EE523: Random Processes for Communication and Signal Processing

Homework #5

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1. A die is rolled repeatedly. Which of the following are Markov Chains. Find their transition matrix.

   a) The largest number X_n shown up to the n^{th} roll.

   b) The number N_n of sixes in n rolls.

   c) At time r, the time C_r since the most recent six.

   d) At time r, the time B_r until the next six.

2. Let $\{X_n, n \geq 1\}$ be i.i.d. integer-valued random variables. Let $S_n = \sum_{r=1}^{n} X_r$, with $S_0 = 0, Y_n = X_n + X_{n-1}$ with $X_0 = 0$ and $Z_n = \sum_{r=0}^{n} S_r$. Which of the following are Markov chains (a) S_n, (b) Y_n, (c) Z_n, and (d) the sequence of pairs (S_n, Z_n).

3. Let X be a Markov chain with a state s that is absorbing, i.e. $p_{ss}(1) = 1$. All other states communicate with s i.e. $i \to s$ for all states $i \in S$. Show that all states in S except s are transient.

4. Classify the states of the following Markov chains with $S = \{1, 2, 3, 4\}$ and transition matrices

   a) $\begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}$

   b) $\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

   In case (a) calculate $f_{34}(n)$ and deduce that the probability of ultimate absorption in state 4, starting from 3 equals $\frac{2}{3}$.

5. Let $\{X_n, n \geq 0\}$ be a Markov chain with $X_0 = i$. Let N be the total number of visits made by the chain to j. Show that

   $$P(N = n) = \begin{cases} 1 - f_{ij} & \text{if } n = 0 \\ f_{ij}(f_{jj})^{n-1}(1 - f_{jj}) & \text{if } n \geq 1. \end{cases}$$

   and deduce that $P(N = \infty) = 1$ if and only if $f_{ij} = f_{jj} = 1$.

6. A particle performs a random walk on the vertices of a cube. At each step it remains where it is with probability 1/4, or moves to one of the neighboring vertices with probability 1/4. Let v and w be two diametrically opposite vertices. If the walk starts at v, find

   a) the mean number of steps until its first return to v.

   b) the mean number of steps until its first return to w.