EE523: Random Processes for Communication and Signal Processing

Homework #5

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1. Let $X_1, X_2 \ldots$ be i.i.d. random variables with common mean μ and finite variance. Show that
$$\binom{n}{2}^{-1} \sum_{1 \le i < j \le n} X_i X_j \xrightarrow{p.} \mu^2 \quad \text{as } n \to \infty.$$
Note that the usual weak law of large numbers approach does not work since the terms of the individual sum are not independent. Moreover, the covariances between the terms is not zero.

2. Let $\{X_n\}$ be a sequence of random variables and let $\{c_n\}$ be a sequence of reals such that $c_n \to c$.
   i) Show that if $X_n \xrightarrow{m.s.} X$ as $n \to \infty$, then $c_n X_n \xrightarrow{m.s.} cX$ as $n \to \infty$.
   ii) Show that if $X_n \xrightarrow{p.} X$ as $n \to \infty$, then $c_n X_n \xrightarrow{p.} cX$ as $n \to \infty$.

3. A die is rolled repeatedly. Which of the following are Markov Chains. Find their transition matrix.
   a) The largest number X_n shown up to the n^{th} roll.
   b) The number N_n of sixes in n rolls.
   c) At time r, the time C_r since the most recent six.

4. Let $\{X_n, n \ge 1\}$ be i.i.d. integer-valued random variables. Let $S_n = \sum_{r=1}^{n} X_r$, with $S_0 = 0, Y_n = X_n + X_{n-1}$ with $X_0 = 0$ and $Z_n = \sum_{r=0}^{n} S_r$. Which of the following are Markov chains (a) S_n, (b) Y_n, (c) Z_n, and (d) the sequence of pairs (S_n, Z_n).

5. Let X be a Markov chain with a state s that is absorbing, i.e. $p_{ss}(1) = 1$. All other states communicate with s i.e. $i \to s$ for all states $i \in S$. Show that all states in S except s are transient.