EE523: Random Processes for Communication and Signal Processing

Homework #2

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1. Show that if $var(X) = 0$ then X is almost surely constant. i.e. there exists a constant a, such that $P(X = a) = 1$.

2. If X takes non-negative integer values, show that

$$E(X) = \sum_{n=0}^{\infty} P(X > n)$$

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3. There are certain cases when our intuition about expected value may not be accurate.

   (a) Give an example of a discrete random variable X whose expectation $EX = \sum_x xP(X = x)$ is infinite. Note that you cannot assign a finite probability to $X = \infty$ i.e. X is such that $P(X < \infty) = 1$.

   (b) Give an example of a discrete random variable X such that $x_{ML} = \arg\max_x P(X = x)$ is different from EX.

4. Let $\mathbf{X} = (X_1, X_2, ..., X_n)^T$ be a column vector of random variables. The covariance matrix of $\mathbf{X}$ is given by

$$cov(\mathbf{X}) = E\{(\mathbf{X} - E\mathbf{X})(\mathbf{X} - E\mathbf{X})^T\}$$

   i.e. it is a square matrix of size $n \times n$. Suppose that the determinant of $cov(\mathbf{X})$ equals 0. Show that there exists $a_i, i = 1, \ldots n$ and b such that

$$P(\sum_{i=1}^{n} a_i X_i = b) = 1$$

   Hint: If $det(A) = 0$ then there exists a vector u such that $u^T Au = 0$.

5. Suppose you buy a package every day. Suppose that there are c different types of objects and each package contains one of those objects. A package is equally likely to contain any of the c objects. Find the expected number of days that elapse before you have a full set of objects.

6. Let X and Y be random variables with mean 0, variance 1 and $E(XY) = \rho$. Show that

$$E(\max(X^2, Y^2)) \leq 1 + \sqrt{1 - \rho^2}$$

7. Show the following

   (a) $E(aY + bZ|X) = aE(Y|X) + bE(Z|X)$

   (b) If X and Y are independent $E(Y|X) = E(Y)$

   (c) $E(E(Y|X, Z)|X) = E(Y|X) = E(E(Y|X)|X, Z)$

8. The conditional variance of Y given X, $var(Y|X)$ is defined by

$$var(Y|X = x) = E((Y - E(Y|X = x))^2 \mid X = x)$$

Note that the conditional variance is a random variable. Show that

$$var(Y) = E(var(Y|X)) + var(E(Y|X))$$