EE523: Random Processes for Communication and Signal Processing

Homework #6

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1. A particle performs a random walk on the vertex set of a connected graph G, that has no loops or multiple edges. At each stage it moves to a neighbor with equal probability. If G has $\eta < \infty$ edges, show that the stationary distribution is given by $\pi_v = d_v/(2\eta)$, where d_v is the degree of vertex v.

2. A chain is called reversible if there exists a distribution π such that $\pi_i p_{ij} = \pi_j p_{ji}$ for all i, j. Show that if a chain is reversible then the stationary distribution of the chain is π. This technique is widely used to generate sample from distributions that we do not know how to sample from in general.

3. Show that a random walk on an infinite binary tree is transient.
Hint: Identify a discrete-time birth-death Markov chain associated with the random walk from which the conclusion is obvious.

4. Let X be a DTMC with a finite number of states and transition matrix $P = (p_{ij})$ where $p_{ij} > 0$ for all i, j. Show that there exists a $\lambda \in (0, 1)$ such that $|p_{ij} - \pi_j| < \lambda^n$, where π is the stationary distribution. Thus for a finite chain the convergence to the stationary distribution is exponentially fast.
Hint: One way of doing this would be to start with the coupling argument used in the proof for showing that for an irreducible, aperiodic chain $p_{ij}(n) \to \pi_j$ as $n \to \infty$. Then identify an appropriate λ.

5. Consider a chain X that models a gambler's wealth. The state j reflects the current wealth of the gambler. The gambler wins with probability p and goes to state $j+1$ and loses with probability $q = 1-p$ and goes to state $j-1$. The gambler is said to be ruined if he hits state 0 and said to satisfied if he hits state N. Both 0 and N are absorbing states i.e. $p_{00} = p_{NN} = 1$. Define $u_j = P(T_0 < \infty | X_0 = j)$.

   a) Find an expression for u_j for a given N.
   b) Find an expression for u_j if the gambler is never satisfied i.e. there is no N such that the gambler stops gambling except 0 when he is forced to.

6. This problem is somewhat open-ended. You may use MATLAB to solve it. The answers could be a mix of analysis and simulation, although insightful analysis perhaps guided by simulation is clearly preferred.

   Consider a simple wireless channel model. The channel is modeled as a two-state Markov chain. It transitions from the *good* state to the *bad state* with probability 0.01 and from the *bad* state to the *good* state with probability 0.1.

   A transmitter uses the following protocol to transmit over this channel. At each slot he transmits a packet. If at that time the channel is in the good state, his transmission goes through, otherwise it is lost. The transmitter receives perfect feedback about the fate of his transmission. If the transmitter realizes that α consecutive transmissions are lost then he decides to wait randomly for some time before transmitting again, because he wants the channel to come back to the good state. He waits randomly with probability $1 - q$ (where q is typically small) and starts transmitting again afresh. Starting afresh means that he forgets everything that happened in the past. Note that since q is small, the transmitter does spend a relatively long time waiting.

Choose some reasonable value of α and q (this is up to you).

- Model the operation of the transmitter as a Markov chain.
- In the steady state, what is the probability that the transmitter is waiting ?
- Over N time slots if K packet transmissions are successful then we call the throughput K/N. What happens to the system throughput as N is large or equivalently the system is in steady state.
- Can you say anything about what an optimal (α, q) pair would be ?

Make any simplifying assumptions that you may want to make as long as they are reasonable. Treat this problem as a case study in Markov chain modeling.