EE523: Random Processes for Communication and Signal Processing

Homework #6

---

1. Let X be a Markov chain with a state s that is absorbing, i.e. $p_{ss}(1) = 1$. All other states communicate with s i.e. $i \to s$ for all states $i \in S$. Show that all states in S except s are transient.

2. Classify the states of the following Markov chains with $S = \{1, 2, 3, 4\}$ and transition matrices

   a) $\begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}$

   b) $\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

   In case (a) calculate $f_{34}(n)$ and deduce that the probability of ultimate absorption in state 4, starting from 3 equals $\frac{2}{3}$.

3. Let $\{X_n, n \geq 0\}$ be a Markov chain with $X_0 = i$. Let N be the total number of visits made by the chain to j. Show that

   $$P(N = n) = \begin{cases} 1 - f_{ij} & \text{if } n = 0 \\ f_{ij}(f_{jj})^{n-1}(1 - f_{jj}) & \text{if } n \geq 1. \end{cases}$$

   and deduce that $P(N = \infty) = 1$ if and only if $f_{ij} = f_{jj} = 1$.

4. A particle performs a random walk on the vertices of a cube. At each step it remains where it is with probability 1/4, or moves to one of the neighboring vertices with probability 1/4. Let v and w be two diametrically opposite vertices. If the walk starts at v, find

   a) the mean number of steps until its first return to v.

   b) the mean number of steps until its first return to w.

5. A particle performs a random walk on the vertex set of a connected graph G, that has no loops or multiple edges. At each stage it moves to a neighbor with equal probability. If G has $\eta < \infty$ edges, show that the stationary distribution is given by $\pi_v = d_v/(2\eta)$, where d_v is the degree of vertex v.

6. A chain is called reversible if there exists a distribution π such that $\pi_i p_{ij} = \pi_j p_{ji}$ for all i, j. Show that if a chain is reversible then the stationary distribution of the chain is π. This technique is widely used to generate sample from distributions that we do not know how to sample from in general.

7. Let X be a DTMC with a finite number of states and transition matrix $P = (p_{ij})$ where $p_{ij} > 0$ for all i, j. Show that there exists a $\lambda \in (0,1)$ such that $|p_{ij}(n) - \pi_j| < \lambda^n$, where π is the stationary distribution. Thus for a finite chain the convergence to the stationary distribution is exponentially fast.
Hint: One way of doing this would be to start with the coupling argument used in the proof for showing that for an irreducible, aperiodic chain $p_{ij}(n) \to \pi_j$ as $n \to \infty$. Then identify an appropriate λ.

8. An irreducible Markov chain X with state space $S = \{0, 1, 2, \dots\}$ has an absorbing state 0, i.e. $p_{00} = 1$ and all other states $1, 2, \dots$ are transient. Let T_0 be the time the chain takes to reach state 0. Let $\mu_{i0} = E(T_0 | X_0 = i)$, i.e. the expected time to hit state 0 given that the chain started in state i.

   a) Find an expression for μ_{i0} in terms of the transition probabilities $\{p_{ij}\}$ for all $i, j \in S$ and $\mu_{j0}, j \in S$.
   b) Let $f_i(n) = P(T_0 > n | X_0 = i)$. Derive a formula for $f_i(n+1)$ in terms of $f_j(n)$ for all $j \in S$ and the transition probabilities $\{p_{ij}\}$.

9. Let X be an irreducible non-null persistent aperiodic Markov chain. Show that X is reversible in equilibrium if and only if

$$p_{j_1 j_2} p_{j_2 j_3} \cdots p_{j_{n-1} j_n} p_{j_n j_1} = p_{j_1 j_n} p_{j_n j_{n-1}} \cdots p_{j_2 j_1}$$

for all n and all finite sequences $j_1, j_2, \dots, j_n$ of states.