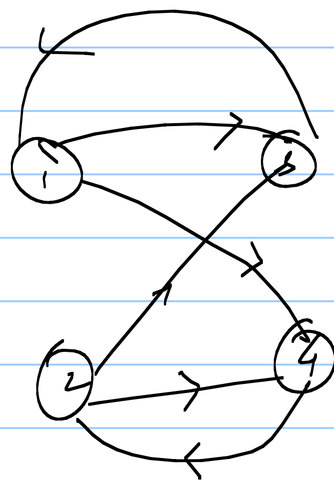
<u>Definition:</u> The period $d(i)$ of a state i is defined by

$d(i) = \gcd\{n : p_{ii}(n) > 0\}$ i.e. the greatest common divisor of all times when the return is possible.

If $d(i) > 1$, then the chain is called periodic
If $d(i) = 1$, " " " " " aperiodic.

E.g.



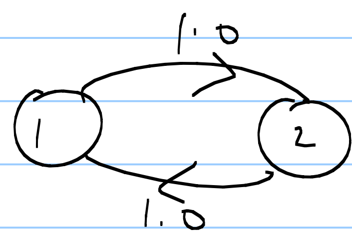
In this chain if $X_0 = 1$, then $X_n = 1$ only if n is even.

$\therefore d(1) = 2.$

<u>Definition:</u> If a state is non-null persistent & aperiodic then it is called Ergodic.

Periodic states present some problems when we want to look at the limiting behavior of Markov chains.

<u>Example</u>



For this chain, we have $p_{11}(n) = p_{22}(n) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$

i.e. $p_{11}(n)$ & $p_{22}(n)$ do not converge when $n \to \infty$.
The reason is periodicity of the states.

We have the following fundamental result.

<u>Theorem</u>: For an irreducible aperiodic chain, we have

$$p_{ij}(n) \longrightarrow \frac{1}{\mu_j} \quad \text{as } n \to \infty \text{ for all } i, j.$$

The main idea is that the chain forgets where it started, since irrespective of starting state i, $p_{ij}(n)$ converges to $\frac{1}{\mu_j}$ that only depends on j.

<u>Proof</u>:
<u>Case 1</u>: If the chain is transient, we have the result

$$p_{ij}(n) \longrightarrow 0 \quad \forall j \quad \text{(shown earlier)}.$$

& $\mu_j = \infty$.

<u>Case 2</u>: If the chain is non-null persistent.

We use the technique of coupling. We construct another chain Z as follows.

Let $(X_n)_{n \geq 1}$ & $(Y_n)_{n \geq 1}$ be independent Markov chains each with state space S and transition matrix P.

Let $Z_n = (X_n, Y_n)$ with state space $S \times S$.

$$p_{ij,kl} = P(Z_{n+1} = (k,l) \mid Z_n = (i,j))$$

$$= P(X_{n+1} = k \mid X_n = i)\, P(Y_{n+1} = l \mid Y_n = j)$$

(using independence)

$$= p_{ij}\, p_{kl}.$$

Since X is irreducible & aperiodic it can be shown that Z is also irreducible.

Since X is non-null persistent $\Rightarrow X$ has a unique stationary distribution π.

$$\pi_j = \sum_{k \in S} \pi_k p_{kj}$$

$$\Rightarrow \pi_i \pi_j = \left(\sum_{k_1 \in S} \pi_{k_1} p_{k_1 j} \right) \left(\sum_{k_2 \in S} \pi_{k_2} p_{k_2 j} \right)$$

$$= \sum_{k_1 \in S} \sum_{k_2 \in S} (\pi_{k_1} \pi_{k_2})\, p_{k_1 j}\, p_{k_2 i}$$

$\Rightarrow$ the stationary distribution of Z exists & is given by $\nu_{ij} = \pi_i \pi_j \Rightarrow Z$ is non-null persistent.

Now suppose that X & Y are started in different states i and j i.e. $X_0 = i$, $Y_0 = j$.

Pick a state $s \in S$ and define

$$T = \min\{n \geq 1 : Z_n = (s,s)\}$$ (first hitting time of (s,s)).

Since Z is persistent $\Rightarrow P(T<\infty)=1$.

Now consider the evolution of Z after time T.
At T, both X_n & Y_n are in state s. $\therefore$ Their evolution is coupled after that point in time and intuitively their starting state cease to matter. We make this intuition more precise below.

Let $(X_0, Y_0) = (i,j)$.

$$p_{ik}(n) = P(X_n=k)$$

$$= P(X_n=k,\ T\leq n) + P(X_n=k,\ T>n)$$

<u>Claim</u>: $P(X_n=k, T\leq n) = P(Y_n=k, T\leq n)$

<u>Proof</u>: $P(X_n=k, T\leq n) = \sum_{\alpha=1}^{n} P(X_n=k, T=\alpha)$

Now $P(X_n=k, T=\alpha) = P(T=\alpha)\cdot P(X_n=k \mid X_1\neq s, \ldots, X_{\alpha-1}\neq s, X_\alpha=s,$
$Y_1\neq s, \ldots, Y_{\alpha-1}\neq s, Y_\alpha=s)$

$= P(T=\alpha)\cdot P(X_n=k \mid X_\alpha=s, Y_\alpha=s)$ (Markov property)
$= P(T=\alpha)\, P(X_n=k \mid X_\alpha=s)$ (independence)
$= P(T=\alpha)\, P(Y_n=k \mid Y_\alpha=s)$

$\vdots \qquad \vdots$

$= P(T=\alpha, Y_n=k)$.

The conclusion follows. $\blacksquare$

Going back to the original proof, we have

$$p_{ik}(n) = P(Y_n = k, T \leq n) + P(X_n = k, T > n)$$

$$\leq P(Y_n = k) + P(T > n)$$
$$= p_{jk}(n) + P(T > n).$$

$$\Rightarrow P(T > n) \geq p_{ik}(n) - p_{jk}(n).$$

In a similar manner by running the previous argument on

$$p_{jk}(n) = P(Y_n = k)$$

we can obtain

$$P(T > n) \geq p_{jk}(n) - p_{ik}(n).$$

$$\Rightarrow \quad |p_{ik}(n) - p_{jk}(n)| \leq P(T > n) \to 0 \text{ as } n \to \infty$$
$$(\text{as } P(T < \infty) = 1).$$

$$\Rightarrow \quad p_{ik}(n) - p_{jk}(n) \to 0 \text{ as } n \to \infty, \ \forall i, j \ \& \ k.$$

$\therefore$ if the limit $p_{ik}(n)$ exists, it does not depend on i.

Next, note that

$$\pi_k - p_{jk}(n) = \sum_{i \in S} \pi_i \left(p_{ik}(n) - p_{jk}(n)\right).$$

Now, we let $n \to \infty$ on both sides.

RHS $\lim_{n\to\infty} \sum_{i\in S} \pi_i \left(p_{ik}(n) - p_{jk}(n)\right)$

$$= \sum_{i\in S} \pi_i \lim_{n\to\infty} \left(p_{ik}(n) - p_{jk}(n)\right)$$

$$= 0$$

The interchange of limit & summation is justified above by using the bounded convergence theorem since

$$|p_{ik}(n) - p_{jk}(n)| \leq 2 \qquad \forall n,$$

$$\& \quad \sum \pi_i \cdot 2 = 2 < \infty.$$

$$\Rightarrow \qquad \lim_{n\to\infty} p_{jk}(n) = \pi_k.$$

But, we know that $\pi_k = \frac{1}{\mu_k}$, so we have the proof.

Case 3: If the chain is null-persistent.
We skip the proof in this case, but the result is true.

$$\lim_{n\to\infty} p_{jk}(n) = 0 \quad \text{since } \mu_k = \infty.$$

More generally, relaxing the assumption of irreducibility we have,

Theorem: For any aperiodic state j of a Markov chain

$$p_{jj}(n) \to \frac{1}{\mu_j} \quad \text{as } n\to\infty$$

$$p_{ij}(n) \to f_{ij}/\mu_j \quad \text{as } n\to\infty$$

Corollary:

Let $X_0 = i$ & $T_{ij}(n) = \frac{1}{n} E\left[\sum_{k=1}^{n} \mathbb{1}_{\{X_k = j\}} \middle| X_0 = i\right]$

$$\Rightarrow T_{ij}(n) = \frac{1}{n} \sum_{k=1}^{n} p_{ij}(k)$$

Then, $T_{ij}(n) \longrightarrow \frac{f_{ij}}{\mu_j}$ as $n \to \infty$.

Proof:

we know that $p_{ij}(k) \longrightarrow \frac{f_{ij}}{\mu_j}$

$$\therefore \frac{1}{n} \sum_{k=1}^{n} p_{ij}(k) \longrightarrow \frac{f_{ij}}{\mu_j} \text{ as } n \to \infty$$

Example:

Discrete-Time Birth & Death Markov Chain (DT-BD-MC).

$S = \{0, 1, 2 \ldots\}$

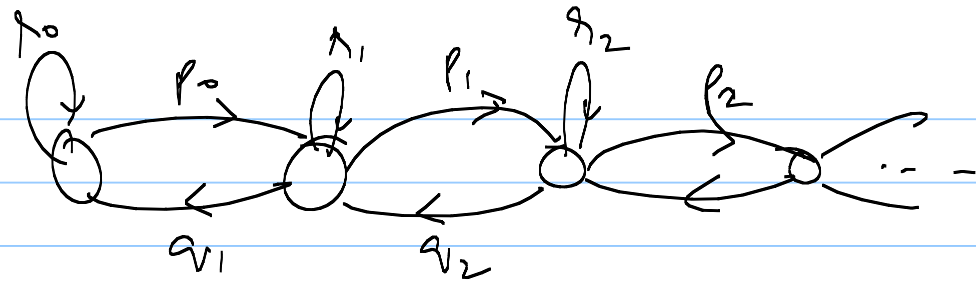
for $i \geq 1$

$$p_{ij} = \begin{cases} q_i & \text{if } j = i-1 \\ p_i & \text{if } j = i+1 \\ r_i & \text{if } j = i \end{cases}$$

where $p_i + q_i + r_i = 1$.

and

$p_{00} = r_0$, $p_{01} = p_0$ s.t. $p_0 + r_0 = 1$.



The transition matrix looks like

$$P = \begin{bmatrix} r_0 & p_0 & 0 & \cdots & & \\ q_1 & r_1 & p_1 & & & \\ 0 & q_2 & r_2 & p_2 & \cdots & \\ \cdots & & & & & \\ & & & & & \end{bmatrix}$$

Now if a stationary distribution exists, we have

$$\pi_0 = \pi_0 r_0 + \pi_1 q_1 \qquad \text{—(1)}$$

& for $j \geq 1$

$$\pi_j = \pi_{j-1} p_{j-1} + \pi_j r_j + \pi_{j+1} q_{j+1}$$

$$\Rightarrow \pi_j (p_j + q_j) = \pi_{j-1} p_{j-1} + \pi_{j+1} q_{j+1}$$

$$\Rightarrow \pi_j p_j - \pi_{j+1} q_{j+1} = \pi_{j-1} p_{j-1} - \pi_j q_j \quad \text{for } j \geq 1$$

$$\& \quad \pi_0 p_0 - \pi_1 q_1 = 0$$

Let $\beta_j = \pi_j p_j - \pi_{j+1} q_{j+1}$. Then we have

$$\beta_0 = 0 \;\&\; \beta_j = \beta_{j-1}, \; j \geq 1.$$

$$\Rightarrow \quad \pi_{j+1} = \left(\frac{p_j}{q_{j+1}}\right)\pi_j \quad , \quad \text{for } j \geq 0.$$

Iterating this we obtain

$$\pi_j = \underbrace{\left(\frac{p_0 p_1 \cdots p_{j-1}}{q_1 q_2 \cdots q_j}\right)}_{\gamma_j} \pi_0.$$

$$= \gamma_j \pi_0$$

& define $\gamma_0 = 1$.

Now, $\sum_{j\geq 0} \pi_j = \pi_0 \left(\sum_{j\geq 0} \gamma_j\right).$

Theorem: If $\sum_{j=0}^{\infty} \gamma_j < \infty$, then a DTBDMC has a unique stationary distribution given by

$$\pi_0 = \frac{1}{\sum_{j=0}^{\infty} \gamma_j} \quad \& \quad \pi_j = \gamma_j \pi_0.$$

If $\sum_{j=0}^{\infty} \gamma_j = \infty$, a stationary distribution does not exist.