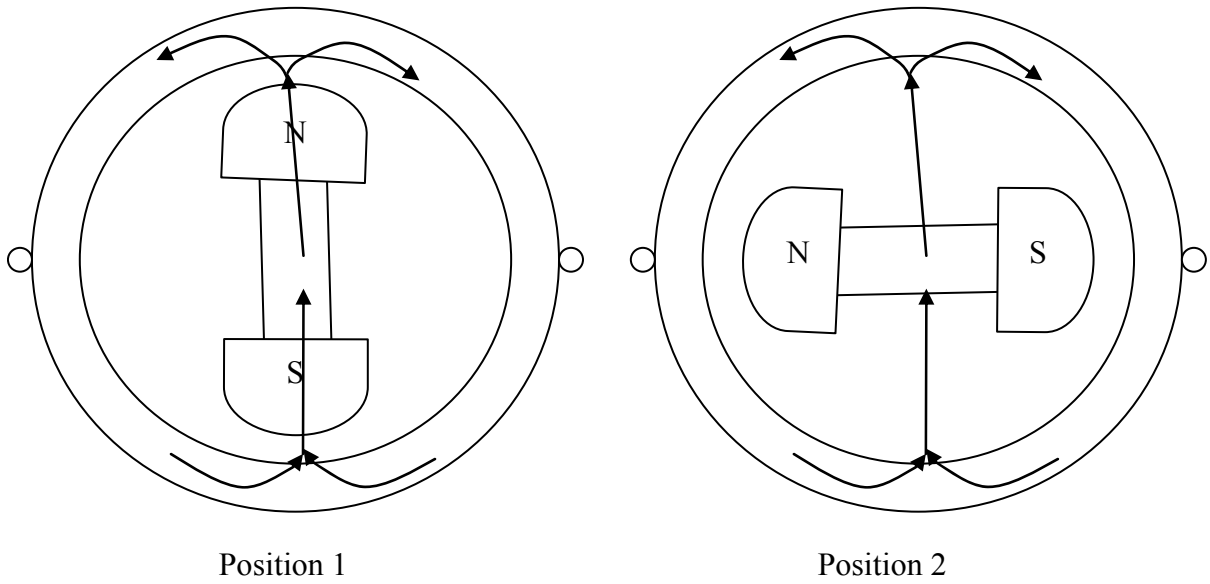
Position 1   Position 2

Fig. 6

So the reluctance varies with θ.

b. *Reluctance and polarity*: The self inductance of the a-phase winding depends on the effect of rotor position on reluctance; however, this effect is not influenced by polarity of the rotor. Therefore, the two positions of Fig. 7 create exactly the same path reluctance as does the two positions of Fig. 6 (note the polarity of the rotor is reversed in Fig. 7 relative to Fig. 6).
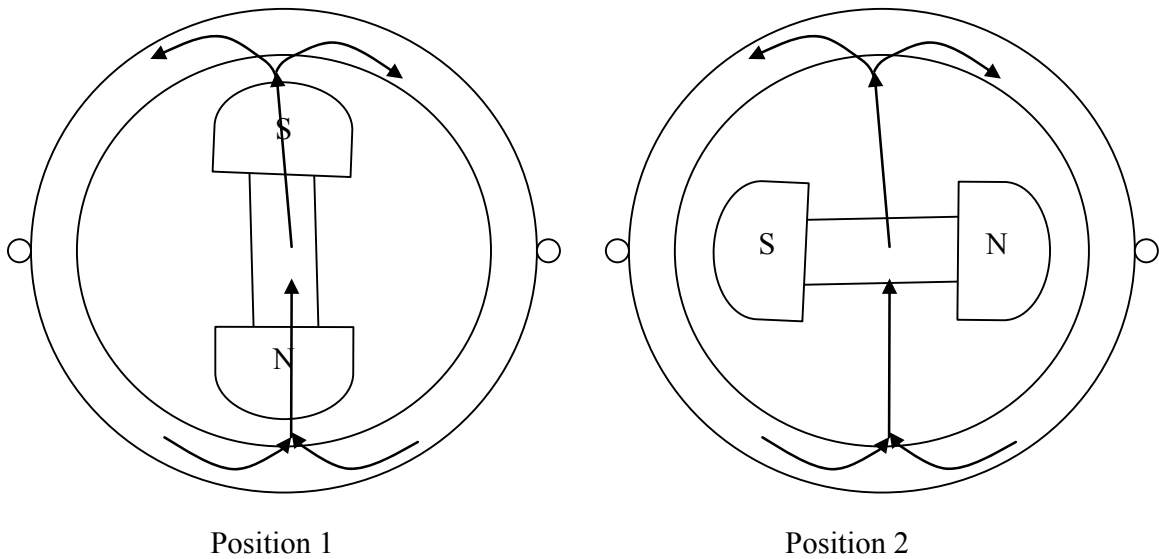
Position 1   Position 2

Fig. 7

As a result, we observe that the reluctance of the phase a flux path will vary with *2θ*.

c. *Angle-independent path*: There is a part of the flux path which has constant reluctance for all *θ*, as illustrated by the colored regions in Fig. 8. These regions correspond to the stator iron (yellow), a portion of the air-gap (green), and a portion of the rotor iron (red).
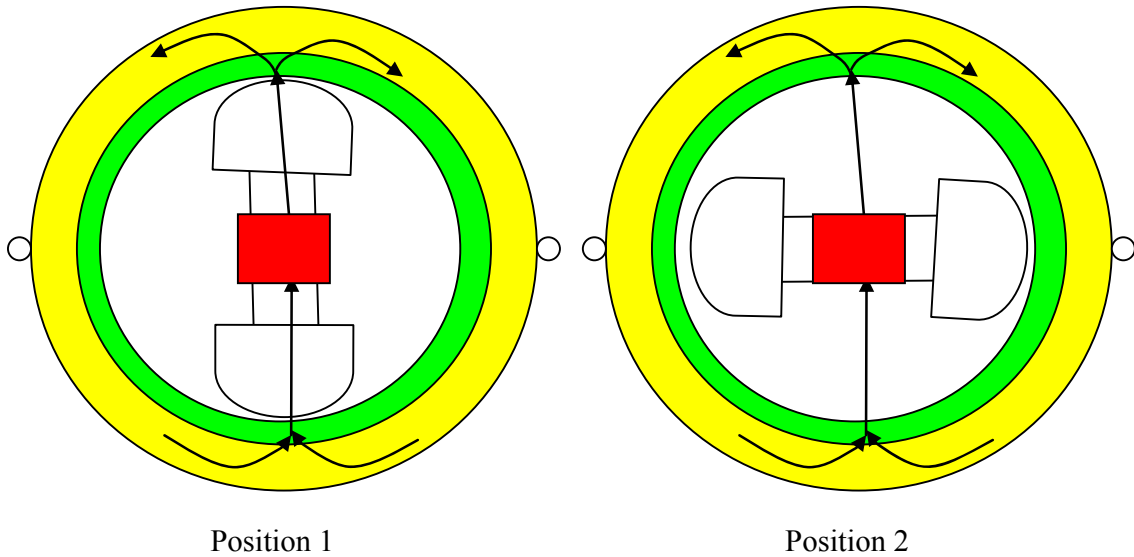
Position 1   Position 2

Fig. 8

The white region of the flux path in Fig. 8, however, has reluctance which varies with θ. The two positions illustrate the positions of minimum reluctance (position 1) and maximum reluctance (position 2).

From the above three concepts, we conclude that the reluctance of the path seen by the a-phase flux has a constant term and a term which varies with 2θ. We will assume the reluctance variation is approximately sinusoidal, so that a reasonable expression for the reluctance is:

$$\mathcal{R} = \mathcal{R}_0 + K \sin(2\theta - \pi/2) \qquad (8)$$

where $\mathcal{R}_0$ is the constant term and K is the amplitude of the reluctance sinusoidal variation.

From eq. (2), we recall that

$$L_{ii} = \frac{\lambda_i}{i_i} = \frac{n_i^2}{\mathcal{R}_i} \qquad (2)$$

Substitution of (8) into (2) yields:

$$L_{ii} = \frac{n_i^2}{\mathcal{R}_0 + K \sin(2\theta - \pi/2)}$$

$$(9)$$

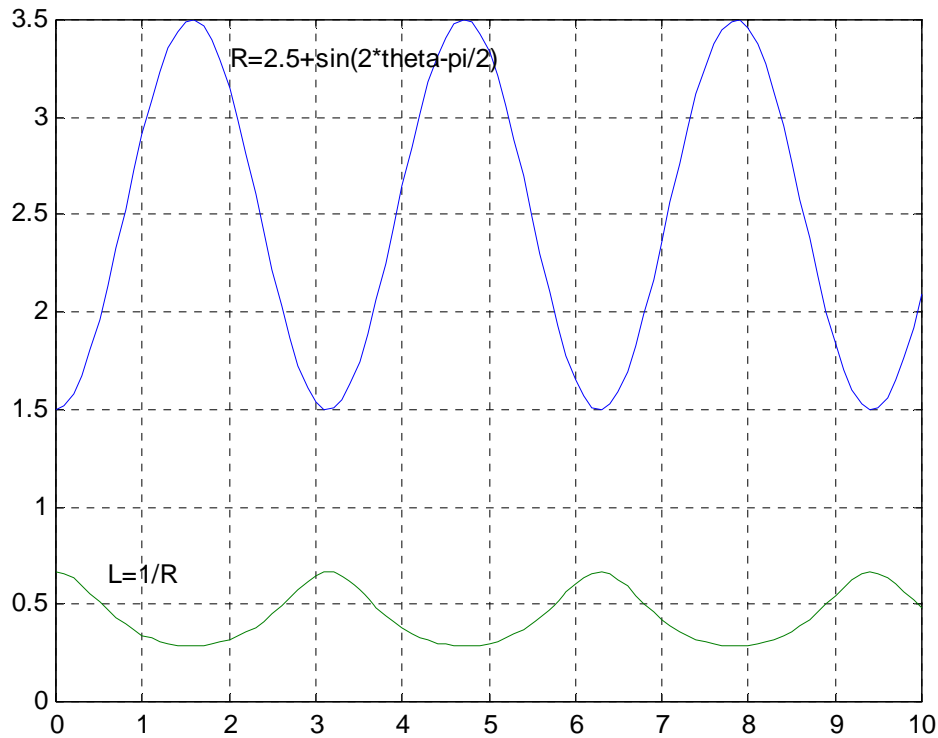Although (9) is not exactly sinusoidal, it is close, as illustrated in Fig. 9.

Fig. 9

We observe in Fig. 9 that both R and L vary with 2θ (two cycles occur between 0 and 2π=6.28). Where R is maximum, L is minimum, so they need to be in antiphase. This occurs for

$$L_a = L_S + L_m \cos 2\theta \qquad (10)$$

This is equivalent to taking only the DC and fundamental terms of (9).

Because the b and c phases are physically displaced from the a-phase by 120 and 240 degrees, respectively, we will also have:

$$L_b = L_S + L_m \cos(2\theta - \frac{2\pi}{3}) \qquad (11)$$

$$L_c = L_S + L_m \cos(2\theta + \frac{2\pi}{3}) \qquad (12)$$

3.2 Stator mutual inductances

The stator-stator mutuals also depend on flux paths that vary with θ. The development of these expressions is similar to that of the stator self inductances, although a bit more involved. The course EE 554 will discuss this in more depth.
Expressions for the mutuals are:

$$L_{ab} = -\left[M_s + L_m \cos 2\left(\theta + \frac{\pi}{6}\right) \right]$$

$$L_{bc} = -\left[M_s + L_m \cos 2\left(\theta + \frac{\pi}{2}\right) \right] \qquad (13)$$

$$L_{ca} = -\left[M_s + L_m \cos 2\left(\theta + \frac{5\pi}{6}\right) \right]$$

My only comment here is with respect to the negative signs. Given that M_s and L_m are positive, and that $M_s > L_m$, the negative signs indicate that the mutual inductances are actually negative numbers.

A mutual inductance between two circuits should be negative if positive currents in the two circuits produce fluxes in opposite directions. One can show that this is the case for the fluxes produced by the stator windings.

The final $\underline{L}_{11}$ submatrix is therefore given by:

$$\underline{L}_{11} = \begin{bmatrix} L_a & L_{ab} & L_{ac} \\ L_{ab} & L_b & L_{bc} \\ L_{ac} & L_{bc} & L_c \end{bmatrix}$$

$$= \begin{bmatrix} L_S + L_m \cos 2\theta & -\left[M_s + L_m \cos 2\left(\theta + \dfrac{\pi}{6}\right) \right] & -\left[M_s + L_m \cos 2\left(\theta + \dfrac{5\pi}{6}\right) \right] \\ -\left[M_s + L_m \cos 2\left(\theta + \dfrac{\pi}{6}\right) \right] & L_S + L_m \cos(2\theta - \dfrac{2\pi}{3}) & -\left[M_s + L_m \cos 2\left(\theta + \dfrac{\pi}{2}\right) \right] \\ -\left[M_s + L_m \cos 2\left(\theta + \dfrac{5\pi}{6}\right) \right] & -\left[M_s + L_m \cos 2\left(\theta + \dfrac{\pi}{2}\right) \right] & L_S + L_m \cos(2\theta + \dfrac{2\pi}{3}) \end{bmatrix}$$

(14)

4.0 Stator-rotor inductances

Here we are interested in the off-diagonal submatrices $\underline{L}_{12}$ and $\underline{L}_{21}$, which are transposes of each other.

$$\underline{L}_{12} = \begin{bmatrix} L_{aF} & L_{aD} & L_{aQ} \\ L_{bF} & L_{bD} & L_{bQ} \\ L_{cF} & L_{cD} & L_{cQ} \end{bmatrix}$$

(15)

Note that ALL of these terms are mutuals (there are no self inductances).

In understanding these terms, it is helpful to first recall why we found stator-stator terms (self and mutuals) dependent on θ.

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With stator-stator self and mutuals, although all windings were locationally fixed (on the stator), and thus the flux path was fixed, the reluctance of that flux path varied due to the rotor motion.

Now, however, although the stator windings are locationally fixed, the field windings are not – they rotate with the rotor. This will give us an inductance that varies with θ, but not due to variation in path reluctance but rather due to the extent to which the fluxes are linked.

Consider L_{aF}, the mutual inductance between the a-phase and the main field winding. Figure 10 illustrates two separate positions for the rotor. The a-phase and the main field winding are shown. Position 1 has the axes of the two windings aligned, which is the position of maximum flux linkage, and maximum inductance.
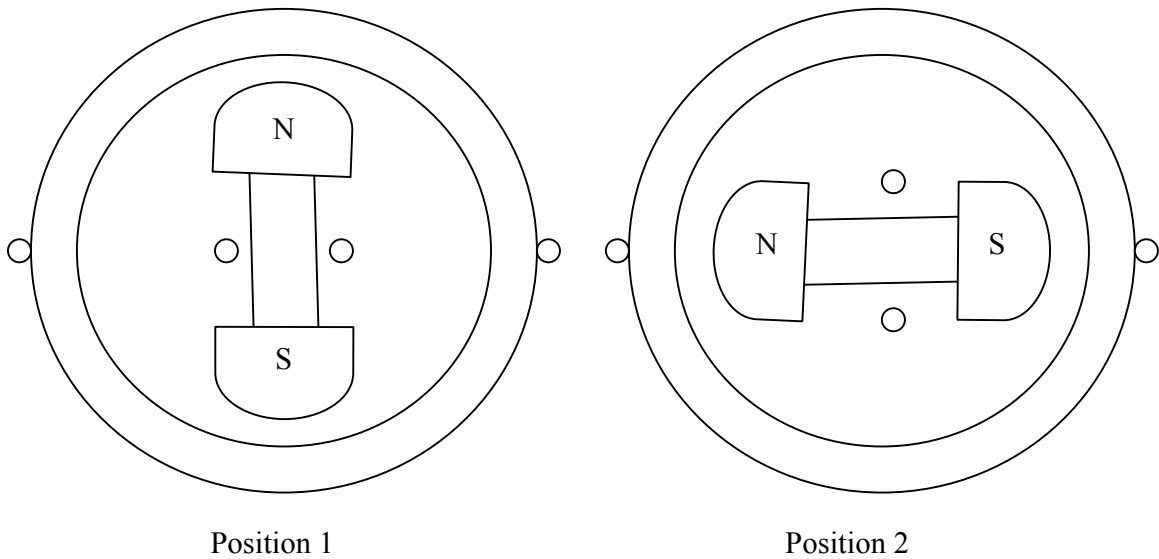
Position 1　　　　　　　　　　　　Position 2

Fig. 10

Position 2, however, has the two windings in quadrature, meaning that no flux from either winding links the other. So this is a position of zero inductance.

We could consider a position 3 as well, which would be the same as position 1, except that the polarity of the rotor would be reversed. Now the two fluxes from the windings are in opposite directions (for positive currents), but the axes are aligned, therefore we get a negative inductance with magnitude equal to that of position 1.

This analysis leads us to the following expression for the mutual inductance between the a-phase and the main field winding:

$$L_{aF} = M_F \cos\theta \qquad (16)$$

The mutual between the a-phase and the D-winding will have exactly the same form, with the only difference being the amplitude:

$$L_{aD} = M_D \cos\theta \qquad (17)$$

The Q-axis damping winding will also have the same form, except it will have different amplitude, and it will be 90° behind:

$$L_{aQ} = M_Q \cos\theta \qquad (18)$$

Expressions for the b and c phases will be exactly like (16), (17), and (18), except that they will be 120° and 240° degrees behind, respectively. Summarizing, we have:

Stator-main field mutuals:

$$L_{aF} = M_F \cos\theta$$

$$L_{bF} = M_F \cos(\theta - \frac{2\pi}{3})$$

$$L_{cF} = M_F \cos(\theta + \frac{2\pi}{3})$$

(19)

Stator-D winding mutuals:

$$L_{aD} = M_D \cos\theta$$

$$L_{bD} = M_D \cos(\theta - \frac{2\pi}{3})$$

$$L_{cD} = M_D \cos(\theta + \frac{2\pi}{3})$$

(20)

Stator-Q winding mutuals:

$$L_{aQ} = M_Q \sin\theta$$

$$L_{bQ} = M_Q \sin(\theta - \frac{2\pi}{3})$$

$$L_{cQ} = M_Q \sin(\theta + \frac{2\pi}{3})$$

(21)

Therefore we can express $\underline{L}_{12}$ as

$$\underline{L}_{12} = \begin{bmatrix} L_{aF} & L_{aD} & L_{aQ} \\ L_{bF} & L_{bD} & L_{bQ} \\ L_{cF} & L_{cD} & L_{cQ} \end{bmatrix}$$

$$= \begin{bmatrix} M_F \cos\theta & M_D \cos\theta & M_Q \sin\theta \\ M_F \cos(\theta - \dfrac{2\pi}{3}) & M_D \cos(\theta - \dfrac{2\pi}{3}) & M_Q \sin(\theta - \dfrac{2\pi}{3}) \\ M_F \cos(\theta + \dfrac{2\pi}{3}) & M_D \cos(\theta + \dfrac{2\pi}{3}) & M_Q \sin(\theta + \dfrac{2\pi}{3}) \end{bmatrix} (22)$$

5.0 Final inductance matrix

We are now in a position to write down the final inductance matrix as follows:

$$\underline{L} = \begin{bmatrix} L_a & L_{ab} & L_{ac} & L_{aF} & L_{aD} & L_{aQ} \\ L_{ab} & L_b & L_{bc} & L_{bF} & L_{bD} & L_{bQ} \\ L_{ac} & L_{bc} & L_c & L_{cF} & L_{cD} & L_{cQ} \\ L_{aF} & L_{bF} & L_{cF} & L_F & L_{FD} & L_{FQ} \\ L_{aD} & L_{bD} & L_{cD} & L_{FD} & L_D & L_{DQ} \\ L_{aQ} & L_{bQ} & L_{cQ} & L_{FQ} & L_{DQ} & L_Q \end{bmatrix}$$

$$= \begin{bmatrix}
L_S + L_m \cos 2\theta & -\left[M_s + L_m \cos 2\left(\theta + \frac{\pi}{6}\right)\right] & -\left[M_s + L_m \cos 2\left(\theta + \frac{5\pi}{6}\right)\right] & M_F \cos\theta & M_D \cos\theta & M_Q \sin\theta \\
-\left[M_s + L_m \cos 2\left(\theta + \frac{\pi}{6}\right)\right] & L_S + L_m \cos(2\theta - \frac{2\pi}{3}) & -\left[M_s + L_m \cos 2\left(\theta + \frac{\pi}{2}\right)\right] & M_F \cos(\theta - \frac{2\pi}{3}) & M_D \cos(\theta - \frac{2\pi}{3}) & M_Q \sin(\theta - \frac{2\pi}{3}) \\
-\left[M_s + L_m \cos 2\left(\theta + \frac{5\pi}{6}\right)\right] & -\left[M_s + L_m \cos 2\left(\theta + \frac{\pi}{2}\right)\right] & L_S + L_m \cos(2\theta + \frac{2\pi}{3}) & M_F \cos(\theta + \frac{2\pi}{3}) & M_D \cos(\theta + \frac{2\pi}{3}) & M_Q \sin(\theta + \frac{2\pi}{3}) \\
M_F \cos\theta & M_F \cos(\theta - \frac{2\pi}{3}) & M_F \cos(\theta + \frac{2\pi}{3}) & L_F & M_R & 0 \\
M_D \cos\theta & M_D \cos(\theta - \frac{2\pi}{3}) & M_Q \sin(\theta - \frac{2\pi}{3}) & M_R & L_D & 0 \\
M_Q \sin\theta & M_Q \sin(\theta - \frac{2\pi}{3}) & M_Q \sin(\theta + \frac{2\pi}{3}) & 0 & 0 & L_Q
\end{bmatrix}$$

We recall, of course, that this matrix is important because of:

$$\underline{\lambda} = \begin{bmatrix} \underline{\lambda}_{abc} \\ \underline{\lambda}_{FDQ} \end{bmatrix} = \begin{bmatrix} \underline{L}_{11} & \underline{L}_{12} \\ \underline{L}_{21} & \underline{L}_{22} \end{bmatrix} \begin{bmatrix} \underline{i}_{abc} \\ \underline{i}_{FDQ} \end{bmatrix}$$ which will be used in $\underline{v} = -\underline{R}\underline{i} - \underline{\dot{\lambda}}$.