EE523: Random Processes for Communication and Signal Processing

Homework #3

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1. Let $X_1, X_2, \ldots X_n$ be i.i.d. random variables. Show that if $m \leq n$ then $E(S_m/S_n) = m/n$ where $S_n = \sum_{i=1}^{n} X_i$. You can assume that $E(X_i)$ and $E(1/X_i)$ exist.

2. Let X have the uniform distribution on $[0,1]$. Find a suitable function g such that $Y = g(X)$ is exponentially distributed with parameter λ.

3. Consider the multivariate Gaussian distribution
$$f_{\vec{X}}(x_1, \ldots x_n) = \frac{1}{\sqrt{(2\pi)^n |\det(C_x)|}} \exp\{-\frac{1}{2}(\mathbf{x}-\mu)^T C_x^{-1}(\mathbf{x}-\mu)\}$$
Show that the covariance between X_i and X_j given by $E(X_i - \mu_i)(X_j - \mu_j) = C_x(i,j)$. Conclude that a multivariate Gaussian distribution is completely specified by its mean and covariance.

4. Let $\mathbf{C}$ be a positive-definite symmetric matrix $n \times n$ matrix and let $\mathbf{L}$ be such that $\mathbf{C} = \mathbf{L}\mathbf{L}^T$. This is called the Cholesky distribution of C. Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a vector of independent random variables distributed $N(0,1)$. Show that the vector $\mathbf{Z} = \mu + \mathbf{X}\mathbf{L}$ has the multivariate Gaussian distribution with mean μ and covariance C. Do you see a method for generating multivariate Gaussian distributions with a specified mean and covariance ?

5. Let $\{X_r : r \geq 1\}$ be independent and uniformly distributed on $[0,1]$. Let $0 < x < 1$ and define
$$N = \min\{n \geq 1 : \sum_{i=1}^{n} X_i > x\}.$$
i.e. you keep generating uniform random variables until their sum first exceeds x and record the total number of such uniform random variables generated. Show that $P(N > k) = x^k/k!$. Find the mean and variance of N.
Hint: Think about setting up a recursive equation.

6. *Chernoff Bound*: Let X be a continuous random variable and let a be some constant. Show that
$$P(X \geq a) \leq e^{-at} E(e^{tX}) \quad \text{for } t > 0$$
Now suppose X is distributed $N(\mu, \sigma^2)$. For a given $a \geq \mu$ find the tightest bound on $P(X \geq a)$.

7. *Monte Carlo Integration*: It is required to estimate $J = \int_0^1 g(x)dx$ where $0 \leq (x) \leq 1$ for all x. Let X and Y be independent random variables with common density function $f(x) = 1$ if $0 < x < 1$, $f(x) = 0$ otherwise. Let $U = I_{\{Y \leq g(X)\}}$ (where I denotes the indicator function), $V = g(X)$, $W = \frac{1}{2}(g(X) + g(1-X))$. Show that
   (a) $E(U) = E(V) = E(W) = J$.
   (b) $var(W) \leq var(V) \leq var(U)$.

   i.e. all three of the estimators have an expected value J but W has the lowest variance.